

Solvable groups which do not possess characters of nontrivial prime power degree

By QINGYUN MENG (Zhengzhou), JIANLI CAO (Zhengzhou) and NI DU (Xiamen)

Abstract. In this paper, we consider solvable groups which do not possess characters of nontrivial prime power degree. In particular, for some of the minimal groups with this property, we characterize them and determine their structure.

1. Introduction

Let G be a finite group and $\text{cd}(G)$ be the set of the irreducible character degrees of G . In recent years, the influence of $\text{cd}(G)$ on the structure of G has been considered by many scholars. In [1], B. HUPPERT pointed out that the structure of a finite group G is controlled to a large extent by the type of the prime-number decomposition of the degrees of the irreducible characters of G over \mathbb{C} . In [9], O. MANZ considered finite solvable groups whose character degrees are powers of primes and characterized them. In this paper, we discuss the opposite situation. The finite solvable groups which do not possess characters of nontrivial prime power degree are investigated.

The study of $\text{cd}(G)$ has been assisted by attaching a graph to $\text{cd}(G)$. We often consider two kinds of graphs connected with these sets. One is called prime character degree graph, written by $\Delta(G)$, whose vertex set is $\rho(G)$, the set of

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primes dividing elements of $\text{cd}(G)$. Two vertices p and q in $\rho(G)$ are adjacent if there is some degree $a \in \text{cd}(G)$ such that pq divides a . The other is called common divisor graph, written by $\Gamma(G)$, whose vertex set is $\text{cd}(G) \setminus \{1\}$. And two vertices a and b are adjacent if they have common divisors. It has been shown in [7] that for a group G , if $\Gamma(G)$ is complete, then G is solvable. Clearly, if all the elements in $\text{cd}(G) \setminus \{1\}$ have a nontrivial common divisor p , then $\Gamma(G)$ is complete. Lewis in [8] proposed to find more groups whose $\Gamma(G)$ is complete while $\text{cd}(G) \setminus \{1\}$ has no common divisor. Obviously, these groups' each nontrivial character degree should have at least two distinct prime divisors. This stimulates us to learn more about solvable groups which do not possess characters of nontrivial prime power degree.

For convenience, a solvable group G is said to be a $(*)$ -group, for every $1 \neq a \in \text{cd}(G)$, a has at least two distinct prime divisors. Note that $(*)$ -property is inherited by epimorphic images, but not by subgroups, even normal subgroups. Call G a critical $(*)$ -group, if G is a $(*)$ -group, but any proper subgroup of G is not a $(*)$ -group. One can imagine that it's not easy to characterize $(*)$ -groups directly. Since every $(*)$ -group has critical $(*)$ -groups as its subgroups, we pay more attention to the properties of the critical $(*)$ -groups.

In Section 2 we give some lemmas about $(*)$ -group and critical $(*)$ -groups. Depending on whether the Frattini subgroup of G contains the socle or not, we consider critical $(*)$ -groups G separately in Section 3. The main results are the following.

Theorem 1.1. *G is a critical $(*)$ -group with $\text{Soc}(G) \not\leq \Phi(G)$ if and only if the following holds:*

- (1) $G' \cap \mathbf{Z}(G) = 1$ and $\mathbf{Z}(G) = \Phi(G)$; In particular, G' is a Sylow subgroup of G .
- (2) $\bar{G} = G/\mathbf{Z}(G)$ is a Frobenius group with a cyclic Frobenius complement of order qr , where q and r are primes. Also, \bar{G}' as the Frobenius kernel is the unique minimal normal subgroup of \bar{G} .

Theorem 1.2. *If G is a critical $(*)$ -group with a normal Sylow subgroup P , then one of the following holds.*

- (1) The character degree graph of G , $\Delta(G)$, is a complete graph;
- (2) $G' = P$ is nilpotent of class 2. In particular, G has derived length 3.

Now we fix some notation which will be used repeatedly. Let m be an integer. The set of prime divisors of m is denoted by $\pi(m)$. Let G be a finite group. $\Phi(G)$ is the Frattini subgroup of G . G_∞ is the nilpotent residual of G . $\text{Soc}(G)$ is the socle of G . Denote $\pi(G)$ for the set of the prime divisors of $|G|$. Write $\text{NL}(G)$

for the set of nonlinear irreducible characters of G . Let N be a normal subgroup of G . We define $\text{Irr}(G|N)$ to be the set of irreducible characters of G whose kernels do not contain N . Also, for a character $\vartheta \in \text{Irr}(N)$, let $\text{Irr}(G|\vartheta)$ denote the set of irreducible constituents of ϑ^G . Correspondingly, write $\text{cd}(G|\vartheta) = \{\chi(1) \mid \chi \in \text{Irr}(G|\vartheta)\}$. For convenience, when we say that x is a prime power we imply that $x \neq 1$.

2. Properties and lemmas

Lemma 2.1. *Let G be a finite group. Then $G_\infty = \bigcap_{p \in \pi(G)} \mathbf{O}^p(G)$.*

PROOF. Because G_∞ is contained in every normal subgroup whose quotient is nilpotent. It follows that $G_\infty \leq \mathbf{O}^p(G)$ for all $p \in \pi(G)$. And this implies that $\mathbf{O}^p(G/G_\infty) = \mathbf{O}^p(G)/G_\infty$. Since G/G_∞ is nilpotent, we have that $\bigcap_{p \in \pi(G)} \mathbf{O}^p(G/G_\infty) = \bar{1}$, which forces that $G_\infty = \bigcap_{p \in \pi(G)} \mathbf{O}^p(G)$. \square

Proposition 2.2. *Let G be a $(*)$ -group. Then $G' = \bigcap_{p \in \pi(G)} \mathbf{O}^p(G)$.*

PROOF. By Lemma 2.1, it is sufficient to show $G' = G_\infty$.

Because G_∞ is contained in every normal subgroup whose quotient is nilpotent. We have that $G_\infty \leq G'$ for any group. Since G is a $(*)$ -group, every nilpotent factor group of G is abelian and so $G' \leq G_\infty$. This forces $G' = G_\infty$ and $G' = \bigcap_{p \in \pi(G)} \mathbf{O}^p(G)$ follows. \square

Before we give another property of $(*)$ -groups, we first present a result related to semi-linear groups. We refer the reader to [10] for our notation relating to semi-linear groups.

Lemma 2.3. *Let V be a vector space over $GF(q)$ of dimension m , where q is a prime power. $\Gamma(V)$ is the semi-linear group of V . Then $\text{cd}(\Gamma(V)) = \{d \mid d|m\}$. In addition, if $((q^m - 1)/(q - 1), m) = 1$, then $\mathbf{Z}(\Gamma(V))$ is the Hall $\pi(q - 1)$ -subgroup of $\Gamma_0(V)$, where $\Gamma_0(V) \leq \Gamma(V)$ consisting of all of the multiplications; and $\text{cd}(K_1) = \text{cd}(\Gamma(V))$ for every $K_1 \geq K$, where K is a complement of $\mathbf{Z}(\Gamma(V))$ in $\Gamma(V)$.*

PROOF. Notice that $\Gamma(V) = \Gamma_0(V) \rtimes G$, where $G = \text{Gal}(GF(q^m)/GF(q))$ is a cyclic group of order m , $\Gamma_0(V) \leq \Gamma(V)$ consisting of all of the multiplications. And observe that $\text{cd}(\Gamma(V)) = \{|G : \mathbf{C}_G(\lambda)| \mid \lambda \in \text{Irr}(\Gamma_0(V))\}$. We know that $\Gamma_0(V) \cong \text{Irr}(\Gamma_0(V))$ as G -sets since G is cyclic. And so $\text{cd}(\Gamma(V)) = \{|G : \mathbf{C}_G(x)| \mid x \in \Gamma_0(V)\} = \{d \mid d|m\}$ (the second “=” is guaranteed by Galois theorem).

Furthermore, if $(|\Gamma_0(V)|/|\mathbf{Z}(\Gamma(V))|, m) = ((q^m - 1)/(q - 1), m) = 1$, then $((q^m - 1)/(q - 1), (q - 1)) = 1$ since $((q^m - 1)/(q - 1), (q - 1)) \mid m$ by Lemma 2.4 (a) of [11]. Thus $\mathbf{Z}(\Gamma(V))$ as the Hall $\pi(q - 1)$ -subgroup of $\Gamma_0(V)$ has a complement in $\Gamma(V)$, say K . That is $\Gamma(V) = K \times \mathbf{Z}(\Gamma(V))$. Now for every K_1 with $K \leq K_1 \leq \Gamma(V)$, we can see that $\text{cd}(K_1) = \text{cd}(\Gamma(V)) = \{d \mid d \mid m\}$. \square

Lemma 2.4. *If G is a $(*)$ -group. Then $\Delta(G)$, the prime character degree graph of G , is connected.*

PROOF. It is enough to prove the fact that if $\Delta(G)$ is non-connected, then $\text{cd}(G)$ contains prime powers. Suppose $\Delta(G)$ is non-connected, then from the Main Theorem of [6], G is as in Example 2.1–2.6 described in Section 2 of [6]. From the results of that paper, it is clear to see that when G is as in Example 2.1, 2.2, 2.3, 2.5, $\text{cd}(G)$ contains prime powers. Actually, when G is as in Example 2.4, $\text{cd}(G)$ also contains prime powers as referred from the statement prior to Lemma 3.4 of [6]. Since the author didn't give the proof in that paper, we write it down here.

Claim. If G is as in Example 2.4, then every divisor of m occurs in $\text{cd}(G)$.

As the definition of Example 2.4 shows: $G/V \cong H$, where V is a vector space over $GF(q)$ of dimension m , q is a prime power. Also, $K/Z \leq H/Z \leq \Gamma(V)$, $K/Z \leq \Gamma_0(V)$ and $H/K \cong \text{Gal}(V)$, where $(q^m - 1)/(q - 1) \mid |K/Z|$, $(m, |K/Z|) = 1$. It follows that $((q^m - 1)/(q - 1), m) = 1$ and so by Lemma 2.3, we have $\text{cd}(H/Z) = \text{cd}(\Gamma(V)) = \{d \mid d \mid m\}$. Finally, notice that $G/V \cong H$, the claim follows. In particular, we get that $\text{cd}(G)$ contains prime powers when G is as in Example 2.4.

At last, by Lemma 3.6 (v) of [6], we also know G has irreducible characters of prime powers degrees when G is as in Example 2.6. Now we complete the proof. \square

Let G be a finite group with $|\rho(G)| \geq 2$. If G satisfies that for every $\chi \in \text{NL}(G)$, $\pi(\chi(1)) = \rho(G)$, then G is a $(*)$ -group. For this kind of $(*)$ -groups, when $|\rho(G)| = 2$, there was a description as the following lemma shows.

Lemma 2.5 ([12]). *Let p and q be two distinct prime integers, and G be a finite group. Then the following two are equivalent:*

- (1) *Any nonlinear irreducible character of G has degree $p^r q^s$ with both $r, s > 0$.*
- (2) *G has abelian normal $\{p, q\}$ -complement A and an abelian Hall $\{p, q\}$ -subgroup $H = P \times Q$, where P is a Sylow p -subgroup and Q is a Sylow q -subgroup, and the centralizers $\mathbf{C}_A(P) = \mathbf{C}_A(Q)$.*

In fact, when $|\rho(G)| \geq 2$, by using the results of paper [12], the conclusion also holds. We write it down here as a theorem.

Theorem 2.6. *Let G be a finite group with $|\rho(G)| \geq 2$. Then the following two are equivalent.*

- (1) *For every $\chi \in \text{NL}(G)$, $\pi(\chi(1)) = \rho(G)$ holds.*
- (2) *G has a nontrivial abelian normal Hall $(\rho(G))'$ -subgroup A and an abelian Hall $\rho(G)$ -subgroup H , where for every $P \in \text{Syl}_p(G)$, $Q \in \text{Syl}_q(G)$ and every pair of primes $\{p, q\} \subseteq \rho(G)$, $\mathbf{C}_A(P) = \mathbf{C}_A(Q)$ holds.*

PROOF. Firstly, we prove that the statement (2) implies statement (1). In fact, for every pair of primes $\{p, q\} \subseteq \rho(G)$ and every $P \in \text{Syl}_p(G)$, $Q \in \text{Syl}_q(G)$, we get that $\mathbf{C}_{\text{Irr}(A)}(P) = \mathbf{C}_{\text{Irr}(A)}(Q)$ by Lemma 3 of [12]. This implies that for any $\varphi \in \text{Irr}(A)$, either $I_H(\varphi) = H$ holds or $|H : I_H(\varphi)|$ can be divided by every prime in $\rho(G)$. Notice that A and H are abelian and $G = A \rtimes H$. By using Gallagher's theorem and Clifford's theorem, we have that for any $\chi \in \text{NL}(G)$, $\pi(\chi(1)) = \rho(G)$ holds.

Now, we prove that the statement (1) implies the statement (2). By hypothesis, for any $\chi \in \text{NL}(G)$, $\pi(\chi(1)) = \rho(G)$ holds. Using Itô's theorem, we know that G has an abelian normal Hall $(\rho(G))'$ -subgroup, say A . Furthermore, by Schur–Zassenhaus theorem we also know that G has a Hall $\rho(G)$ -subgroup, say H . We claim that H is abelian. In fact, $H \cong G/A$ is a quotient group of G . It is easy to see that the degree of any nonlinear irreducible character of H satisfies the statement(1). In particular, for any $p \in \rho(G)$, by Thompson's theorem H has a normal p -complement. Notice that $\rho(G) = \pi(H)$. By the arbitrariness of p , we conclude that H is nilpotent. Suppose that H is nonabelian. Then there exists some prime $p \in \pi(H)$ and $P \in \text{Syl}_p(H)$ such that P is nonabelian. Hence, H has at least one nonlinear irreducible character of p -power degree. This contradicts with the statement (1). Thus, H must be abelian. Observe that A and H are abelian and $G = A \rtimes H$. If we take any pair of primes $\{p, q\} \subseteq \rho(G)$ and $P \in \text{Syl}_p(G)$, $Q \in \text{Syl}_q(G)$. Then $A \rtimes (P \times Q)$ is the Hall $(\rho(G))' \cup \{p, q\}$ -subgroup of G . And for any $\varphi \in \text{NL}(A \rtimes (P \times Q))$, $\pi(\varphi(1)) = \{p, q\}$ holds. Implying Lemma 2.5 to $A \rtimes (P \times Q)$, we have that $\mathbf{C}_A(P) = \mathbf{C}_A(Q)$ holds. \square

Lemma 2.7. *Let G be a critical $(*)$ -group and $1 < N \triangleleft G$. If N has a complement in G , then $G' \leq N$. Furthermore, if N is a minimal normal subgroup of G , then $G' = N$.*

PROOF. Let H be a complement of N in G . Suppose $G' \not\leq N$, then $H \cong G/N$ is nonabelian. And so $\text{cd}(G/N) = \text{cd}(H)$ contains prime powers since G is a

critical $(*)$ -group. Notice that $\text{cd}(G/N) \subseteq \text{cd}(G)$. This contradicts with $\text{cd}(G)$ has no prime powers. Now $G' \leq N$ holds. Recall that G is nonabelian and hence $G' \neq 1$. Thus the minimality of N forces $G' = N$. \square

Lemma 2.8. *Let G be a critical $(*)$ -group. Then G has at most one normal Sylow subgroup. In particular, $|\rho(G)| \leq |\pi(G)| + 1$.*

PROOF. Suppose there exist distinct primes p and q belonging to $\pi(G)$ such that G has a normal Sylow p -subgroup P and a normal Sylow q -subgroup Q . Then by Schur–Zassenhaus theorem and Lemma 2.7, we get that $G' \leq P \cap Q = 1$, contradicting with that G is nonabelian. Thus, G has at most one normal Sylow subgroup. Particularly, in light of Itô's theorem (see Corollary 12.34 in [4]), the inequality $|\pi(G)| \leq |\rho(G)| + 1$ holds. \square

3. Critical $(*)$ -groups

Let G be a critical $(*)$ -group. The following theorem is a description of G when $\text{Soc}(G) \not\leq \Phi(G)$.

Theorem 3.1. *G is a critical $(*)$ -group with $\text{Soc}(G) \not\leq \Phi(G)$ if and only if the following two hold:*

- (1) $G' \cap \mathbf{Z}(G) = 1$ and $\mathbf{Z}(G) = \Phi(G)$; In particular, G' is a Sylow subgroup of G .
- (2) $\bar{G} = G/\mathbf{Z}(G)$ is a Frobenius group with a cyclic Frobenius complement of order qr , where q and r are two distinct primes. Also, \bar{G}' as the Frobenius kernel is the unique minimal normal subgroup of \bar{G} .

PROOF. We first prove that (1) and (2) hold in a critical $(*)$ -group with $\text{Soc}(G) \not\leq \Phi(G)$. Since $\text{Soc}(G)$ is not contained in $\Phi(G)$, it follows that there exists some minimal normal subgroup of G , say N , such that $N \cap \Phi(G) = 1$. Then by Hilfsatz III. 4.4 of [3], we know that N has a complement in G , say H . And so by Lemma 2.7, $N = G'$ follows. Now $G = G' \rtimes H$. Since G' is a minimal normal subgroup of G , we may assume G' is a p -group, and let $P \in \text{Syl}_p(G)$. Then $G' \leq P$ and so $P \triangleleft G$. Note that $P' \leq \Phi(G) \cap G' = \Phi(G) \cap N = 1$. This is $P' = 1$ and so P is abelian. Recall that $G' \leq P$, we have that every Sylow subgroup of G is abelian. That is to say, G is an A -group. We get $P = G' \times (P \cap \mathbf{Z}(G))$ from a theorem of Taunt (see Theorem 14.6, Chapter 6 in [3]). In particular, we have $G' \cap \mathbf{Z}(G) = 1$.

Consider factor group $\bar{G} = G/\mathbf{Z}(G)$. Observing that $\mathbf{Z}(G) = \mathbf{C}_H(G')$, it follows that $H/\mathbf{Z}(G)$ acts faithfully, coprimely and irreducibly on \bar{G}' . Thus, \bar{G}' is the unique minimal normal subgroup of \bar{G} . Using Theorem 12.3 of [4], we know that \bar{G} is a Frobenius group with \bar{G}' as its Frobenius kernel.

Now we show that \bar{G} is a critical $(*)$ -group. It is clear that \bar{G} is a $(*)$ -group since G is a $(*)$ -group. For any nonabelian subgroup $\bar{K} = K/\mathbf{Z}(G) < \bar{G}$, it suffices to show that $\text{cd}(\bar{K})$ contains prime powers. Note that $\text{Irr}(K) = \text{Irr}(\bar{K}) \cup \text{Irr}(K|\mathbf{Z}(G))$. For every $\lambda \in \text{Irr}(\mathbf{Z}(G))$, since $G' \cap \mathbf{Z}(G) = 1$, by Proposition 19.12 (a) of [2], λ extends to K . Now using Gallagher's theorem it follows that $\text{cd}(K|\lambda) = \text{cd}(\bar{K})$. And thus, $\text{cd}(K) = \text{cd}(\bar{K})$. Recall that G is a critical $(*)$ -group and $K < G$, therefore, $\text{cd}(K) = \text{cd}(\bar{K})$ contains prime power.

On the other hand, because \bar{G} is a Frobenius group, we have $\text{cd}(\bar{G}) = \{1, |\bar{H}|\}$. Since we have known that \bar{G} is a critical $(*)$ -group, it forces $|\bar{H}| = qr$, where q and r are distinct primes.

At last, we prove $\mathbf{Z}(G) = \Phi(G)$. On one hand, $\Phi(G) \leq \mathbf{Z}(G)$ since $\Phi(G) \cap G' = 1$. On the other hand, we claim that $\mathbf{Z}(G) \leq \Phi(H)$, and so $\mathbf{Z}(G) \leq \Phi(G)$ by Hilfsatz III. 3.3 of [3], which implies that $\mathbf{Z}(G) = \Phi(G)$.

Now we prove $\mathbf{Z}(G) \leq \Phi(H)$. It is enough to show that for any $L < H$, $L\mathbf{Z}(G) < H$. If this is not true, then there exists some $L < H$ such that $L\mathbf{Z}(G) = H$. Let $G_1 = G'L$, then $G_1 < G$. Note that $\bar{G} \cong G_1/L \cap \mathbf{Z}(G)$ is a Frobenius group with $\text{cd}(G_1) = \text{cd}(G_1/L \cap \mathbf{Z}(G)) = \text{cd}(\bar{G}) = \{1, qr\}$, which contradicts with G is a critical $(*)$ -group (the first “=” is also guaranteed by Proposition 19.12 (a) of [2]). Thus, we have $\mathbf{Z}(G) \leq \Phi(H)$. In particular, $P \cap \mathbf{Z}(G) \leq \Phi(H)$. Observe that $P = G' \times (P \cap \mathbf{Z}(G))$, $P \cap \mathbf{Z}(G)$ is the Sylow p -subgroup of H , this forces that $P \cap \mathbf{Z}(G) = 1$ and $P = G' \times (P \cap \mathbf{Z}(G)) = G'$ is the Sylow p -subgroup of G .

Now we prove the sufficiency of the theorem. Note that $G' \cap \mathbf{Z}(G) = 1$, by Proposition 19.12 of [2], we get $\text{cd}(G) = \text{cd}(\bar{G}) = \{1, qr\}$ and so G is a $(*)$ -group. Clearly, \bar{G} is a critical $(*)$ -group. Now for any nonabelian subgroup $K < G$, consider $\text{cd}(K)$. Since $\mathbf{Z}(G) = \Phi(G)$, it follows that $K\mathbf{Z}(G) < G$. For $\bar{K} < \bar{G}$, by the critical property of \bar{G} , we get that $\text{cd}(\bar{K}) = \text{cd}(K/K \cap \mathbf{Z}(G)) = \text{cd}(K)$ (the second “=” is also guaranteed by Proposition 19.12 (a) of [2]) contains prime powers. And so G is a critical $(*)$ -group.

Note that \bar{G}' is the unique minimal normal subgroup of \bar{G} , and $G' \cap \mathbf{Z}(G) = 1$. It follows that G' is a minimal normal subgroup of G which is not contained in $\Phi(G) = \mathbf{Z}(G)$. That is $\text{Soc}(G) \not\leq \Phi(G)$. □

Corollary 3.2. *Suppose G is a critical $(*)$ -group with $\text{Soc}(G) \not\leq \Phi(G)$. Then*

- (1) $\text{cd}(G) = \{1, qr\}$, where $q \neq r$ are two distinct primes;

$$(2) |\pi(G)| = |\rho(G)| + 1.$$

PROOF. Let $\bar{G} = G/\mathbf{Z}(G)$. Note that $\text{cd}(G) = \text{cd}(\bar{G}) \cup \text{cd}(G|\mathbf{Z}(G))$. From Theorem 3.1 we know $G' \cap \mathbf{Z}(G) = 1$, then by using Proposition 19.12 (a) of [2] and Gallagher's theorem, we obtain $\text{cd}(G|\mathbf{Z}(G)) = \text{cd}(\bar{G})$ and so $\text{cd}(G) = \text{cd}(\bar{G})$. Again by Theorem 3.1, (1) follows.

From Theorem 3.1, we know that G' is an abelian normal Sylow subgroup of G . In light of Ito's theorem and Lemma 2.8, $|\rho(G)| + 1 = |\pi(G)|$ follows. \square

In the following, we consider critical $(*)$ -group G when $\text{Soc}(G) \leq \Phi(G)$.

Proposition 3.3. *Suppose G is a critical $(*)$ -group with $\text{Soc}(G) \leq \Phi(G)$, then there exists $\chi \in \text{NL}(G)$ such that $\pi(\chi(1)) \neq \rho(G)$. In particular, $|\rho(G)| \geq 3$ and $|\text{cd}(G)| \geq 3$.*

PROOF. Suppose that for every $\theta \in \text{NL}(G)$, $\pi(\theta(1)) = \rho(G)$. By using Theorem 2.6, we get that $G = A \rtimes H$, where A is the abelian normal Hall $(\rho(G))'$ -subgroup of G , H is a Hall $\rho(G)$ -subgroup of G and $\mathbf{C}_A(Q) = \mathbf{C}_A(R)$ for every pair of primes $\{q, r\} \subseteq \rho(G)$, $Q \in \text{Syl}_q(G)$, $R \in \text{Syl}_r(G)$. Now it's not difficult to see that $\mathbf{C}_A(Q) = \mathbf{C}_A(H)$. Using Fitting's lemma, we get $[A, Q] = [A, H]$. Clearly, $[A, H] \neq 1$. Now we show that $[A, H] = A$. Suppose $[A, H] < A$, again by Fitting's lemma, $G = [A, H]H \times \mathbf{C}_A(H)$ and so $\text{cd}(G) = \text{cd}([A, H]H)$. This implies $[A, H]H$ is a $(*)$ -group, which contradicts with G is a critical $(*)$ -group. Now we have $[A, H] = A$. And $1 = \mathbf{C}_A(H) = \mathbf{C}_A(Q) = \mathbf{C}_A(R)$ holds.

By Lemma 2.8, we know that A is a Sylow subgroup of G , say A is a Sylow p -subgroup. Since $\text{Soc}(G) \leq \Phi(G)$, we have $1 \neq A \cap \text{Soc}(G) \leq A \cap \Phi(G) \leq M$, where M is any maximal subgroup of G . This fact tells us that for any maximal subgroup of G , say M , the p -part of M , i.e. $M \cap A$ is nontrivial. Since A is the Sylow p -subgroup of G , we have $A \not\leq \Phi(G)$. This implies that there exists a maximal subgroup of G , say M_0 , such that $M_0 \cap A$ is not A . By the maximality of M_0 , we also know that M_0 contains a subgroup H_0 which is conjugating to H . Now $M_0 = (A \cap M_0) \rtimes H_0$. Recall that $1 = \mathbf{C}_A(Q) = \mathbf{C}_A(R)$ for every $Q \in \text{Syl}_q(G)$, $R \in \text{Syl}_r(G)$ and every pair of primes $\{q, r\} \subseteq \rho(G)$. In light of Itô's theorem, we get that $\rho(G) = \rho(M_0)$. Applying the statement (2) of Theorem 2.6 to M_0 , we see that M_0 is a $(*)$ -group which contradicts our hypothesis that G is a critical $(*)$ -group. Thus, there must exist a character $\chi \in \text{NL}(G)$ with $\pi(\chi(1)) \neq \rho(G)$. In particular, since G is a $(*)$ -group, we have that $|\rho(G)| \geq 3$ and $|\text{cd}(G)| \geq 3$ hold. \square

Now we give an example of critical $(*)$ -group G with $\text{Soc}(G) \leq \Phi(G)$.

Example 3.4. Let F be the Galois field of order 2^6 . Define a 2-group $P = \{x = (x_1, x_2) \mid x_1, x_2 \in F\}$ with the operation “ \cdot ”:

$$x \cdot y = (x_1, x_2) \cdot (y_1, y_2) = (x_1 + y_1, x_2 + y_2 + x_1 y_1^2), \quad \forall x, y \in P$$

It is easy to see (P, \cdot) is a group. Fix two elements of $F^*(F^* \cong C_{63})$ of order 3 and 7, say s and t respectively. Now let s and t act on P as follows:

$$s : P \rightarrow P, \quad x = (x_1, x_2) \mapsto (x_1 s, x_2), \quad \forall x \in P.$$

$$t : P \rightarrow P, \quad x = (x_1, x_2) \mapsto (x_1 t, x_2 t^3), \quad \forall x \in P.$$

We can see that $s, t \in \text{Aut}(P)$ are of order 3 and 7 respectively. Let $C = \langle s, t \rangle$, then $C \leq \text{Aut}(P)$ is a cyclic group of order 21. Now let G denote the semi-direct product of C over P , then G is a critical $(*)$ -group with $\text{Soc}(G) \leq \Phi(G)$.

PROOF. First, we claim $P' = \Phi(P) = \mathbf{Z}(P) = \{(0, a) \mid a \in F\}$. In fact, we can obtain $\mathbf{Z}(P) = \{(0, a) \mid a \in F\}$ by computing. Observe that $P/\mathbf{Z}(P) \cong F$ is abelian and so $P' \leq \mathbf{Z}(P)$. And it is not hard to get $|\mathbf{Z}(P) : P'| \leq 2$ by computing. Notice that $\langle t \rangle \cong C_7$ acts Frobeniusly on $P' = \mathbf{Z}(P) \cap P'$. Then $|P'| \equiv 1 \pmod{7}$. This forces $P' = \mathbf{Z}(P)$. Notice that $P/P' = P/\mathbf{Z}(P) \cong F$. This forces that $P' = \Phi(P)$ since $P' \leq \Phi(P)$. Now we have $P' = \Phi(P) = \mathbf{Z}(P)$.

In the following, we give 3 steps to show G is a critical $(*)$ -group with $\text{Soc}(G) \leq \Phi(G)$.

Step 1. G is a $(*)$ -group.

Note that $P/\mathbf{Z}(P) \cong F$ and C acts Frobeniusly on $P/\mathbf{Z}(P)$, which equals to say that $G/\mathbf{Z}(P)$ is a Frobenius group with $\text{cd}(G/\mathbf{Z}(P)) = \{1, 21\}$. On the other hand, for nonprincipal character $\lambda \in \text{Irr}(\mathbf{Z}(P))$, $\chi \in \text{Irr}(G \mid \lambda)$, by Theorem 6.11 of [4], we know there exists a $\theta \in \text{Irr}(\mathbf{C}_G(\lambda) \mid \lambda)$ such that $\chi(1) = |G : \mathbf{C}_G(\lambda)|\theta(1)$. Observe that $\mathbf{C}_G(\lambda) = P\langle s \rangle$, and so $\chi(1) = 7\theta(1)$. Moreover, since $P' = \mathbf{Z}(P)$ and $\mathbf{Z}(P) = P' \cap \mathbf{Z}(P) \not\leq \text{Ker}(\lambda)$, it follows that λ can't extend to P by Proposition 19.12 (a) of [2]. This fact implies that $2 \mid \theta(1)$ and so $14 \mid \chi(1)$. Now we get that G is a $(*)$ -group.

Step 2. G is a critical $(*)$ -group.

First, we claim that $\bar{P} = P/\mathbf{Z}(P)$ is irreducible under C . Suppose this is not true and $\bar{P}_1 < \bar{P}$ is invariant under C . Since C act Frobeniusly on \bar{P}_1 , it follows that $|\bar{P}_1| \equiv 1 \pmod{21}$. This forces that $\bar{P}_1 = \bar{P}$ and so the claim holds. Let $H \leq G$ be a $(*)$ -group, then by Corollary 12.2 of [4], we know $|\pi(H)| \geq 3$. And so H contains a Hall $\{3, 7\}$ -subgroup of G . Conjugating if necessary, we may assume $C \leq H$. We show that $H = G$ and so complete the proof of Step 2.

In fact, suppose $H \cap P \leq \mathbf{Z}(P)$, then $\text{cd}(H) = \{1, 7\}$, contradicting with H is a $(*)$ -group. So we have $H \cap P \not\leq \mathbf{Z}(P)$. Notice that C normalize $H \cap P$ since $H \cap P \triangleleft H$, and C normalize $\mathbf{Z}(P)$. It follows that $(H \cap P)\mathbf{Z}(P)/\mathbf{Z}(P)$ is invariant under C . This forces that $(H \cap P)\mathbf{Z}(P) = P$ since $\bar{P} = P/\mathbf{Z}(P)$ is irreducible under C . In addition, since $\mathbf{Z}(P) = \Phi(P)$, we have $P = (H \cap P)\mathbf{Z}(P) = H \cap P$ and $H = G$. Now we obtain that G is a critical $(*)$ -group.

Step 3. $\text{Soc}(G) \leq \Phi(G)$.

It is not hard to see $\text{Soc}(G) \leq P$. Let N be any minimal normal subgroup of G , we show $N \leq \mathbf{Z}(P)$. Suppose $N \not\leq \mathbf{Z}(P)$. Consider factor group $\bar{N} = N\mathbf{Z}(P)/\mathbf{Z}(P)$. Clearly, $\bar{1} \neq \bar{N} \leq P/\mathbf{Z}(P)$, which is invariant under C . Observe the fact that $\bar{P} = P/\mathbf{Z}(P)$ is irreducible under C in Step 2, this forces $N\mathbf{Z}(P) = P$. Recall that N is a minimal normal subgroup which is abelian, we get that P is abelian, a contradiction. Now, $\text{Soc}(G) \leq \mathbf{Z}(P)$. Since $\mathbf{Z}(P) = \Phi(P)$, we have $\text{Soc}(G) \leq \Phi(G)$. \square

We study critical $(*)$ -groups under the following hypothesis throughout the rest of this paper.

Hypothesis 1. Let G be a critical $(*)$ -group with $\text{Soc}(G) \leq \Phi(G)$. Suppose G has a normal Sylow p -subgroup P and let H be a p -complement in G .

Lemma 3.5. *Assume Hypothesis 1. Then P is nonabelian.*

PROOF. Suppose P is abelian, then we claim that P is a minimal normal subgroup of G . Or else, let $P_1 < P$ be H -invariant, then $P_1H < G$. Since G is a critical $(*)$ -group, it follows that $\text{cd}(P_1H)$ contains prime powers and so there exists $\lambda \in \text{Irr}(P_1)$ with $|H : \mathbf{C}_H(\lambda)|$ is a prime power. Using Theorem 13.28 of [4], there exists a character $\xi \in \text{Irr}(P \mid \lambda)$ with $\mathbf{C}_H(\lambda) \leq \mathbf{C}_H(\xi)$. Recall that P is abelian and so $\xi_{P_1} = \lambda$ with $\mathbf{C}_H(\lambda) \geq \mathbf{C}_H(\xi)$, which implies that $\mathbf{C}_H(\lambda) = \mathbf{C}_H(\xi)$. Now by Clifford's theorem and Gallagher's theorem, there exists a character $\chi \in \text{Irr}(G \mid \xi)$ with $\chi(1) = |H : \mathbf{C}_H(\lambda)|$ which is a prime power; a contradiction. Now the claim holds. In this case, $G' = P \leq \text{Soc}(G) \leq \Phi(G)$ by hypothesis. And so G is nilpotent, contradicting with that G is a $(*)$ -group. Thus P is nonabelian. \square

Corollary 3.6. *Let G be a critical $(*)$ -group with $\text{Soc}(G) \leq \Phi(G)$. Then $|\rho(G)| = |\pi(G)|$.*

In fact, if there is no normal Sylow subgroup, then in light of Itô's theorem, we know $|\rho(G)| = |\pi(G)|$; if G has a normal Sylow subgroup, then by Lemma 3.5, we also have $|\rho(G)| = |\pi(G)|$.

Lemma 3.7. *Assume Hypothesis 1. Then the following holds:*

- (1) $\mathbf{C}_{G'}(H) = 1$;
- (2) $G' = [P, H]$ and $G = G' \rtimes \mathbf{N}_G(H)$, where $\mathbf{N}_G(H) = \mathbf{C}_G(H) = \mathbf{C}_P(H) \times H$.

PROOF. Since there is no $\theta \in \text{NL}(P)$ extending to G , by Lemma 3.2 of [13] we have that $\mathbf{C}_{P'}(H) = 1$. Clearly, $P' \leq G' \leq P$ and $G'/P' = [P/P', H] = [P/P', H, H] = [G'/P', H]$. By Fitting's lemma, we get that $\mathbf{C}_{G'/P'}(H) = \bar{1}$, and so $\mathbf{C}_{G'}(H) \leq P'$. Recall that $\mathbf{C}_{P'}(H) = 1$. It forces $\mathbf{C}_{G'}(H) = 1$. Now, $G' = G' \cap P = G' \cap ([P, H]\mathbf{C}_P(H)) = [P, H](G' \cap \mathbf{C}_P(H)) = [P, H]$. In particular, we have $P = G' \rtimes \mathbf{C}_P(H)$ and so $G = G' \rtimes (\mathbf{C}_P(H) \times H)$. It is not hard to see $\mathbf{N}_G(H) = \mathbf{C}_G(H) = \mathbf{C}_P(H) \times H$, which is a complement of G' in G . \square

Lemma 3.8. *Assume Hypothesis 1. Then the following holds:*

- (1) For any character $\varphi \in \text{NL}(P)$, $\mathbf{C}_H(\varphi) < H$.
- (2) For any $\lambda \in \text{Irr}(P/P')$ with G' not contained in $\text{Ker}(\lambda)$, $\mathbf{C}_H(\lambda) < H$. In particular, $|H : \mathbf{C}_H(\lambda)|$ has at least two distinct prime divisors.
- (3) Let $P_1 \triangleleft P$ be any H -invariant subgroup of P . If $\varphi \in \text{NL}(P_1)$, then $\mathbf{C}_H(\varphi) < H$.

PROOF. It is trivial for (1). For (2), let $\lambda \in \text{Irr}(P/P')$ such that G' is not contained in $\text{Ker}(\lambda)$. If $\mathbf{C}_H(\lambda) = H$, then λ is extendible to G . Suppose η is an extension of λ . Clearly, $G' \leq \text{Ker}(\eta)$. Then $G' = G' \cap P \leq \text{Ker}(\eta) \cap P \leq \text{Ker}(\lambda)$, a contradiction. In particular, since G is a critical (*)-group, we have that $|H : \mathbf{C}_H(\lambda)|$, as an irreducible character degree, has at least two distinct prime divisors. For (3), suppose there exists a character $\varphi \in \text{NL}(P_1)$ such that $\mathbf{C}_H(\varphi) = H$. Since $P_1 \triangleleft P$ is H -invariant, then by using Theorem 13.28 of [4], there exists $\theta \in \text{Irr}(P | \varphi)$ such that θ is H -invariant, which contradicts with (1). Now we complete the proof. \square

Under Hypothesis 1, by Lemma 2.7 we know $G' \leq P$ and so $G'' \leq P'$. In fact, if $G'' = P'$, we also have $G' = P$.

Lemma 3.9. *Assume Hypothesis 1. If $G'' = P'$, then $G' = P$.*

PROOF. Suppose $G'' = P'$ and $G' < P$. We show there will be a contradiction and so complete the proof.

Clearly, $G'H \triangleleft G$. Note that $\text{cd}(G'H) = \text{cd}(G'H/P') \cup \text{cd}(G'H|P')$. For $\varphi \in \text{NL}(G'H/P')$, take $\theta \in \text{Irr}(G'/P')$ be an irreducible constituent of $\varphi_{G'/P'}$. It's clear that θ is nonprincipal. Using Theorem 8.16 of [4], θ extends to $G'\mathbf{C}_H(\theta)/P'$. By Gallagher's theorem, we know that for any $\psi \in \text{Irr}(G'\mathbf{C}_H(\theta)/P' | \theta)$, $\psi(1) = \theta(1) = 1$. According to Clifford's theorem, we have $\varphi(1) = |H : \mathbf{C}_H(\theta)|\theta(1) = |H :$

$\mathbf{C}_H(\theta)$. On the other hand, using Theorem 13.28 of [4], there exists a character $\vartheta \in \text{Irr}(P/P')$ with $[\vartheta_{G'/P'}, \theta] \neq 0$ such that $\mathbf{C}_H(\theta) \leq \mathbf{C}_H(\vartheta)$. Observe that $\vartheta_{G'/P'} = \theta$, thus $\mathbf{C}_H(\theta) \geq \mathbf{C}_H(\vartheta)$. And so $\mathbf{C}_H(\theta) = \mathbf{C}_H(\vartheta)$. Clearly, $G' \not\leq \text{Ker } \vartheta$. By Lemma 3.8 (2), we know $|H : \mathbf{C}_H(\vartheta)| = |H : \mathbf{C}_H(\theta)|$ has two distinct primes. And so $\varphi(1) = |H : \mathbf{C}_H(\theta)|$ is not prime power for any $\varphi \in \text{NL}(G'H/P')$.

For $\varphi \in \text{Irr}(G'H | P')$, let $\xi \in \text{Irr}(G')$ with $[\varphi_{G'}, \xi] \neq 0$. If ξ is linear, then $\varphi_{G'} = e(\xi_1 + \cdots + \xi_t)$ and $\text{Ker } \varphi \geq \bigcap_{i=1, \dots, t} \text{Ker } \xi_i \geq G'' = P'$, contradicting with that P' is not contained in $\text{Ker } \varphi$. Thus ξ is nonlinear. Now using Lemma 3.8 (3), we have $\mathbf{C}_H(\xi) < H$. By Clifford's theorem, we get that $\varphi(1)$ is not a prime power.

Now we can see that $G'H$ is a $(*)$ -group, a contradiction, as desired. \square

Lemma 3.10. *Assume Hypothesis 1 and suppose $\Delta(G)$ is not complete. Then there exists a prime $r \in \pi(H)$ which is not adjacent to p , and $G' = P = [P, R]$ is nilpotent of class 2, where $R \in \text{Syl}_r(G)$.*

PROOF. From Lemma 3.8 (2), we know that $G' \leq P \leq \mathbf{F}(G)$ and P is the unique normal Sylow subgroup of G , and so $\pi(H) = \pi(|G : \mathbf{F}(G)|)$. By Theorem 18.1 of [10], we know that $|G : \mathbf{F}(G)| \in \text{cd}(G)$. Thus, the derived subgraph of $\pi(H)$ is complete. Since $\Delta(G)$ is not complete by hypothesis, it must be that p is not adjacent to some $r \in \pi(H)$.

Let R be a Sylow r -subgroup of G . We consider subgroup PR . Clearly, $PR \triangleleft G$ since $G' \leq PR$ and R acts coprimely on P by fixing every nonlinear character of P . Applying Theorem 19.3 of [10], we have that $[P, R]$ is nilpotent of class 2 with $[P, R]' = P'$ and $P' \leq \mathbf{Z}(PR)$. Note that $P' = [P, R]' \leq G''$ and it forces that $P' = G''$. By Lemma 3.9, we have $G' = P$ which is nilpotent of class 2 since $P' \leq \mathbf{Z}(PR) \leq \mathbf{Z}(P)$.

Next, we claim that $[P, R] = P$. In fact, if $[P, R] < P$, then consider subgroup $[P, R]H$. We show $[P, R]H$ is a $(*)$ -group and get a contradiction. For $\varphi \in \text{NL}([P, R]H)$, let $\xi \in \text{Irr}([P, R])$ with $[\varphi_{[P, R]}, \xi] \neq 0$, then ξ is a nonprincipal character. If ξ is nonlinear, then $\varphi(1)$ is not a prime power by Lemma 3.8 (3). If ξ is linear, observe that

$$\xi \in \text{Irr}([P, R]/P') \subseteq \text{Irr}([P, R]/P' \times \mathbf{C}_{P/P'}(R)) = \text{Irr}(P/P')$$

and $\text{Ker } \xi$ does not contain $G' = P$. Applying Lemma 3.8 (2), we also get that $\varphi(1)$ is not a prime power. Now we proved the claim. \square

Corollary 3.11. *Assume Hypothesis 1 and suppose that $\Delta(G)$ is not complete. If $r \in \pi(H)$ is not adjacent to p , then the Sylow r -subgroups of G are cyclic.*

Moreover, if there exists another prime s such that s is not adjacent to p either, then $P' = \mathbf{Z}(P)$.

PROOF. By Lemma 3.10, we know $P = [P, R]$ and so $\mathbf{C}_{P/P'}(R) = \bar{1}$ by Fitting's Lemma. This is to say the principal character is the unique R -invariant character in $\text{Irr}(P/P')$. Observing that G is a $(*)$ -group, we conclude that the r -complement of H , $\mathbf{O}^r(H)$, does not fix any nonprincipal character of $\text{Irr}(P/P')$. In the proof of Lemma 3.10, we also know R fixes every nonlinear character of P . By the same reason, $\mathbf{O}^r(H)$ does not fix any element of $\text{NL}(P)$. Now we get that $\mathbf{O}^r(H)$ does not fix any nontrivial element of $\text{Irr}(P)$.

Suppose R is not cyclic, then there exists $R_0 < R$ such that $[P, R_0] \neq 1$. In fact, if every proper subgroup of R centralizes P , since R is not cyclic, we can find two different maximal subgroups of R , say R_1, R_2 , such that $R = R_1R_2$. And then $[P, R] = [P, R_1R_2] = [P, R_1][P, R_2] = 1$, this is a contradiction. Now consider subgroup PR_0 . Clearly, $PR_0 \triangleleft G$. And it is not hard to see that $\Delta(PR_0)$ is not complete since p is not adjacent to r in $\Delta(G)$. By replacing $[P, R]$ with $[P, R_0]$, repeat the argument for $[P, R]$ in the proof of Lemma 3.10, we can also get $[P, R_0] = P$. Applying Fitting's lemma, $\mathbf{C}_{P/P'}(R_0) = \bar{1}$. Thus R_0 fixes no nontrivial element in $\text{Irr}(P/P')$.

Consider subgroup $P\mathbf{O}^r(H)R_0$. Clearly, $P\mathbf{O}^r(H)R_0 < G$. Since we have shown that $\mathbf{O}^r(H)$ does not fix any nontrivial element of $\text{Irr}(P)$ and R_0 fix every member of $\text{NL}(P)$ but no nontrivial one in $\text{Irr}(P/P')$, so we obtain that $P\mathbf{O}^r(H)R_0$ is a $(*)$ -group, contradicting with G is a critical $(*)$ -group. Thus, R must be cyclic.

Suppose neither r nor s is adjacent to p . Let RS be a Hall $\{r, s\}$ -subgroup of H . We have known that $P' \leq \mathbf{Z}(P)$ in the proof of Lemma 3.10. Suppose P' is a proper subgroup of $\mathbf{Z}(P)$. Notice that $\mathbf{C}_P(RS) = \mathbf{C}_P(R) \cap \mathbf{C}_P(S) = P'$ since $\mathbf{C}_{P/P'}(R) = \mathbf{C}_{P/P'}(S) = \bar{1}$ and $P' \leq \mathbf{Z}(PR) \cap \mathbf{Z}(PS)$, it follows that $\mathbf{Z}(P)RS$ is nonabelian.

Now consider $\text{cd}(\mathbf{Z}(P)RS)$. For $\varphi \in \text{Irr}(\mathbf{Z}(P)RS/P')$, let $\theta \in \text{Irr}(\mathbf{Z}(P)/P')$ such that $[\varphi_{\mathbf{Z}(P)/P'}, \theta] \neq 0$, then $rs \mid |RS : \mathbf{C}_{RS}(\theta)|$ since $\mathbf{C}_{\mathbf{Z}(P)/P'}(R) = \mathbf{C}_{\mathbf{Z}(P)/P'}(S) = \bar{1}$. Using Clifford's theorem, we know $rs \mid \varphi(1)$. For every nonprincipal character ξ of P' , ξ is RS -invariant since $\mathbf{C}_P(RS) = P'$. According to Theorem 13.28 of [4], we may let λ_ξ be an extension of ξ in $\mathbf{Z}(P)$ such that λ_ξ is RS -invariant. We have $\text{Irr}(\mathbf{Z}(P) \mid \xi) = \{\lambda\lambda_\xi \mid \lambda \in \text{Irr}(\mathbf{Z}(P)/P')\}$. Observe that $\mathbf{C}_{RS}(\lambda\lambda_\xi) = \mathbf{C}_{RS}(\lambda)$ since $\mathbf{C}_{RS}(\lambda_\xi) = RS$. On the other hand, recall that $\mathbf{C}_{P/P'}(RS) = \bar{1}$. And so for every nonprincipal character $\lambda \in \text{Irr}(\mathbf{Z}(P)/P')$, we have that $rs \mid |RS : \mathbf{C}_{RS}(\lambda)| = |RS : \mathbf{C}_{RS}(\lambda\lambda_\xi)|$. Now for $\psi \in \text{Irr}(\mathbf{Z}(P)RS \mid P')$,

by Clifford's theorem we have $rs \mid \psi(1)$. Thus, $\mathbf{Z}(P)RS$ is a $(*)$ -group, contradicting with G is a critical $(*)$ -group. Now $P' = \mathbf{Z}(P)$ holds. \square

Theorem 3.12. *Let G be a critical $(*)$ -group. Suppose G has a normal Sylow subgroup P . Then one of the following holds:*

- (1) $\Delta(G)$ is a complete graph.
- (2) $G' = P$ is of nilpotent class 2. In particular, G has derived length 3.

PROOF. If $\text{Soc}(G) \not\leq \Phi(G)$, then $\text{cd}(G/\mathbf{Z}(P)) = \{1, rs\}$ by Corollary 3.2 and (1) holds. Suppose $\text{Soc}(G) \leq \Phi(G)$ and $\Delta(G)$ is not complete, then by Lemma 3.10, we know (2) holds. \square

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References

- [1] B. HUPPERT, Research in representation theory at Mainz (1984–1990), *Progr. Math.* **95** (1991), 17–36.
- [2] B. HUPPERT, Character Theory of Finite Groups, de Gruyter Expositions in Mathematics, **25**, Walter de Gruyter & Co., Berlin, 1998.
- [3] B. HUPPERT, Endliche Gruppen, Vol. 1, Springer-Verlag, Berlin, 1967.
- [4] I. M. ISAACS, Character Theory of Finite Groups, Academic Press, New York, 1976.
- [5] I. M. ISAACS, Recovering information about a group from its complex group algebra, *Arch. Math.* **47** (1986), 293–295.
- [6] M. L. LEWIS, Solvable groups whose degree graphs have two connected components, *J. Group Theory* **4** (2001), 255–275.
- [7] M. BIANCHI, D. CHILLAG, M. L. LEWIS and E. PACIFIC, Character degree graphs that are complete graphs, *Proc. Amer. Math. Soc.* **135** (2007), 671–676.
- [8] M. L. LEWIS, An overview of graphs associated with character degrees and conjugacy class sizes in finite groups, *Rocky Mountain J. Math.* **38** (2008), 175–211.
- [9] O. MANZ, Endliche auflösbare Gruppen, deren sämtliche Charaktergrade Primzahlpotenzen sind, *J. Algebra.* **94** (1985), 211–255.
- [10] O. MANZ and T. R. WOLF, Representation of Solvable Groups, Cambridge University Press, Cambridge, 1993.
- [11] T. NORITZSCH, Groups having three complex irreducible character degrees, *J. Algebra* **175** (1995), 767–798.
- [12] SHENGAN CHEN, YUN FAN and KARPING SHUM, Finite groups with character degrees of two distinct primes, *Wuhan Univ. J. Nat. Sci.* **11**, no. 2 (2006), 343–345.

- [13] M. L. LEWIS and QINGYUN MENG, Square character degree graphs yield direct products, *J. Algebra* **349** (2012), 185–200.

QINGYUN MENG
COLLEGE OF SCIENCE
HENAN UNIVERSITY OF TECHNOLOGY
ZHENGZHOU, 450001
CHINA

E-mail: meng19821030@live.cn

JIANLI CAO
COLLEGE OF SCIENCE
HENAN UNIVERSITY OF TECHNOLOGY
ZHENGZHOU, 450001
CHINA

E-mail: cao.j.l@163.com

NI DU
SCHOOL OF MATHEMATICAL SCIENCES
XIAMEN UNIVERSITY
XIAMEN, 361005
CHINA

E-mail: duni@xmu.edu.cn

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