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The shuffle variant of Terai's conjecture on exponential Diophantine equations

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Abstract. Let p, q and r be positive integers with $p, q, r \geq 2$, and let a, b and c be pair-wise relatively prime positive integers such that $a^p + b^q = c^r$. Terai's conjecture states that apart from a handful of exceptions, the exponential Diophantine equation $a^x + b^y = c^z$ in positive integers x, y and z, has the unique solution $(x, y, z) = (p, q, r)$. In this paper we consider a similar problem (which we call the shuffle variant of Terai's problem). Our problem states that apart from a handful of exceptions, the exponential Diophantine equation $c^x + b^y = a^z$ in positive integers x, y and z, has the unique solution $(x, y, z) = (1, 1, p)$ if $q = r = 2$ and $c = b + 1$, and no solutions otherwise. We establish several results on our problem by the theory of linear forms in two archimedean and non-archimedean logarithms with various elementary techniques. In particular we prove that the shuffle variant of Terai's problem is true if $q = r = 2$ and $c = b + 1$.

1. Introduction

We consider the exponential Diophantine equation

$$a^x + b^y = c^z \tag{1.1}$$

in positive integers x, y and z, where a, b and c are fixed pair-wise relatively prime positive integers. There is an interesting problem on equation (1.1) (cf. [18], [19], [23]):

Conjecture 1. *Let p, q and r be positive integers with $p, q, r \geq 2$, and let a, b and c be pair-wise relatively prime positive integers such that $a^p + b^q = c^r$.*

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Assume that (a,b,c) *is not any of the following cases* (*up to permutation of* a *and* b)*:* $(2,7,3),(2,2^{p-2}-1,2^{p-2}+1)$*;* $p\geq 3$*. Then* (1.1) *has the unique solution* $(x,y,z)=(p,q,r)$*.*

In what follows, we call this Terai's conjecture. Most known results on Terai's conjecture concern the case of $p=q=2$. For $r\geq 2$ we can find that all of the relatively prime positive integers a,b and c satisfying $a^2+b^2=c^r$ are given by (cf. [8, p. 466]):

$$a=|A|,\quad b=|B|,\quad c=m^2+n^2, \tag{I}$$

where m, n are relatively prime positive integers of different parities with $m>n$, and A, B are the integers defined by $A+B\sqrt{-1}=(m+n\sqrt{-1}\,)^r$. There are a number of partial results in this case. Many of them concern the case where $m\equiv 2 \pmod 4$ or $n=1$ (see for example [4], [5], [6], [7], [9], [13], [14], [16], [19], [24] and their references).

In [18] the author considered the case of $q=r=2$ and obtained results. For $p\geq 3$ we can find that all of the relatively prime positive integers a, b and c satisfying $a^p+b^2=c^2$ are given by (cf. [8, p. 465]):

$$a=m^2-n^2,\quad b=\frac{(m+n)^p-(m-n)^p}{2},\quad c=\frac{(m+n)^p+(m-n)^p}{2}, \tag{II}$$

where m, n are relatively prime positive integers of different parities with $m>n$, or

$$a=2mn,\quad b=\left|2^{p-2}m^p-n^p\right|,\quad c=2^{p-2}m^p+n^p, \tag{III}$$

where m,n are relatively prime positive integers with $n\equiv 1 \pmod 2$.

In case $p=q=r=2$, Terai's conjecture coincides with Jeśmanowicz' conjecture [10], which is the origin of Terai's conjecture (cf. [3, 4, 15] for Jeśmanowicz' conjecture). Let (a,b,c) be a primitive Pythagorean triple, that is, a, b, c are relatively prime positive integers satisfying $a^2+b^2=c^2$ (we may assume that b is even). In this case, we consider the equation

$$c^x+b^y=a^z \tag{1.2}$$

where $x,y,z\in\mathbb{N}$. In [20] we proposed an analogue of Jeśmanowicz' conjecture as follows.

Conjecture 2. *Let* (a,b,c) *be a primitive Pythagorean triple such that* $a^2+b^2=c^2$ *and* b *is even. Then* (1.2) *has the unique solution* $(x,y,z)=(1,1,2)$ *if* $c=b+1$*, and no solutions if* $c>b+1$*.*

We call this the *shuffle* variant of Jeśmanowicz' problem. In this paper we propose a similar problem for much more general cases as follows.

Conjecture 3. *Let p, q and r be positive integers with $p, q, r \geq 2$, and let a, b and c be pair-wise relatively prime positive integers such that $a^p + b^q = c^r$. Assume that (a, b, c) is not any of the following cases: $(2, 7, 3), (2, 2^{p-2} - 1, 2^{p-2} + 1)$; $p \geq 3$. Then (1.2) has the unique solution $(x, y, z) = (1, 1, p)$ if $q = r = 2$ and $c = b + 1$, and no solutions otherwise.*

We call this the *shuffle* variant of Terai's problem. In case where $q = r = 2$ and $c = b + 1$, since $a^p = c^2 - b^2 = (c + b)(c - b) = c + b$, we find that (1.2) always has the solution $(x, y, z) = (1, 1, p)$. Remark that

$$2^5 + 7^2 = 3^4; \quad 3^2 + 7 = 2^4,$$

$$2^p + (2^{p-2} - 1)^2 = (2^{p-2} + 1)^2; \quad (2^{p-2} + 1) + (2^{p-2} - 1) = 2^{p-1} \ (p \geq 3).$$

We also remark that $q = r = 2$ and $c = b + 1$ if and only if a, b and c are given by (II) with $m = n + 1$. It seems that our problem, as well as Terai's conjecture, is very difficult to solve.

In this paper we first prove three results concerning the case where a, b and c are given by (I), (II) and (III).

Theorem 1. *Let r be a positive integer such that $r \equiv 2 \pmod 8$, and let a, b, c be given by (I). Assume that $m > 2r/\pi$ and $n = 1$. Then Conjecture 3 is true.*

Theorem 2. *Let p be a positive integer such that $p \equiv \pm 2 \pmod{12}$, and let a, b, c be given by (II). Assume that $n = 1$. Then Conjecture 3 is true.*

Theorem 3. *Let p be a positive integer with $p \geq 3$, and let a, b, c be given by (III). Assume that $n = 1$. Then (1.2) has a solution only if $m = 1$. If $m = 1$, then all of the solutions of (1.2) are given by*

$$(x, y, z) = \begin{cases} (1, k, 2); \ k \geq 1 & \text{if } p = 3, \\ (1, 1, 3), (1, 3, 5), (3, 1, 7) & \text{if } p = 4, \\ (1, 1, p - 1) & \text{if } p \geq 5. \end{cases}$$

Finally we prove that the first part of Conjecture 3 is true.

Theorem 4. *Let p be a positive integer with $p \geq 2$, and let a, b, c be pair-wise relatively prime positive integers such that $a^p + b^2 = c^2$ and $c = b + 1$. Then (1.2) has the unique solution $(x, y, z) = (1, 1, p)$.*

2. Linear forms in two logarithms

In this section we will quote preliminary results on linear forms in two archimedean and non-archimedean logarithms. We denote the sets of positive integers, integers, rational numbers and real numbers by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively.

For any algebraic number α of degree d over $\mathbb{Q}$, we define as usual the absolute logarithmic height of α by

$$\mathrm{h}(\alpha) = \frac{1}{d}\left(\log|c_0| + \sum_{i=1}^{d} \log\max\bigl\{1, |\alpha^{(i)}|\bigr\}\right),$$

where c_0 is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and the $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(d)}$ are the conjugates of α in the field of complex numbers.

Let α_1 and α_2 be two non-zero algebraic numbers with $|\alpha_1| \geq 1$ and $|\alpha_2| \geq 1$, and let $\log\alpha_1$ and $\log\alpha_2$ be any determination of their logarithms. We consider the linear form in two logarithms

$$\Lambda = b_2 \log\alpha_2 - b_1 \log\alpha_1,$$

where $b_1, b_2 \in \mathbb{N}$. Put $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}]$. Define $b' = b_1/(D\log A_2) + b_2/(D\log A_1)$, where $A_1, A_2 > 1$ are real numbers such that

$$\log A_i \geq \max\bigl\{\mathrm{h}(\alpha_i),\ |\log\alpha_i|/D,\ 1/D\bigr\} \quad (i = 1, 2).$$

We choose to use a result due to Laurent [12, Corollary 2] with $m = 10$ and $C_2 = 25.2$.

Proposition 1. *With the above notation, suppose that $\alpha_1, \alpha_2, \log\alpha_1, \log\alpha_2$ are real and positive. If α_1 and α_2 are multiplicatively independent, then we have the lower estimate*

$$\log|\Lambda| \geq -25.2\bigl(\max\{\log b' + 0.38, 10\}\bigr)^2 \log\alpha_1 \log\alpha_2.$$

Applying Proposition 1 to (1.2), we show the following lemma.

Lemma 2.1. *Let (x, y, z) be a solution of (1.2). Then*

$$\frac{x}{\log a} < \frac{y\log b}{\log a \log c} + 25.2\left(\max\left\{\log\left(\frac{x}{\log a} + \frac{z}{\log c}\right) + 0.38, 10\right\}\right)^2.$$

Proof. Since $z\log a = \log(c^x+b^y) = x\log c+\log(1+b^y c^{-x}) < x\log c+b^y c^{-x}$, we see that

$$(0<)\ \Lambda := z\log a - x\log c < b^y c^{-x}.$$

Remark that a and c are relatively prime positive integers greater than 1. Therefore, they are multiplicatively independent. To use Proposition 1, we set $\alpha_1 = c$, $\alpha_2 = a$, $b_1 = x, b_2 = z$. Then $D = 1, \mathrm{h}(a) = a, \mathrm{h}(c) = c$. We may take $A_1 = a$ and $A_2 = c$. It follows from Proposition 1 that

$$-25.2\left(\max\left\{\log\left(\frac{x}{\log a}+\frac{z}{\log c}\right)+0.38, 10\right\}\right)^2 \log a\log c < \log\Lambda < y\log b - x\log c.$$

The desired conclusion follows from this. □

Next, we shall quote a result on linear forms in ℓ-adic logarithms due to Bugeaud [2]. Here we consider the case where $y_1 = y_2 = 1$ in the notation from [2, p.375]

Let ℓ be a prime number. Let a_1 and a_2 be non-zero integers prime to ℓ. Let g be the least positive integer such that

$$\mathrm{ord}_\ell(a_1^g - 1) \geq 1, \quad \mathrm{ord}_\ell(a_2^g - 1) \geq 1,$$

where we denote the ℓ-adic valuation by $\mathrm{ord}_\ell(\,\cdot\,)$. Assume that there exists a real number E such that

$$1/(\ell-1) < E \leq \mathrm{ord}_\ell(a_1^g - 1).$$

We consider the integer

$$\Lambda = a_1^{b_1} - a_2^{b_2},$$

where $b_1, b_2 \in \mathbb{N}$. We let $A_1, A_2 > 1$ be real numbers such that

$$\log A_i \geq \max\{\log|a_i|, E\log\ell\} \quad (i=1,2),$$

and we put $b' = b_1/\log A_2 + b_2/\log A_1$.

Proposition 2. *With the above notation, if a_1 and a_2 are multiplicatively independent, then we have the upper estimates*

$$\mathrm{ord}_\ell(\Lambda) \leq \frac{36.1g}{E^3(\log\ell)^4}\left(\max\{\log b' + \log(E\log\ell)+0.4, 6E\log\ell, 5\}\right)^2 \log A_1\log A_2,$$

$$\mathrm{ord}_\ell(\Lambda) \leq \frac{53.8g}{E^3(\log\ell)^4}\left(\max\{\log b' + \log(E\log\ell)+0.4, 4E\log\ell, 5\}\right)^2 \log A_1\log A_2,$$

if ℓ is odd or if $\ell = 2$ and $\mathrm{ord}_2(a_2-1) \geq 2$. Else, we have

$$\mathrm{ord}_2(\Lambda) \leq 208\left(\max\{\log b' + 0.04, 10\}\right)^2 \log A_1\log A_2.$$

3. Proof of Theorem 1

Let r be a positive integer such that $r \equiv 2 \pmod 8$, and let m be a positive even integer m. We define integers a, b and c by (I) with $n = 1$. Then

$$A = m^r - \binom{r}{2} m^{r-2} + \cdots + \binom{r}{r-2} m^2 - 1,$$
$$B = \binom{r}{1} m^{r-1} - \binom{r}{3} m^{r-3} + \cdots - \binom{r}{r-3} m^3 + \binom{r}{r-1} m.$$

Lemma 3.1. *If $m > 2r/\pi$, then the following* (i) *and* (ii) *hold.*

(i) $a = A$ *and* $b = B$.

(ii) $\max\{a, b\} < c^{r/2} < \min\{a^2, b^2\}$.

Proof. (i) We define the real number θ $(0 < \theta < \pi/2)$ by $\tan\theta = 1/m$. Since $A = c^{r/2}\cos(r\theta)$ and $B = c^{r/2}\sin(r\theta)$, if $m > 2r/\pi$, then

$$r\theta = r\arctan(1/m) < r/m < \pi/2.$$

Hence both A and B are positive.

(ii) This follows from the fact that $\{a, b, c^{r/2}\}$ forms a Pythagorean triple. $\square$

We consider the equation

$$(m^2+1)^x + b^y = a^z \tag{3.1}$$

where $m, x, y, z \in \mathbb{N}$ and m is even. Let (m, x, y, z) be a solution of (3.1). Assume that $m > 2r/\pi$. Then $a = A$ and $b = B$ by (i) in Lemma 3.1. We prepare several lemmas.

Lemma 3.2. *x is odd.*

Proof. Since

$$\binom{r}{1} - \binom{r}{3} + \cdots - \binom{r}{r-3} + \binom{r}{r-1} = \Im\big((1+\sqrt{-1}\,)^r\big) = 2^{r/2}\sin(\pi r/4) = 2^{r/2},$$

we observe that

$$(m^2+1)^x \equiv 2^x \pmod{m^2-1},$$
$$b \equiv \left(\binom{r}{1} - \binom{r}{3} + \cdots - \binom{r}{r-3} + \binom{r}{r-1}\right) m \equiv 2^{r/2} m \pmod{m^2-1},$$
$$a \equiv 1 - \binom{r}{2} + \cdots + \binom{r}{r-2} - 1 \equiv 0 \pmod{m^2-1}.$$

It follows from (3.1) that $2^x \equiv -(2^{r/2}m)^y \pmod{m^2-1}$. Since $r/2$ is odd and m is even, we see that $\left(\frac{2}{m^2-1}\right)^x = -\left(\frac{2m}{m^2-1}\right)^y = -1^y = -1$, where we denote the Jacobi symbol by $\left(\frac{*}{*}\right)$. Hence x is odd. □

Lemma 3.3. *z is even.*

Proof. Taking (3.1) modulo $2m$, we find that $(-1)^z \equiv 1 \pmod{2m}$. Hence z is even since $2m \geq 3$. □

By Lemma 3.3, we can write $z = 2Z$, where $Z \in \mathbb{N}$.

Lemma 3.4. *$m = r$ and $y = 1$.*

Proof. Since x is odd by Lemma 3.2, we observe that

$$(m^2+1)^x \equiv m^2+1, \quad b \equiv rm, \quad a^{2Z} \equiv \left(\binom{r}{r-2}m^2 - 1\right)^{2Z} \equiv 1 \pmod{2m^2}.$$

It follows from (3.1) that $(rm)^y \equiv m^2 \pmod{2m^2}$. If $y > 1$, then, since r is even, we find that $m^2 \equiv 0 \pmod{2m^2}$. This is clearly absurd. Hence $y = 1$, so $rm \equiv m^2 \pmod{2m^2}$, that is, $r \equiv m \pmod{2m}$. In particular, m divides r. Therefore, we obtain $m = r$ since $r/m < \pi/2 < 2$. □

From Lemma 3.4 we see that

$$\begin{aligned} a &= r^r - \binom{r}{2}r^{r-2} + \cdots + \binom{r}{r-2}r^2 - 1, \\ b &= \binom{r}{1}r^{r-1} - \binom{r}{3}r^{r-3} + \cdots - \binom{r}{r-3}r^3 + \binom{r}{r-1}r, \\ c &= r^2+1 \end{aligned}$$

and $r\theta = r\arctan(1/r) < 1$. In particular, $b/a = \tan(r\theta) < 1.6$.

Lemma 3.5. *The following* (i)–(iv) *hold.*

(i) $x + 1 \leq rZ$.

(ii) $x + 1 \equiv rZ \pmod{r^2/d(r)}$, *where*

$$d(r) = \begin{cases} 1 & \text{if } r \not\equiv 0 \pmod 3, \\ 3 & \text{if } r \equiv 0 \pmod 3. \end{cases}$$

(iii) $c^x > b$.

(iv) $x \geq rZ/2$.

Proof. (i) From (ii) in Lemma 3.1 we see that $c^x < c^x + b = a^{2Z} < c^{rZ}$, so $x + 1 \le rZ$.

(ii) Since $d(r)(r-1)(r-2)$ is a multiple of 6, we observe that

$$(r^2+1)^x \equiv r^2x + 1 \pmod{r^4},$$

$$b \equiv -\frac{d(r)(r-1)(r-2)}{6}\frac{r^4}{d(r)} + r^2 \equiv r^2 \pmod{r^4/d(r)},$$

$$a^{2Z} \equiv \left(\binom{r}{r-2}r^2 - 1\right)^{2Z} \equiv -r^3(r-1)Z + 1 \equiv r^3Z + 1 \pmod{r^4}.$$

It follows from (3.1) that $x + 1 \equiv rZ \pmod{r^2/d(r)}$.

(iii) Suppose that $c^x \le b$. Then $c^x \le b-1$ since b is even and c is odd. Hence $a^2 \le a^{2Z} = c^x + b \le 2b - 1 < 3.2a - 1$. But this does not hold.

(iv) From (ii) in Lemma 3.1 and (iii) in this lemma we see that $c^{rZ/2} < a^{2Z} = c^x + b < 2c^x$, so $3^{rZ/2-x} \le c^{rZ/2-x} < 2$, which implies that $x \ge rZ/2$. □

Lemma 3.6. *We have the upper estimate* $x < 2521\log a$.

Proof. From (iii) in Lemma 3.5 we find that $a^{2Z} = c^x + b < 2c^x$. Since $b < 1.6a$ and $c \ge 5$, it follows from Lemma 2.1 that

$$\frac{x}{\log a} < 1 + 25.2\left(\max\left\{\log\left(\frac{2x}{\log a} + 1\right) + 0.38, 10\right\}\right)^2.$$

This implies that $x/\log a < 2521$. □

In what follows, we put $\Lambda_1 = z\log a - x\log c$ (> 0). Since $\Lambda_1 < b/c^x < 1/c^{x-r/2}$, it follows from Lemma 3.6 that

$$\left|\frac{\log c}{\log a} - \frac{z}{x}\right| < \frac{1}{xc^{x-r/2}\log a} < \frac{2521}{x^2c^{x-r/2}}.$$

In the proof of the following lemma, we use a reduction method via continued fraction expansions.

Lemma 3.7. $x + 1 = rZ$.

Proof. In case $r = 2$, (3.1) is $5^x + 4 = 3^{2Z}$. Since $(3^Z+2)(3^Z-2) = 5^x$ and $\gcd(3^Z+2, 3^Z-2) = 1$, we see that $3^Z - 2 = 1$. Hence $Z = 1$, so $x = 1$.

Suppose that $x + 1 \ne rZ$. We will observe that this leads to a contradiction. Then, by the first remark, we see that $r \ne 2$, so $r \ge 10$ since $r \equiv 2 \pmod 8$.

Furthermore, (i) and (ii) in Lemma 3.5 yield $rZ \geq r^2/d(r)+x+1$. Since we know from (iv) in Lemma 3.5 that $x \geq rZ/2$, we see that $rZ \geq r^2/d(r) + rZ/2 + 1$, hence $x \geq rZ/2 \geq r^2/d(r) + 1$. It follows from Lemma 3.6 that

$$r^2/d(r) + 1 < 2521 \log a = 2521 \log \big((r^2+1)^{r/2} \cos(r \arctan(1/r))\big).$$

This implies that $r \leq 25586$ if $r \not\equiv 0 \pmod 3$, and $r \leq 85914$ if $r \equiv 0 \pmod 3$.

On the other hand, since $r \geq 10$, we see that $c^{x-r/2} \geq (r^2+1)^{r^2/d(r)-r/2+1} > 5042$, hence

$$\left|\frac{\log c}{\log a} - \frac{z}{x}\right| < \frac{1}{2x^2}.$$

Therefore, $\frac{z}{x}$ is a convergent in the simple continued fraction expansion of $\frac{\log c}{\log a}$. Hence we can write $\frac{z}{x} = \frac{p_s}{q_s}$, which is the s-th such convergent. Then

$$\left|\frac{\log c}{\log a} - \frac{p_s}{q_s}\right| > \frac{1}{(a_{s+1}+2)q_s^2},$$

where a_{s+1} is the $(s+1)$-st partial quotient to $\frac{\log c}{\log a}$ (cf. [11]). Since $q_s \leq x$, it follows that $a_{s+1} + 2 > x q_s^{-2} c^{x-r/2} \log a \geq x^{-1} c^{x-r/2} \log a$, so

$$a_{s+1} + 2 > \frac{c^{r^2/d(r)-r/2+1} \log a}{r^2/d(r)+1}.$$

We can numerically check, for each r under consideration, that the above inequality does not hold for any s satisfying $q_s < 2521 \log a$. This is a contradiction. We conclude that $x + 1 = rZ$. □

Proof of Theorem 1. It suffices to show that $r = 2$. From Lemma 3.7 we know that $c^{rZ-1} + b = a^{2Z}$. Since $a^2 + b^2 = c^r$, we observe that $c^{rZ-1} \equiv a^{2Z} \equiv (c^r - b^2)^Z \equiv c^{rZ} \pmod b$. Since $\gcd(b, c) = 1$, it follows that $c \equiv 1 \pmod b$, that is, b divides r^2. Hence $b = r^2$ since b is a multiple of r^2. Then

$$\binom{r}{1} r^{r-1} - \binom{r}{3} r^{r-3} + \cdots - \binom{r}{r-3} r^3 = b - r^2 = 0.$$

If $r > 2$, taking this modulo r^5, we find that $\binom{r}{r-3} r^3 = r^4(r-1)(r-2)/6 \equiv 0 \pmod{r^5}$. This implies that $2 \equiv 0 \pmod r$, in contradiction with $r \geq 10$, hence $r = 2$. We complete the proof of Theorem 1. □

4. Proof of Theorem 2

Let p be a positive integer such that $p \equiv \pm 2 \pmod{12}$, and let m be a positive even integer m. We define integers a, b and c by (II) with $n = 1$. Then

$$b=\binom{p}{1}m^{p-1}+\binom{p}{3}m^{p-3}+\cdots+\binom{p}{p-1}m,\ c=m^p+\binom{p}{2}m^{p-2}+\cdots+\binom{p}{p-2}m^2+1.$$

We consider the equation

$$c^x + b^y = (m^2-1)^z \tag{4.1}$$

where $m, x, y, z \in \mathbb{N}$ and m is even. Let (m, x, y, z) be a solution of (4.1). We prepare several lemmas.

Lemma 4.1. *x is odd and z is even.*

Proof. This can be proved similarly to the proofs of Lemmas 3.2 and 3.3. □

By Lemma 4.1, we can write $z = 2Z$, where $Z \in \mathbb{N}$.

Lemma 4.2. *We have $y = 1$, $p \equiv 0 \pmod{m}$, $4Z - px + 2p/m \equiv 0 \pmod{m^2}$, $m^2(m-1) < 5044p$ and*

$$x < 2521\log(m^2-1),\ \max\{px/4, p/2\} \le Z < (2521/2)p\log(m+1).$$

Proof. Since $\binom{p}{2}$ is odd and x is odd by Lemma 4.1, we observe that

$$c^x \equiv \binom{p}{2}m^2x + 1 \equiv m^2+1, \quad b \equiv pm, \quad (m^2-1)^{2Z} \equiv 1 \pmod{2m^2}.$$

It follows from (4.1) that $(pm)^y \equiv m^2 \pmod{2m^2}$. Similarly to the proof of Lemma 3.4, we may conclude that $y = 1$ and $p \equiv 0 \pmod{m}$.

Since $p \equiv 0 \pmod{m}$ and $(p-1)(p-2)$ is a multiple of 6, we see that

$$\binom{p}{p-3}m^3 \equiv 0 \pmod{m^4}, \quad \binom{p}{p-2}m^2 \equiv -pm^2/2 \pmod{m^4/2}.$$

So we observe that

$$c^x \equiv -pm^2x/2 + 1, \quad b \equiv pm, \quad (m^2-1)^{2Z} \equiv -2m^2Z + 1 \pmod{m^4/2}.$$

It follows from (4.1) that $4Z - px + 2p/m \equiv 0 \pmod{m^2}$.

Since $b < c$ and $(m^2-1)^z = c^x + b < 2c^x$, using a similar observation in Lemma 3.6, we find that $x < 2521\log(m^2-1)$. Since $(m^2-1)^{2Z} = c^x + b \le (c+b)^x = (m+1)^{px}$, we see that $2Z \le px\log(m+1)(\log(m^2-1))^{-1} < 2521p\log(m+1)$. On the other hand, since $c \ge (m^2-1)^{p/2}$, we see that $(m^2-1)^{2Z} > c^x \ge (m^2-1)^{px/2}$, so $Z > px/4$.

If $x = 1$, then $(m+1)^p = c + b = (m^2-1)^{2Z} = (m+1)^{2Z}(m-1)^{2Z}$. Since $\gcd(m+1, m-1) = 1$, we see that $m = 2$, so $Z = p/2$. In particular, we always find that $Z \ge p/2$.

Since $4Z - px > 0$, $px \ge 2Z\log(m^2-1)\big(\log(m+1)\big)^{-1}$ and $4Z - px + 2p/m \equiv 0 \pmod{m^2}$, we may conclude that

$$m^2 \le 4Z - px + \frac{2p}{m} \le \frac{2Z\log\left(1 + 2/(m-1)\right)}{\log(m+1)} + \frac{2p}{m} < \left(\frac{5042}{m-1} + \frac{2}{m}\right)p.$$

This implies that $m^2(m-1) < 5044p$. □

Lemma 4.3. *The following* (i) *and* (ii) *hold.*

(i) $m-1$ *does not have any prime factors congruent to* 3 *modulo* 4*, and we can write* $m = 3k+2$ *for some non-negative integer* k.

(ii) *If* $x > 1$*, then we have the lower estimate* $x \ge 1 + 2\cdot 3^{p-e(p)}$*, where*

$$e(p) = \frac{36.1}{3(\log 3)^3}\log(5044p)\big(\max\{\log(p+1)+0.4, 6\log 3\}\big)^2.$$

Proof. By Lemma 4.2, we know that $z = 2Z \ge p$. We observe that

$$2^x c^x = ((m+1)^p + (m-1)^p)^x \equiv (m-1)^{px}, \quad 2^x b \equiv -2^{x-1}(m-1)^p \pmod{(m+1)^p}.$$

It follows from (4.1) that $(m-1)^{px} \equiv 2^{x-1}(m-1)^p \pmod{(m+1)^p}$. Since $\gcd(m+1, m-1) = 1$, we find that

$$(m-1)^{p(x-1)} \equiv 2^{x-1} \pmod{(m+1)^p}.$$

Similarly, multiplying (4.1) by 2^x and taking modulo $(m-1)^p$, we may show that

$$(m+1)^{p(x-1)} + 2^{x-1} \equiv 0 \pmod{(m-1)^p}.$$

Since $x-1$ is even by Lemma 4.1, we see from the above congruence that $m-1$ is not divisible by any prime congruent to 3 modulo 4, in particular, by 3. Also, since $p \not\equiv 0 \pmod 3$ and m is a divisor of p by Lemma 4.2, we see that m is not divisible by 3. Therefore, $m \equiv 2 \pmod 3$, that is, $m = 3k+2$ for some non-negative integer k. Since $m+1$ is divisible by 3, it follows that $(m-1)^{p(x-1)} \equiv 2^{x-1} \pmod{3^p}$. From (P1.2) in [21, p.11] we observe that

$$\begin{aligned} p &\le \mathrm{ord}_3\left((m-1)^{p(x-1)} - 2^{x-1}\right) = \mathrm{ord}_3\left((m-1)^{2p\left(\frac{x-1}{2}\right)} - 2^{2\left(\frac{x-1}{2}\right)}\right) \\ &= \mathrm{ord}_3\left(\frac{(m-1)^{2p\left(\frac{x-1}{2}\right)} - 2^{2\left(\frac{x-1}{2}\right)}}{(m-1)^{2p} - 2^2}\right) + \mathrm{ord}_3\left((m-1)^{2p} - 2^2\right) \\ &= \mathrm{ord}_3\left(\frac{x-1}{2}\right) + \mathrm{ord}_3\left((m-1)^{2p} - 2^2\right) = \mathrm{ord}_3(x-1) + \mathrm{ord}_3(\Lambda_2), \end{aligned}$$

where $\Lambda_2 = (m-1)^p - (-2)$. Then $\Lambda_2 = (3k+1)^p + 2 \equiv 3(kp+1) \pmod 9$. We remark that $\{0, p, -p\}$ is a complete residue system modulo 3. If $k \not\equiv -p \pmod 3$, then $\mathrm{ord}_3(\Lambda_2) = 1$, so $\mathrm{ord}_3(x-1) \ge p-1$, hence $x \equiv 1 \pmod{2 \cdot 3^{p-1}}$.

Finally, we will consider the case where $k \equiv -p \pmod 3$. In this case, since $k \not\equiv 0 \pmod 3$, we see that $m > 2$, so $m \ge 4$. We use Proposition 2 to find an upper bound for $\mathrm{ord}_3(\Lambda_2)$. For this we put $\ell = 3$, $a_1 = m-1$, $a_2 = -2$, $b_1 = p$, $b_2 = 1$. Then $g = 1$, and we may take $E = 1$, $A_1 = m-1$, $A_2 = 3$. We put $b' = p/\log 3 + 1/\log(m-1)$ $(\le (p+1)/\log 3)$. Since $(m-1)^3 < m^2(m-1) < 5044p$ by Lemma 4.2, it follows from Proposition 2 that we may take e_p as desired. □

Proof of Theorem 2. Suppose that $x > 1$. Then Lemma 4.2 and (ii) in Lemma 4.3 yield

$$(p - e(p))\log 3 < \log(x/2) < \log(2521\log m) < \log\bigl(2521\log\bigl((5044p)^{1/3} + 1\bigr)\bigr).$$

This implies that $p \le 17066$. Hence m, x and Z are also bounded and reduced by Lemmas 4.2 and 4.3. In these cases, we can find a contradiction by using continued fraction expansion similar to the proof of Theorem 1. We conclude that $x = 1$. Hence $(m, z) = (2, p)$ as we observed in the proof of Lemma 4.2. This completes the proof of Theorem 2. □

5. Proof of Theorem 3

Let p be a positive integer with $p \ge 3$, and let m be a positive integer. Then we consider the equation

$$(2^{p-2}m^p + 1)^x + (2^{p-2}m^p - 1)^y = (2m)^z \qquad (5.1)$$

where $x, y, z \in \mathbb{N}$. In case $m = 1$, (5.1) is $(2^{p-2}+1)^x + (2^{p-2}-1)^y = 2^z$. If $p = 3$, then $3^x + 1 = 2^z$. Since $z > 1$ and any power of 3 is congruent to 1 or 3 modulo 8, it follows that $z = 2$, so $x = 1$. If $p \geq 4$, then by the result of Scott [22, Theorem 6], we may conclude that all of the solutions are given by $(x, y, z) = (1, 1, 3), (1, 3, 5), (3, 1, 7)$ if $p = 4$, and $(x, y, z) = (1, 1, p-1)$ if $p \geq 5$.

In case $m = 2$, (5.1) is $(4^{p-1}+1)^x + (4^{p-1}-1)^y = 2^{2z}$. Taking this modulo 3, we have $(-1)^x \equiv 1 \pmod 3$. Hence x is even. Then taking the above modulo 4, we have $(-1)^y \equiv -1 \pmod 4$. Hence y is odd. Since $(4^{p-1}-1)^y = (2^z + (4^{p-1}+1)^{x/2})(2^z - (4^{p-1}+1)^{x/2})$, and the two factors on the right-hand side are relatively prime, we can write $2^z + (4^{p-1}+1)^{x/2} = u^y$ and $2^z - (4^{p-1}+1)^{x/2} = v^y$ for some positive odd integers u and v. We note that $y > 1$. Adding the first equation and the second one, we have $(u+v)w = 2^{z+1}$, where $w = (u^y + v^y)/(u+v) = u^{y-1} - u^{y-2}v + \cdots - uv^{y-2} + v^{y-1}$ is a positive integer. Since w is a sum of y odd integers, we see that w is odd. Hence $w = 1$, so $y = 1$. This is a contradiction.

In what follows, we consider the case of $m \geq 3$. We define integers a, b and c by (III) with $n = 1$. Remark that $c > b \geq (2m)^{p-1}$ $(= a^{p-1})$.

Suppose that there exists a solution (x, y, z) of (5.1). We will observe that this leads to a contradiction. For this we prepare several lemmas.

Lemma 5.1. *$z \geq p$.*

Proof. Since $a^z = c^x + b^y \geq c + b = (2m)^{p-1}m > (2m)^{p-1} = a^{p-1}$, the lemma holds. □

Lemma 5.2. *Both x and y are odd.*

Proof. We observe that

$$c^x \equiv 2^{p-2}m^p x + 1, \quad b^y \equiv (-1)^{y-1}2^{p-2}m^p y + (-1)^y \pmod{2^{p-1}m^p}.$$

It follows from (5.1) and Lemma 5.1 that

$$2^{p-2}m^p x + (-1)^{y-1}2^{p-2}m^p y + 1 + (-1)^y \equiv 0 \pmod{2^{p-1}m^p}.$$

Reducing this modulo $2m$, we have $(-1)^y \equiv -1 \pmod{2m}$. Hence y is odd since $2m \geq 3$. Therefore, the above congruence gives that $x + y \equiv 0 \pmod 2$, so x is odd. □

Lemma 5.3. *$z \geq (p-1)\max\{x, y\}$ and $z \geq 2p$.*

Proof. Since $c > b \geq a^{p-1}$, it follows from (5.1) that

$$z > (\log \max\{c^x, b^y\})/\log a \geq \max\{x, y\}(\log b)/\log a \geq (p-1)\max\{x, y\}.$$

Suppose that $z < 2p$. Then $\max\{x, y\} \leq (2p-1)/(p-1) < 3$, so $x = y = 1$ by Lemma 5.2, hence $(2m)^z = 2^{p-1}m^p$. This contradicts Lemma 5.1. We conclude that $z \geq 2p$. □

Lemma 5.4. $x + y \equiv 0 \pmod{2^{p-2}m^p}$.

Proof. From Lemmas 5.2 and 5.3 we observe that

$$c^x \equiv 2^{p-2}m^p x + 1, \quad b^y \equiv 2^{p-2}m^p y - 1, \quad a^z \equiv 0 \pmod{2^{2p-4}m^{2p}}.$$

It follows from (5.1) that $x + y \equiv 0 \pmod{2^{p-2}m^p}$. □

Toward a contradiction we will use Proposition 2. We remark that both b, c are odd and $b \not\equiv c \pmod 4$. We will consider the cases $p \geq 4$ and $p = 3$ separately.

- The case of $p \geq 4$.

We assume that $p \geq 4$. Then $c \equiv -b \equiv 1 \pmod 4$. We put $\ell = 2$, $a_1 = c$, $a_2 = -b$, $b_1 = x$, $b_2 = y$ and $\Lambda_3 = c^x - (-b)^y = (2m)^z$. Then $g = 1$. Since $c - 1 = 2^{p-2}m^p$ and $c > b \geq (2m)^{p-1} > 2^{p-2}$, we may take $E = p - 2$, $A_1 = c$, $A_2 = b$. We put $b' = x/\log b + y/\log c$. We write $M = \max\{x, y\}$. Then Lemma 5.4 yields $M \geq (x + y)/2 \geq 2^{p-3}m^p$ (> 32), and so $c - 1 = b + 1 = 2^{p-2}m^p \leq 2M$. Since $c > b \geq (2m)^{p-1}$, we see that

$$b' \leq \frac{2M}{\log b} \leq \frac{2M}{(p-1)\log(2m)} \leq \frac{M}{(p-1)\log 2}.$$

Combining Proposition 2 with Lemma 5.3, we have

$$(p-1)M \leq z \leq \frac{36.1\log(2M-1)\log(2M+1)}{(\log 2)^4(p-2)^3}\big(\max\{\log M + 0.4, 6(p-2)\log 2\}\big)^2.$$

If $\log M + 0.4 \leq 6(p-2)\log 2$, then

$$(\log 2)^2(p-1)(p-2) \leq 1299.6M^{-1}\log(2M-1)\log(2M+1).$$

The right-hand side of the above inequality is a decreasing function on $M \geq 4$. Since $M \geq 2^{p-3}m^p$, it follows that

$$(\log 2)^2(p-1)(p-2)2^{p-3}m^p \leq 1299.6\log(2^{p-2}m^p - 1)\log(2^{p-2}m^p + 1).$$

This implies that $p \leq 6$ and $m \leq 13$. In these cases, we may observe that (5.1) has no solutions. This is a contradiction. Similarly, in the case where $\log M + 0.4 > 6(p-2)\log 2$, using the fact that $M \geq 2^{p-3}m^p \geq 32$, we can find a contradiction.

- The case of $p = 3$.

We assume that $p = 3$. First, we suppose that m is even. Then $c \equiv -b \equiv 1 \pmod 4$. Hence we may put the values of ℓ, a_1, a_2, b_1, b_2, Λ_3, g, A_1, A_2 as in the case of $p \geq 4$. Since $c - 1 = 2m^3 \equiv 0 \pmod{2^4}$ and $b > 16$, we may take $E = 4$. Similarly to the case of $p \geq 4$, combining Proposition 2 with Lemma 5.3, we have

$$4M \leq 2z \leq \frac{36.1}{64(\log 2)^4}\Big(\max\big\{\log(2M)+0.4, 24\log 2\big\}\Big)^2 \log(2M-1)\log(2M+1).$$

This implies that $M \leq 18753$. Hence $m \leq 26$. In these cases, we may observe that (5.1) has no solutions.

Finally, we suppose that m is odd. Then $b \equiv -c \equiv 1 \pmod 4$. We put $\ell = 2$, $a_1 = b$, $a_2 = -c$, $b_1 = y$, $b_2 = x$ and $\Lambda_3 = b^y - (-c)^x = (2m)^z$. Then $g = 1$, and we may take $E = 2$. Similarly to the preceding case we find an upper bound for m. In these cases, we may observe that (5.1) has no solutions. We conclude that (5.1) has no solutions in the case of $m \geq 3$, and complete the proof of Theorem 3.

6. Proof of Theorem 4

Let p be a positive integer with $p \geq 2$, and let a, b, c be pair-wise relatively prime positive integers such that $a^p + b^2 = c^2$ and $c = b + 1$. Then from (I), (II) and (III) we see that a, b, c are given by

$$a = 2m - 1, \quad b = \frac{(2m-1)^p - 1}{2}, \quad c = \frac{(2m-1)^p + 1}{2} \ (= b + 1),$$

where m is a positive integer with $m \geq 2$. We consider the equation

$$(b+1)^x + b^y = (2m-1)^z \tag{6.1}$$

where $x, y, z \in \mathbb{N}$. In what follows, let (x, y, z) be a solution of (6.1). First we prove an important lemma.

Lemma 6.1. *z is divisible by p.*

Proof. Let R be the least non-negative residue of z modulo p. Since $(2m-1)^z \equiv (2m-1)^R \pmod b$, it follows from (6.1) that $(2m-1)^R \equiv 1 \pmod b$. If $R > 0$, then $b + 1 \leq (2m-1)^R \leq (2m-1)^{p-1}$, which implies that $(2m-1)^{p-1}(2m-3) \leq -1$. This is a contradiction. □

By Lemma 6.1, we can write $z = pZ$, where $Z \in \mathbb{N}$. Then we rewrite (6.1) as

$$(b+1)^x + b^y = (2b+1)^Z. \tag{6.2}$$

It suffices to show that $x = y = Z = 1$.

If p is even, then we may rewrite (6.2) as

$$(2N^2-2N+1)^x+\big(2N(N-1)\big)^y=(2N-1)^{2Z},$$

where $N=((2m-1)^{p/2}+1)/2$ is a positive integer with $N\geq 2$. By the same method as in [20, Section 5], we may conclude that $x=y=Z=1$.

In what follows, we consider the case where p is odd. Remark that $b\geq 13$.

Lemma 6.2. *The following* (i)–(v) *hold.*

(i) *Write* $M=\max\{x,y\}$*. Then* $Z\leq M<1.3Z$*, where the first equality is attained if and only if* $x=y=Z=1$*.*

(ii) $y\equiv Z \pmod 2$.

(iii) $y\equiv 2Z+(-1)^y \pmod{b+1}$ *if* $x=1$*, and* $y\equiv 2Z \pmod{b+1}$ *if* $x>1$*.*

(iv) $x\equiv 2Z-1 \pmod b$ *if* $y=1$*, and* $x\equiv 2Z \pmod b$ *if* $y>1$*.*

(v) $Z\geq b+1$ *if* $\min\{x,y\}>1$*.*

Proof. (i) Since $b^M<(b+1)^x+b^y=(2b+1)^z$, we find that $M<\frac{\log(2b+1)}{\log b}Z<1.3Z$. Since $(2b+1)^Z=(b+1)^x+b^y\leq(b+1)^M+b^M\leq(2b+1)^M$, we find that $Z\leq M$, where the equality is attained if and only if $M=Z=1$.

(ii, iii) We observe that

$$b^y\equiv(-1)^{y-1}(b{+}1)y{+}(-1)^y,\ (2b{+}1)^Z{\equiv}(-1)^{Z-1}2(b{+}1)Z{+}(-1)^Z \pmod{(b{+}1)^2}.$$

It follows from (6.2) that

$$(b+1)^x+(-1)^{y-1}(b+1)y+(-1)^y\equiv(-1)^{Z-1}2(b+1)Z+(-1)^Z \pmod{(b+1)^2}.$$

Reducing this modulo $b+1$, we have $(-1)^y\equiv(-1)^Z \pmod{b+1}$. Hence $y\equiv Z \pmod 2$. Then $(b+1)^{x-1}+(-1)^{y-1}y\equiv(-1)^{y-1}2Z \pmod{b+1}$. Statement (iii) follows from this.

(iv) We observe that $(b+1)^x\equiv bx+1 \pmod{b^2}$ and $(2b+1)^Z\equiv 2bZ+1 \pmod{b^2}$. It follows from (6.2) that $x+b^{y-1}\equiv 2Z \pmod b$. The desired conclusion follows from this.

(v) Suppose that $\min\{x,y\}>1$. From (iii) and (iv) we see that $y\equiv 2Z \pmod{b+1}$ and $x\equiv 2Z \pmod b$. Since $M<2Z$ by (i), we can write $2Z=y{+}(b{+}1)U=x{+}bV$ for some positive integers U and V. Suppose that $U=V=1$. Then $x=y+1$. Multiplying (6.2) by 2^{x+y} and taking it modulo $2b+1$, we find that $2^y(2b+2)^x+2^x(2b)^y\equiv 0 \pmod{2b+1}$, so $2^y+(\pm1)^y2^x\equiv 0 \pmod{2b+1}$, which implies that $1+(\pm1)^y2\equiv 0 \pmod{2b+1}$. This is clearly absurd. It follows that $U\geq 2$ or $V\geq 2$, hence $Z\geq b+1$. □

We will only consider the case where b is even (the case where b is odd is similar). Remark that m is odd and b is not a power of 2. Let (x, y, Z) be a solution of (6.2). Using Lemma 2.1, we have

$$\begin{cases} \dfrac{x}{\log(2b+1)} < \dfrac{y \log b}{\log(b+1)\log(2b+1)} + 25.2\big(\max\{\log b' + 0.38, 10\}\big)^2, \\ \dfrac{y}{\log(2b+1)} < \dfrac{x \log(b+1)}{\log b \log(2b+1)} + 25.2\big(\max\{\log b'' + 0.38, 10\}\big)^2, \end{cases} \tag{6.3}$$

where $b' = x/\log(2b+1) + Z/\log(b+1)$ and $b'' = y/\log(2b+1) + Z/\log b$.

Suppose that $y > 1$. We will observe that this yields an absolute upper bound for b, hence for p and m. For this we use the method based on the works of Le (cf. [15, 16]). Since $y > 1$, it follows from (iv) in Lemma 6.2 that x is even, particularly, $\min\{x, y\} > 1$, hence $Z \geq b+1$ by (v) in Lemma 6.2. By (i) in Lemma 6.2, we find that $M \geq Z+1 \geq b+2$. Therefore, we also have an upper estimate $b' < 2M/\log(b+1)$. From this we observe that if y/M (≤ 1) is not close to 1, that is, $y/M < \delta$ for some $\delta < 1$, then by the first inequality in (6.3) we may deduce an absolute upper bound (which depends only on δ) for $M/\log(b+1)$ $(= x/\log(b+1))$. This yields an absolute upper bound for b (since $M > b$). We remark that if y/M is sufficiently close to 1, then we are not able to bound M from the above, since, in each of two inequalities in (6.3), the value of the left-hand side is almost the same as the first term on the right-hand side. Here, we take $\delta = 0.93$. If $y/M < \delta$, then the first inequality in (6.3) implies that $x < 60859 \log(b+1)$. Since $b+2 \leq M = x$, we find that $b \leq 829414$.

It remains to consider the case where $\delta < y/M$. We apply Proposition 2 to (6.2) with $\ell = 2$, $a_1 = 2b+1$, $a_2 = (-1)^{b/2}(b+1)$, $b_1 = Z$, $b_2 = x$. Then $g = 1$. Since $e := \mathrm{ord}_2(b) = \mathrm{ord}_2(m-1)$, we may take $E = e+1$, $A_1 = 2b+1$, $A_2 = b+1$. Hence

$$ey \leq \frac{36.1}{E^3(\log 2)^4}\Big(\max\big\{\log b' + \log(E\log 2) + 0.4, 6E\log 2\big\}\Big)^2 \log(b+1)\log(2b+1).$$

Since $\delta M < y, b' < 2M/\log(b+1)$ and $2^E < b \leq M-2$, it follows that

$$\delta E^3(E-1)M < \frac{36.1}{(\log 2)^4}\Big(\max\big\{\log(2M) + 0.4, 6E\log 2\big\}\Big)^2 \log(M-1)\log(2M-3).$$

This implies that $M \leq 913320$. Therefore, p and m are bounded above. It is not hard to see that for any (p, m) under consideration, (6.2) has no solutions with $y > 1$. This is a contradiction. We conclude that $y = 1$. Hence $M = x$.

Since $(2b+1)^Z = (b+1)^x + b < 2(b+1)^x$, we observe from the first inequality in (6.3) that $x < 2521\log(2b+1)$. Suppose that $x > 1$. Then (i) and (iv) in Lemma 6.2 yield $Z \le M - 1 = x - 1$ and $b + x \le 2Z - 1$, so $(b+3)/2 \le x < 2521\log(2b+1)$. This implies that $b \le 58868$. Hence p, m, x and Z are also bounded above. It is not hard to see that there is no (p, m, x, Z) under consideration satisfying all of the conditions in Lemma 6.2. This is a contradiction. Therefore, $x = 1$, hence $Z = 1$. This completes the proof of Theorem 4.

Remark 1. It is proved that Conjecture 2 is true if $c \equiv 1 \pmod b$ (cf. [20]). So it is natural to ask whether we can extend Theorem 4 to the case where $q = r = 2$ and $c \equiv 1 \pmod b$. But this question seems not worth to consider. In fact, it is likely that there are very few triples (a, b, c) fulfilling the condition that $p \ge 3$, $q = r = 2$, $c \equiv 1 \pmod b$ and $c > b + 1$. We will give a reason. In such case, we know that b and c are given by (II) or (III). We write $c = 1 + tb$, where $t \in \mathbb{N}$ and $t > 1$. In case of (II), we have

$$(t+1)(m-n)^p - (t-1)(m+n)^p = 2. \tag{6.4}$$

In case of (III), we have

$$(1 \pm t)2^{p-2}m^p - (-1 \pm t)n^p = 1. \tag{6.5}$$

We may apply the celebrated theorem on binomial Thue equations due to Bennett [1, Theorem 1.1].

Theorem B. *If A, B and N are integers with $AB \ne 0$ and $N \ge 3$, then the equation*

$$|AX^N - BY^N| = 1$$

has at most one solution in positive integers X and Y.

By Theorem B, we see that (6.4) does not hold if t is odd, and that (6.5) holds for at most one pair (m, n). In case where $p \equiv 0 \pmod 4$, we can observe from a result in [17] that (6.4) does not hold if t is even.

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References

[1] M. A. Bennett, Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$, *J. Reine Angew. Math.* **535** (2001), 1–49.

[2] Y. Bugeaud, Linear forms in p-adic logarithms and the Diophantine equation $(x^n - 1)/(x - 1) = y^q$, *Math. Proc. Cambridge Phil. Soc.* **127** (1999), 373–381.

[3] Z. F. Cao, Introduction to Diophantine Equations, *Harbin Institute Technology Press, Harbin*, 1989.

[4] Z. F. Cao, A note on the Diophantine equation $a^x + b^y = c^z$, *Acta Arith.* **91** (1999), 85–93.

[5] Z. F. Cao and X. L. Dong, On the Terai-Jeśmanowicz conjecture, *Publ. Math. Debrecen* **61** (2002), 253–265.

[6] Z. F. Cao and X. L. Dong, An application of a lower bound for linear forms in two logarithms to the Terai-Jeśmanowicz conjecture, *Acta Arith.* **110** (2003), 153–164.

[7] M. Cipu and M. Mignotte, On a conjecture on exponential Diophantine equations, *Acta Arith.* **140** (2009), 251–270.

[8] H. Cohen, Number Theory, Volume II: Analytic and Modern Tools, *Springer-Verlag, New York*, 2007.

[9] Y. Z. Hu and P. Z. Yuan, The exponential Diophantine equation $a^x + b^y = c^z$, *Acta Math. Sin.* **48** (2005), 1175–1178.

[10] L. Jeśmanowicz, Several remarks on Pythagorean numbers, *Wiadom. Mat.* **1** (1955/1956), 196–202.

[11] A. Y. Khinchin, Continued Fractions, (3rd edition), *P. Noordho Ltd., Gorningen*, 1963.

[12] M. Laurent, Linear forms in two logarithms and interpolation determinants II, *Acta Arith.* **133** (2008), 325–348.

[13] M.-H. Le, An open problem concerning the Diophantine equation $a^x + b^y = c^z$, *Publ. Math. Debrecen* **68** (2006), 283–295.

[14] M.-H. Le, A note on the Diophantine system $a^2 + b^2 = c^r$ and $a^x + b^y = c^z$, *Acta Math. Sinica (Chin. Ser.)* **51** (2008), 677–684.

[15] M.-H. Le, A note on Jeśmanowicz' conjecture concerning primitive Pythagorean triplets, *Acta Arith.* **138** (2009), 137–144.

[16] M.-H. Le, The pure exponential Diophantine equation $a^x + b^y = c^z$ for generalized Pythagorean triples, *Acta Math. Sinica (Chin. Ser.)* **53** (2010), 1239–1248.

[17] W. Ljunggren, Ein Satz über die diophantische Gleichung $Ax^2 - By^4 = C$ $(C = 1, 2, 4)$, *Tolfte Skandinaviska Matematikerkongressen, Lund* **12** (1953/1954), 188–194.

[18] T. Miyazaki, Exceptional cases of Terai's conjecture on Diophantine equations, *Arch. Math. (Basel)* **95** (2010), 519–527.

[19] T. Miyazaki, Terai's conjecture on exponential Diophantine equations, *Int. J. Number Theory* **4** (2011), 981–999.

[20] T. Miyazaki, The shuffle variant of Jeśmanowicz' conjecture concerning Pythagorean triples, *J. Austral. Math. Soc.* **90** (2011), 355–370.

[21] P. Ribenboim, Catalan's Conjecture: Are 8 and 9 the only Consecutive Powers?, *Academic Press, Boston, MA*, 1994.

[22] R. Scott, On the equations $p^x - b^y = c$ and $a^x + b^y = c^z$, *J. Number Theory* **44** (1993), 153–165.

[23] N. Terai, Applications of a lower bound for linear forms in two logarithms to exponential Diophantine equations, *Acta Arith.* **90** (1999), 17–35.

[24] J. J. Xia and P. Z. Yuan, On the Terai-Jeśmanowicz conjecture, *Acta Math. Sin. (Engl. Ser.)* **24** (2008), 2061–2064.

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