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Grünwald shift spaces

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Abstract. An n-dimensional differentiable shift space $\mathcal{S}$ for which in case $n = 2$ there exists an affine connection if $\mathcal{S}$ is a Grünwald plane (cf. [13, § 4]) admits for $n \geq 3$ no affine connection. In contrast to this the set of all images of the system of curves arising by shifting the argument from a Grünwald curve $\mathcal{C}$ under the translation group of $\mathbb{R}^n$ is a system of geodesics with respect to a metrizable affine connection if and only if $\mathcal{C}$ is a curve corresponding to parabolas in a suitable coordinate system.

1. Introduction

The investigation of systems $\mathfrak{S}$ of curves in the plane $\mathbb{R}^2$ such that any two different points are incident with precisely one curve of $\mathfrak{S}$ has a long tradition (see e.g. [17]). In particular since the second half of the previous century the systems $\mathfrak{S}$ has been studied intensively as natural generalisations of the real affine plane and the 2-dimensional hyperbolic geometry. These geometries, now called $\mathbb{R}^2$-planes, are classified if they admit an at least 3-dimensional Lie group of automorphisms [15, Chapter 3].

Although already E. Beltrami has shown that a differentiable curve is a local geodesic with respect to an affine connection ∇ precisely if it is a solution of an Abelian differential equation having as coefficients expressions in Christoffel symbols associated with ∇, the use of differential geometry for study of $\mathbb{R}^2$-planes having differentiable curves as lines started only 2000 by G. Gerlich [5], [6], [7].

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He asked for which $\mathbb{R}^2$-planes A with differentiable lines there exists an affine connection ∇ generating the lines of A and for affine planes A with an at least three-dimensional collineation group he proved that ∇ exists if and only if A is either desarguesian or a Moulton plane. Moreover, in [13] it is shown that the differentiable lines of a generalized shift $\mathbb{R}^2$-plane A are geodesics with respect to an affine connection ∇ precisely if A is either the Euclidean plane or a Grünwald model of the real affine plane (see [8]).

The extension of the investigation from $\mathbb{R}^2$-planes to geometries on $\mathbb{R}^n$ having as lines a system $\mathfrak{S}$ of curves such that any two different points are incident with precisely one curve of $\mathfrak{S}$ surprisingly turns out to be difficult as one can see in the papers [1], [2], [3], where D. Betten created a theory of 3-dimensional topological incidence geometries.

If one tries to extend the characterization of differentiable shift spaces having as lines geodesics with respect to an affine connection starting with a Grünwald plane (cf. [13, § 4]), then one meets also great difficulties. Namely, we show that for at least 3-dimensional differentiable shift spaces $\mathcal{S}$ generalizing in a natural way the 2-dimensional shift spaces corresponding to Grünwald planes there exists no affine connection ∇ such that the lines of $\mathcal{S}$ are geodesics of ∇. This is surprising since there exist n-dimensional shift spaces if the derivatives of their generating functions are homeomorphisms of $\mathbb{R}$ (Proposition 1).

In contrast to a shift space the set of all images of the system of curves arising by shifting the argument from a Grünwald curve $\mathcal{C}$ under the translation group of $\mathbb{R}^n$ is a system of geodesics with respect to a natural affine connection if and only if $\mathcal{C}$ is a curve corresponding to parabolas in a suitable coordinate system (Theorem 2). Moreover, ∇ is metrizable and for $n = 2$ we get the metric tensor (4.4) of [13].

2. Grünwald shift spaces

An n-dimensional line space $\mathcal{S} = (\mathbb{R}^n, \mathcal{L})$, $n \geq 2$, is an incidence geometry such that the point set is the Euclidean space $\mathbb{R}^n$, the set $\mathcal{L}$ of lines consists of closed subsets of $\mathbb{R}^n$ homeomorphic to $\mathbb{R}$ and any two different points are incident with precisely one line.

We call an n-dimensional line space S an n-dimensional shift space if there exist continuous functions $f_i^{(k)} : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, n-1$, $i = k+1, \dots, n$, such that

$$\begin{aligned}\ell^{(k)}_{(u_1,\dots,u_n,v_{k+2},\dots,v_n)} = \{(u_1, \dots, u_{k-1}, t + u_k, f^{(k)}_{k+1}(t) + u_{k+1}, f^{(k)}_{k+2}(t + v_{k+2}) \\ + u_{k+2}, \dots, f^{(k)}_n(t + v_n) + u_n),\ t \in \mathbb{R}\}, \quad (1)\end{aligned}$$

$$\text{where} \quad u_1, \ldots, u_n, v_{k+2}, \ldots, v_n \in \mathbb{R},$$

$$\text{and} \quad \{(u_1, \ldots, u_{n-1}, t),\ t \in \mathbb{R}\} \quad \text{with} \quad u_1, \ldots, u_{n-1} \in \mathbb{R}$$

form the set of lines for the line space $S(f_i^{(k)})$. The functions $f_i^{(k)}$ we will call *generating functions* of the shift space $S(f_i^{(k)})$.

Clearly, the group T of translations of $\mathbb{R}^n$ is a group of collineations of the shift spaces $S(f_i^{(k)})$.

We call an n-dimensional line space S, respectively n-dimensional shift space $S(f_i^{(k)})$, *differentiable* if the lines of S, respectively of $S(f_i^{(k)})$, are two times differentiable curves.

Shift spaces of the following proposition give for $n = 2$ Grünwald planes if their lines are geodesics with respect to an affine connection (cf. [13, § 4]). For this reason we call the shift spaces of the following proposition *Grünwald shift spaces.*

Proposition 1. *Let $f_i^{(k)} : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, n-1$, $i = k+1, \ldots, n$, be differentiable functions such that the derivatives $f_i^{(k)\prime}$ are homeomorphisms of R for all $2 \le i \le n$. Then the functions $f_i^{(k)}$ are generating functions for an n-dimensional shift space $S(f_i^{(k)})$.*

Proof. The lines of $S(f_i^{(k)})$ are the sets of form (1). Let

$$a = (a_1, a_2, \ldots, a_n) \quad \text{and} \quad b = (b_1, b_2, \ldots, b_n)$$

be two different points of $\mathbb{R}^n$.

Let $a_r = b_r$ for $r \le k-1 < n-1$ and $a_k \ne b_k$. For a line through a and b we have $u_p = a_p = b_p$, $p = 1, \ldots, k-1$. Moreover, the coordinates a_k, b_k, a_{k+1}, b_{k+1} satisfy the following system of equations:

$$\begin{aligned} a_k &= t_a + u_k, \quad a_{k+1} = f_{k+1}^{(k)}(t_a) + u_{k+1}; \\ b_k &= t_b + u_k, \quad b_{k+1} = f_{k+1}^{(k)}(t_b) + u_{k+1}. \end{aligned} \tag{2}$$

Since the derivative of the function $f_p^{(k)}$, $p = k+1, \ldots, n$, is a homeomorphism of $\mathbb{R}$, the function

$$t \longmapsto f_p^{(k)}(t+d) - f_p^{(k)}(t), \tag{3}$$

is a homeomorphism of $\mathbb{R}$ for any fixed $d \in \mathbb{R}\backslash\{0\}$ (cf. [15, § 3, p. 161]).

Now, from (2) we obtain $t_b = t_a - (a_k - b_k)$ and

$$a_{k+1} = f^{(k)}_{k+1}(t_a) + u_{k+1}; \quad b_{k+1} = f^{(k)}_{k+1}(t_a + (b_k - a_k)) + u_{k+1}.$$

This yields

$$a_{k+1} - b_{k+1} = f^{(k)}_{k+1}(t_a) - f^{(k)}_{k+1}(t_a + (b_k - a_k)).$$

Because $a_k \neq b_k$ relation (3) gives that there exists precisely one solution t_a of the last equation. Then we have $u_k = a_k - t_a$, $t_b = b_k - u_k$, and $u_{k+1} = a_{k+1} - f(t_a)$.

For $p = k+2, \dots, n$ the coordinates a_p and b_p fulfill the following system of equations

$$a_p = f^{(k)}_p(t_a + v_p) + u_p; \quad b_p = f^{(k)}_p(t_b + v_p) + u_p.$$

Since the function $f^{(k)}_p$ satisfy (3) this system has precisely one solution u_p, v_p.

If $a_r = b_r$ for $r \le n-1$, and $a_n \neq b_n$ then the line $\{(a_1, \dots, a_{n-1}, t),\ t \in \mathbb{R}\}$ is the unique line joining a and b, and the proposition is proved. □

3. Riccati and Abelian differential equations

For a later use we consider the special Riccati differential equations with unknown function $y = y(x)$:

$$y' + a_1 y^2 + a_2 y + a_3 = 0, \tag{4}$$

where a_i are constants (cf. [9, A4.9] or [10, pp. 33 and 41]).

(4.1) If $a_i = 0$ for $i \in \{1, 2\}$, then $y = -a_3 x + c$ for $c \in \mathbb{R}$.

(4.2) If $a_1 = 0$ and $a_2 \neq 0$, then $y = -\frac{a_3}{a_2} + c\mathrm{e}^{-a_2 x}$ for $c \in \mathbb{R}$.

(4.3) If $a_1 \neq 0$ and $a_2^2 = 4a_1a_3$, then we have

$$y = -\frac{a_2}{2a_1} + \frac{c_1}{c_1 a_1 x + c_2} \quad \text{with} \quad c_1, c_2 \in \mathbb{R} \quad \text{and} \quad (c_1, c_2) \neq (0, 0).$$

(4.4) If $a_1 \neq 0$ and $\lambda^2 = 4a_1a_3 - a_2^2 > 0$, then

$$y = -\frac{a_2}{2a_1} + \frac{\lambda}{2a_1}\operatorname{cotan}\frac{\lambda}{2}(x + c) \quad \text{with} \quad c \in \mathbb{R}.$$

(4.5) If $a_1 \neq 0$ and $\lambda^2 = a_2^2 - 4a_1a_3 > 0$, then

$$y = -\frac{a_2}{2a_1} + \frac{\lambda}{2a_1}\frac{c_1\mathrm{e}^{\frac{\lambda}{2}x} - c_2\mathrm{e}^{-\frac{\lambda}{2}x}}{c_1\mathrm{e}^{\frac{\lambda}{2}x} + c_2\mathrm{e}^{-\frac{\lambda}{2}x}} \quad \text{with} \quad c_1, c_2 \in \mathbb{R} \quad \text{and} \quad (c_1, c_2) \neq (0, 0).$$

Also in (4.3) and (4.5) the solution depends only on one parameter which is defined on the projective line.

Lemma 1. *If f is a differentiable function such that its derivative f' is a homeomorphism of $\mathbb{R}$ and a solution of an equation* (4), *then f has the form $f = -1/2\, a_3 x^2 + cx + d$ with $a_3 \neq 0$, $c, d \in \mathbb{R}$.*

Proof. Solutions (4.1) with $a_3 = 0$ and (4.2) are excluded since in this case f' is not a homeomorphism. Solutions (4.1) with $a_3 \neq 0$ give the functions in the assertion.

Since in case of solutions (4.3) and (4.4) the function f' is not a homeomorphism of $\mathbb{R}$ we have to consider solutions (4.5). But also in this case the function f' is not a homeomorphism of $\mathbb{R}$ since we have $\lim_{x\to\pm\infty} f' = -\frac{a_2 \pm \lambda}{2a_1}$. □

We consider Abelian differential equations

$$y' = \alpha + \beta y + \gamma y^2 + \varepsilon y^3 \quad \text{with} \quad \varepsilon \neq 0, \tag{5}$$

where $\alpha, \beta, \gamma, \varepsilon \in \mathbb{R}$, and we are interested in real functions f such that $f' = y$ is a homeomorphism of $\mathbb{R}$.

To differential equation (5) is associated the cubic algebraic equation

$$\alpha + \beta y + \gamma y^2 + \varepsilon y^3 = 0. \tag{6}$$

Because $\varepsilon \neq 0$, the cubic equation (6) has a real solution $y = y_1$ and hence equation (5) has a solution $y(t) = y_1$ for all $t \in \mathbb{R}$. According to the existence and uniqueness Theorem applied to (5), any other solution $f' = y$ of equation (5) satisfies either $y(t) > y_1$ or $y(t) < y_1$ for all $t \in \mathbb{R}$. Hence it follows

Lemma 2. *There exists no real function f with $f' = y$ satisfying* (5) *such that f' is a homeomorphism of the real line $\mathbb{R}$.*

4. Affine connections

Since we apply results of differential geometry only for the n-dimensional space $\mathbb{R}^n$ there exist global coordinates and the components Γ^h_{ij}, $h,i,j \in \{1, 2, \dots, n\}$, of any affine connection ∇ can be written in a unique way in these coordinates.

An affine connection ∇ is called symmetric if $\nabla_X Y = \nabla_Y X - [X, Y]$, where $[X, Y]$ is the Lie bracket, i.e. if for its components Γ^h_{ij} one has $\Gamma^h_{ij} = \Gamma^h_{ji}$ for all $h, i, j \in \{1, 2, \dots, n\}$.

By a *geodesic* of ∇ we mean a piecewise C^2-curve γ: $I \to \mathbb{R}^n$ satisfying $\nabla_{\dot\gamma}\dot\gamma = \varrho \cdot \dot\gamma$, where $\varrho : I \to \mathbb{R}$ is a continuous function, and $I \subset \mathbb{R}$ is an open interval (cf. [4, p. 3], [14, p. 122]).

Using the components of ∇ the system of differential equations for geodesics has the form (cf. [14, p. 144])

$$\ddot\gamma^h + \sum_{i,j=1}^{n} \Gamma^h_{ij}\dot\gamma^i\dot\gamma^j = \varrho(t)\dot\gamma^h, \quad h \in \{1, 2, \dots, n\}. \tag{7}$$

From this it follows that the geodesics depend only on the symmetric part of the connection ∇. Hence we will always assume that ∇ is symmetric.

Let $\mathfrak{g}$ be a Lie algebra of a group G of diffeomorphisms and let $\nabla = \{\Gamma^h_{ij}\}$ be an affine connection. The Lie derivative $\mathcal{L}_\xi\nabla$ along an element $\xi(\neq 0) \in \mathfrak{g}$ is given with respect to components of ∇ by

$$\mathcal{L}_\xi\Gamma^h_{ij} \equiv \frac{\partial^2\xi^h}{\partial x_i \partial x_j} + \sum_{\alpha=1}^{n}\left(\xi^\alpha\frac{\partial\Gamma^h_{ij}}{\partial x_\alpha} - \frac{\partial\xi^h}{\partial x_\alpha}\Gamma^\alpha_{ij} + \frac{\partial\xi^\alpha}{\partial x_i}\Gamma^h_{\alpha j} + \frac{\partial\xi^\alpha}{\partial x_j}\Gamma^h_{\alpha i}\right),$$

where $h, i, \dots = 1, 2, \dots, n$.

The group G preserves geodesics with respect to ∇ if and only if

$$\mathcal{L}_\xi\Gamma^h_{ij} = \delta^h_i\psi_j + \delta^h_j\psi_i, \tag{8}$$

where δ^h_i is the Kronecker symbol and ψ_i are differentiable functions [11], [12, p. 143], [18].

The group G consists of affine mappings with respect to ∇ precisely if $\mathfrak{L}_\xi\Gamma^h_{ij} = 0$ or, equivalently, if and only if ψ_i vanishes. Moreover, if $\mathbb{R}^n$ is a (pseudo-) Riemannian space with respect to the metric tensor g, then the Lie group G is a group of isometries precisely if $\mathfrak{L}_\xi g = 0$ (cf. [18, p. 43], [12, p. 100]).

Proposition 2. *Let S be a system of geodesics with respect to an affine connection ∇. If the translation group T of $\mathbb{R}^n$ consists of geodesic maps for S, then the affine connection ∇ may be chosen in such a way that the components Γ^h_{ij} are constant. Moreover, the components $\Gamma^\sigma_{\sigma\sigma}$, $\sigma = 1, \dots, n$, are zero.*

Proof. Since T consists of geodesic maps for the Lie derivative $\mathcal{L}_\xi\Gamma^h_{ij}$ along any element $\xi \neq 0$ of the Lie algebra of T one has (8). Taking in particular $\xi = (\delta^h_\sigma)^n_{h=1}$ one obtains

$$\mathcal{L}_\xi\Gamma^h_{ij} \equiv \frac{\partial\Gamma^h_{ij}}{\partial x_\sigma} = \delta^h_i\psi_j + \delta^h_j\psi_i.$$

Integrating these equations for any $\sigma = 1, \dots, n$, we get

$$\Gamma^h_{ij} = \overset{\circ}{\Gamma}{}^h_{ij} + \delta^h_i \Psi_j + \delta^h_j \Psi_i,$$

where $\overset{\circ}{\Gamma}{}^h_{ij}$ are constants, and $\Psi_j(x)$ are suitable differentiable functions.

If with respect to the affine connection ∇ having the components Γ^h_{ij} the system S consists of geodesics, then the same holds for any connection with the components $\bar{\Gamma}^h_{ij}$ satisfying the equations $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \bar{\psi}_j + \delta^h_j \bar{\psi}_i$, where $\bar{\psi}_i$ are differentiable functions ([11], [12], [18]). Choosing the functions $\bar{\psi}_i$ in such a way that $\bar{\psi}_i = -\Psi_i - \frac{1}{2} \overset{\circ}{\Gamma}{}^i_{ii}$ we see that then $\bar{\Gamma}^h_{ij}$ are constants, and $\bar{\Gamma}^\sigma_{\sigma\sigma} = 0$, $\sigma = 1, \dots, n$. □

As representatives of affine connections for which the lines of a shift space S are geodesics we will take henceforth affine connections ∇° having constant components such that $\bar{\Gamma}^\sigma_{\sigma\sigma} = 0$, $\sigma = 1, \dots, n$. We shall call such connections *natural connections of* S. With respect to a natural connection ∇° the translation group of $\mathbb{R}^n$ consists of affine transformations of S. Namely, for $\xi = (\delta^h_\sigma)^n_{h=1}$ one has $\mathcal{L}_\xi \Gamma^h_{ij} \equiv \frac{\partial}{\partial x_\sigma} \Gamma^h_{ij} = 0$.

If a connection ∇ has the components Γ^h_{ij}, $h, i, j, \in \{1, \dots, n\}$, the components R^h_{ijk}, $h, i, j, k \in \{1, \dots, n\}$ of the curvature tensor R of ∇ are given by (cf. [4, p. 8], [16, p. 27])

$$R^h_{ijk} = \frac{\partial}{\partial x_j} \Gamma^h_{ik} - \frac{\partial}{\partial x_k} \Gamma^h_{ij} + \sum_{\alpha=1}^n (\Gamma^\alpha_{ik} \Gamma^h_{\alpha j} - \Gamma^\alpha_{ij} \Gamma^h_{\alpha k}). \tag{9}$$

The Ricci tensor belonging to R has components $R_{ij} = \sum_{\alpha=1}^n R^\alpha_{i\alpha j}$.
The curvature tensor R of ∇ is often called the Riemannian tensor of ∇.

In particular, ∇ is the *Levi–Civita connection* of a (pseudo-) Riemannian space with the metric $g = (g_{ij})$ if $\nabla g = 0$, i.e.

$$\frac{\partial}{\partial x_k} g_{ij} = \sum_{\alpha=1}^n (g_{i\alpha} \Gamma^\alpha_{jk} + g_{j\alpha} \Gamma^\alpha_{ik}), \tag{10}$$

where the components Γ^h_{ij} (called Christoffel symbols) are given by

$$\Gamma^h_{ij} = \frac{1}{2} \sum_{\alpha=1}^n g^{h\alpha} \left(\frac{\partial}{\partial x_i} g_{j\alpha} + \frac{\partial}{\partial x_j} g_{i\alpha} - \frac{\partial}{\partial x_\alpha} g_{ij} \right); \tag{11}$$

thereby $(g^{h\alpha})$ denotes the inverse matrix of (g_{ij}). For g then there exists a unique symmetric affine connection ∇ such $\nabla g = 0$.

The integrability conditions of (10) have the following form [4, p. 79]:

$$\sum_{\alpha=1}^n (g_{i\alpha} R^\alpha_{jkl} + g_{j\alpha} R^\alpha_{ikl}) = 0. \tag{12}$$

5. Geometry of Grünwald shift spaces

Theorem 1. *Let $S(f_i^{(k)})$ be an n-dimensional Grünwald shift space. If the set of lines of $S(f_i^{(k)})$ forms the set of geodesics with respect to a natural connection ∇°, then $S(f_i^{(k)})$ is a Grünwald plane.*

Proof. For a line

$$x_{(k)} = (u_1, \dots, u_{k-1}, t + u_k, f_{k+1}^{(k)}(t) + u_{k+1}, f_{k+2}^{(k)}(t + v_{k+2}) + u_{k+2}, \dots, f_n^{(k)}(t + v_n) + u_n),\ t \in \mathbb{R},$$

one has

$$\dot{x}_{(k)} = (0, \dots, 0, 1, f_{k+1}^{(k)}{}'(t), f_{k+2}^{(k)}{}'(t + v_{k+2}), \dots, f_n^{(k)\prime}(t + v_n)),$$

$$\ddot{x}_{(k)} = (0, \dots, 0, 0, f_{k+1}^{(k)}{}''(t), f_{k+2}^{(k)}{}''(t + v_{k+2}), \dots, f_n^{(k)\prime\prime}(t + v_n)).$$

This line is a geodesic if and only if relation (7) holds. We put in this relation $t_{k+1} \equiv t$ and $t_\lambda \equiv t + v_\lambda$ for $\lambda > k + 1$.

For $h = k = n - 1$ one has $\varrho(t_n) = 2\Gamma^{n-1}_{n-1\,n}\, f_n^{(n-1)\prime}(t_n) + \Gamma^{n-1}_{n\,n}\,(f_n^{(n-1)\prime}(t_n))^2$ and for $h = n$, $k = n - 1$ using $\Gamma^n_{nn} = 0$ (Proposition 2) we get

$$f_n^{(n-1)\prime\prime} + \Gamma^n_{n-1\,n-1} + 2\Gamma^n_{n-1\,n}\, f_\sigma^{(k)\prime} = \varrho(t_n) f_n^{(n-1)\prime}. \tag{13}$$

Substituting $\varrho(t_n)$ into (13) we obtain

$$f_n^{(n-1)\prime\prime} = -\Gamma^n_{n-1\,n-1} - 2\Gamma^n_{n-1\,n}\, f_\sigma^{(k)\prime} + 2\Gamma^{n-1}_{n-1\,n}\,(f_n^{(n-1)\prime})^2 + \Gamma^{n-1}_{n\,n}\,(f_n^{(n-1)\prime})^3. \tag{14}$$

This equation with constant coefficients is an Abelian differential equation with respect to $f_n^{(n-1)\prime}(t_n)$. By Lemma 1 and 2 it follows that $f_n^{(n-1)}(t_n) = \alpha\, t_n^2 + \beta\, t_n + \gamma$ with constants $\alpha \neq 0$, β and γ. Putting this in (14) we get

$$\Gamma^{n-1}_{n\,n} = \Gamma^{n-1}_{n-1\,n} = \Gamma^n_{n-1\,n} = 0, \text{ but } \Gamma^n_{n-1\,n-1} = -2\alpha \neq 0. \tag{15}$$

If $n > 2$ then from (7) for $h = k = n - 2$ one has

$$\varrho(t_{n-1}) = 2\Gamma^{n-2}_{n-2\ n-1} f_{n-1}^{(n-2)\prime}(t_{n-1}) + 2\Gamma^{n-2}_{n-2\ n} f_n^{(n-2)\prime}(t_n) + \Gamma^{n-2}_{n-1 n-1}(f_{n-1}^{(n-2)\prime}(t_{n-1}))^2$$
$$+ 2\Gamma^{n-2}_{n-1\ n} f_{n-1}^{(n-2)\prime}(t_{n-1}) f_n^{(n-2)\prime}(t_n) + \Gamma^{n-2}_{n\,n}(f_n^{(n-2)\prime}(t_n))^2$$

and for $h = n$, $k = n - 2$ using $\Gamma^n_{nn} = \Gamma^n_{n-1\,n} = 0$ we get

$$f_h^{(n-2)\prime\prime}(t_n) + \Gamma^n_{n-2\,n-2} + 2\Gamma^n_{n-2\,n-1}\, f_{n-1}^{(n-2)\prime}(t_{n-1}) + 2\Gamma^n_{n-2\,n}\, f_n^{(n-2)\prime}(t_n)$$
$$+ \Gamma^n_{n-1\,n-1}\,(f_{n-1}^{(n-2)\prime}(t_{n-1}))^2 = \varrho(t_{n-1}) f_n^{(n-2)\prime}(t_n).$$

Substituting into this $\varrho(t_{n-1})$ we obtain

$$(f_{n-1}^{(n-2)\prime}(t_{n-1}))^2 \cdot (\Gamma^n_{n-1\,n-1} - \Gamma^{n-2}_{n-1\,n-1}\, f_n^{(n-2)\prime}(t_n))$$
$$+ f_{n-1}^{(n-2)\prime}(t_{n-1}) \cdot A(t_n) + B(t_n) = 0,$$

where $A(t_n)$ and $B(t_n)$ are functions of the variable t_n. Since the variables t_{n-1} and t_n are independent and $f_{n-1}^{(n-2)\prime} \neq 0$ the coefficient functions $A(t_n)$ and $B(t_n)$ vanish and

$$\Gamma^n_{n-1\,n-1} - \Gamma^{n-2}_{n-1\,n-1}\, f_n^{(n-2)\prime}(t_n) = 0. \tag{16}$$

From (16) it follows $\Gamma^{n-2}_{n-1\,n-1} = 0$ and $\Gamma^n_{n-1\,n-1} = 0$. This contradicts relations (15). Hence n must be 2 and $S(f_i^{(k)})$ is a Grünwald plane (cf. [13]). □

Remark. If $n = 2$ then the proof of Theorem 1 yields that $\Gamma^2_{11} = -2\alpha \neq 0$ and all other components are zero. Hence this shift space is the Grünwald plane M_α having a metric tensor g with corresponds to the Levi–Civita connection ∇ of the form (4.4) in [13].

6. Translation shell of a Grünwald curve

Let $\mathcal{C}$ be a curve homeomeomorphic to $\mathbb{R}$ which is a closed subset of in $\mathbb{R}^n$, $n \geq 2$. The translation shell $\mathcal{C}^T$ of $\mathcal{C}$ is the set of all images of $\mathcal{C}$ under the translation group T of $\mathbb{R}^n$. We consider a curve of the form

$$\mathcal{C} = \{(t, f_2(t), f_3(t), \dots, f_n(t)),\ t \in \mathbb{R}\}, \tag{17}$$

where $f_i(t)$ are two times differentiable functions such that the derivatives $f_i'(t)$ are homeomorphisms of $\mathbb{R}$ for all $i = 2, \dots, n$. The translation shell of $\mathcal{C}$ is the set

$$\mathcal{C}^T = \{(t + u_1, f_2(t) + u_2, f_3(t) + u_3, \dots, f_n(t) + u_n),\ t \in \mathbb{R}\},$$
$$\text{where} \quad u_1, \dots, u_n \in \mathbb{R}.$$

The extended translation shell $\hat{\mathcal{C}}^T$ is the set

$$\hat{\mathcal{C}}^T = \{(t + u_1, f_2(t) + u_2, f_3(t + v_3) + u_3, \dots, f_n(t + v_n) + u_n),\ t \in \mathbb{R}\},$$
$$\text{where} \quad u_1, \dots, u_n, v_3, \dots, v_n \in \mathbb{R}. \tag{18}$$

We search for affine connections ∇ for which the extended translation shell $\hat{\mathcal{C}}^T$ or the translation shell $\mathcal{C}^T$ consists of geodesics with respect to ∇. If $n = 2$ then the extended shell $\hat{\mathcal{C}}^T$ is a Grünwald plane if we adjoin to $\hat{\mathcal{C}}^T$ the lines $\{(u, t);\ t \in \mathbb{R}\}$, $u \in \mathbb{R}$. For this reason we call such curves $\mathcal{C}$ *Grünwald curves.*

Theorem 2. *For a Grünwald curve $\mathcal{C}$ the extended translation shell $\hat{\mathcal{C}}^T$ consists of geodesics with respect to a natural affine connection ∇° with components Γ^h_{ij} if and only if the functions $f_i(t)$ may be chosen as $f_i(t) = -\frac{1}{2}\,\Gamma^{\,i}_{11}\,t^2 + \beta_i\,t$ with $\Gamma^{\,i}_{11} \neq 0$, $\beta_i \in \mathbb{R}$, $i = 2, 3, \ldots, n$, whereas all other components of ∇° are zero.*

Proof. Let $x(t)$ be a curve in (18). Then we have

$$\dot{x}(t) = (1, f_2'(t), f_3'(t + v_3), \ldots, f_n'(t + v_n)),$$
$$\ddot{x}(t) = (0, f_2''(t), f_3''(t + v_3), \ldots, f_n''(t + v_n)).$$

The curve $x(t)$ is a geodesic if and only if relation (7) holds. We put in this relation $t_2 \equiv t$, and $t_\lambda \equiv t + v_\lambda$ for $\lambda > 2$.

For $h = 1$ one has

$$\varrho(t_2) = 2\sum_{\sigma=2}^{n} \Gamma^1_{1\sigma}\, f_\sigma'(t_\sigma) + \sum_{\sigma,\tau=2}^{n} \Gamma^1_{\sigma\tau}\, f_\sigma'(t_\sigma)\, f_\tau'(t_\tau),$$

and for $h > 1$ we get

$$f_h''(t_h) + \Gamma^h_{11} + 2\sum_{\sigma=2}^{n} \Gamma^h_{1\sigma}\, f_\sigma'(t_\sigma) + \sum_{\sigma,\tau=2}^{n} \Gamma^h_{\sigma\tau}\, f_\sigma'(t_\sigma)\, f_\tau'(t_\tau) = \varrho(t_2) f_h'(t_h). \tag{19}$$

Putting $\varrho(t_2)$ into (19) and fixing all variables t_σ different from t_h we obtain with respect to function $f'(t_h)$ an Abelian differential equation with constant coefficients since ∇° is a natural affine connection. By Lemmas 1 and 2 it follows $f_h(t_h) = \alpha_h {t_h}^2 + \beta_h t_h + \gamma_h$, with constants $\alpha_h \neq 0$, β_h and γ_h.

Substituting $\varrho(t_2)$ and $f_h(t_h)$ in (19) we obtain

$$\begin{aligned} &\alpha_h + \Gamma^h_{11} + 2\sum_{\sigma=2}^{n} (\Gamma^h_{1\sigma} - \Gamma^1_{1\sigma}\,(\alpha_h\, t_h + \beta_h)) \cdot (\alpha_\sigma\, t_\sigma + \beta_\sigma) \\ &+ \sum_{\sigma,\tau=2}^{n} (\Gamma^h_{\sigma\tau} - \Gamma^1_{\sigma\tau}\,(\alpha_h\, t_h + \beta_h))\,(\alpha_\sigma\, t_\sigma + \beta_\sigma)\,(\alpha_\tau\, t_\tau + \beta_\tau) \equiv 0, \end{aligned} \tag{20}$$

where $h = 2, 3, \ldots, n$ and $t_2, t_3, \ldots, t_n$ are independent variables.

Since (20) is a cubic polynomial the coefficients at monomials are zero. This yields $\alpha_h = -\Gamma^h_{11} \neq 0$ and all other components of ∇° are zero. □

Remark. The metric tensor g with components

$$g_{11} = 1 + (x_1)^2 \cdot \sum_{\alpha=2}^{n} (\Gamma^\alpha_{11})^2, \quad g_{1b} = \Gamma^b_{11} \cdot x_1, \quad g_{ab} = \delta_{ab}, \qquad a, b \neq 1,$$

where δ_{ab} is the Kronecker symbol, determines (see (11)) the Levi–Civita connection ∇ with components as in Theorem 2 (having $\Gamma^h_{11}, h = 2, \dots, n$, as the only non zero components). Moreover, the Riemannian tensor vanishes, hence the space $(\mathbb{R}^n, \Gamma^h_{ij})$ is locally Euclidean (cf. [14]).

If we strength the hypothesis on the Grünwald curve $\mathcal{C}$ we obtain the same system of functions $f_i(t)$ as in Theorem 2, but for a given system of functions $f_i(t)$ there are more natural affine connections having the curves of the translation shell $\mathcal{C}^T$ as geodesics.

Theorem 3. *Let $\mathcal{C}$ be a Grünwald curve such that the derivatives of all its functions $f_i(t)$ satisfy Abelian differential equations with constant coefficients. Then the translation shell $\mathcal{C}^T$ of $\mathcal{C}$ consists of geodesics with respect to a natural affine connection ∇° with components Γ^h_{ij} if and only if the functions $f_i(t)$ may be chosen as $f_i(t) = -\frac{1}{2}\,\Gamma^{\,i}_{11}\,t^2 + \beta_i\,t$ with $\Gamma^{\,i}_{11} \neq 0$, $\beta_i \in \mathbb{R}$, $i = 2, 3, \dots, n$, whereas all other components of ∇° are zero with exception of $\Gamma^h_{h\sigma} = \Gamma^h_{\sigma h} = \Gamma^1_{1\sigma} = \Gamma^1_{\sigma 1}$ for $h > 1, \sigma > 1$.*

Proof. Let $x(t)$ be a curve in $\mathcal{C}^T$. Since $f_i'(t)$, $i = 2, \dots, n$, satisfy Abelian differential equations with constant coefficients it follows from Lemma 1 and 2 that $f_i(t) = \frac{1}{2}\,\alpha_i\,t^2 + \beta_i t + \gamma_i$ with constants $\alpha_i \neq 0$, β_i and γ_i. Therefore for $x(t)$ we have

$$\dot{x}(t) = (1, \alpha_2 t + b_2, \alpha_3 t + b_3, \dots, \alpha_n t + b_n) \quad \text{and} \quad \ddot{x}_{(k)} = (0, \alpha_2, \alpha_3, \dots, \alpha_n).$$

The curve $x(t)$ is a geodesic if and only if relation (7) holds. For $h = 1$ in (7) one has

$$\varrho(t) = 2\sum_{\sigma=2}^{n} \Gamma^1_{1\sigma}\,(\alpha_\sigma t + \beta_\sigma) + \sum_{\sigma,\tau=2}^{n} \Gamma^1_{\sigma\tau}\,(\alpha_\sigma t + \beta_\sigma)\,(\alpha_\tau t + \beta_\tau),$$

and for $h > 1$ we get

$$\alpha_h + \Gamma^h_{11} + 2\sum_{\sigma=2}^{n} \Gamma^h_{1\sigma}\,(\alpha_\sigma t + \beta_\sigma) + \sum_{\sigma,\tau=2}^{n} \Gamma^h_{\sigma\tau}\,(\alpha_\sigma t + \beta_\sigma)\,(\alpha_\tau t + \beta_\tau) = \varrho(t)(\alpha_h t + \beta_h). \tag{21}$$

Putting $\varrho(t)$ into (21) we obtain a polynomial which is identically zero. It follows immediately that $\Gamma^1_{\sigma\tau} = 0$ for $\sigma > 1, \tau > 1$ and $\alpha_h = -\,\Gamma^h_{11}$ as well as $\Gamma^h_{\sigma 1} = \Gamma^h_{1\sigma} = 0$ for $h > 1, \sigma > 1$. Finally, we have

$$\sum_{\sigma,\tau=2}^{n} \Gamma^h_{\sigma\tau}\,(\alpha_\sigma t + \beta_\sigma)\,(\alpha_\tau t + \beta_\tau) - 2\,(\alpha_h t + \beta_h) \cdot \sum_{\sigma=2}^{n} \Gamma^1_{1\sigma}\,(\alpha_\sigma t + \beta_\sigma) = 0.$$

From this relation it follows that $\Gamma^h_{h\sigma} = \Gamma^h_{\sigma h} = \Gamma^1_{1\sigma} = \Gamma^1_{\sigma 1}$ for $h > 1, \sigma > 1$ and all other components $\Gamma^h_{\sigma\tau}$ vanish. □

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