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Fixed point theorems on generalized b-metric spaces

By IOAN-RADU PETRE (Cluj-Napoca) and MONICA BOTA (Cluj-Napoca)

Abstract. In this paper we will present some fixed and strict fixed point theorems in generalized b-metric spaces using the Picard and weak Picard operators technique. Also, we give an application for a system of Volterra-type equations.

1. Introduction

The concept of b-metric space or generalizations of it appeared in some works, such as N. Bourbaki [8], I. A. Bakhtin [1], S. Czerwik [9], J. Heinonen [11], etc. Some examples of b-metric spaces and some fixed point theorems in b-metric spaces can also be found in M. Boriceanu, A. Petruşel and I. A. Rus [4], M. Boriceanu [5], [6], M. Bota [7]. The purpose of this paper is to present some fixed and strict fixed point results in generalized b-metric spaces and to give an application for a system of Volterra-type equations.

2. Notations and auxiliary results

The aim of this section is to present some notions and terminology used in the paper. We first give the definition of a generalized b-metric space.

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Definition 2.1. Let X be a set and let $S \geq I$ be a square $m \times m$ matrix of nonnegative real numbers, where I denotes the identity matrix. A functional $d : X \times X \to \mathbb{R}_+^m$ is said to be a generalized b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq S[d(x, y) + d(y, z)]$.

Then the pair (X, d) is called a generalized b-metric space.

The class of generalized b-metric spaces is larger then the class of generalized metric spaces, since a generalized b-metric space is a generalized metric space when $S = I$ in the third assumption of the above definition. We say that $\|\cdot\| : X \to \mathbb{R}_+^m$ is a generalized norm if (in a similar way to the generalized metric) it satisfies the classical axioms of a norm. In this case, the pair $(X, \|\cdot\|)$ is called a generalized normed space. If the generalized metric generated by the norm $\|\cdot\|$ (i.e., $d(x, y) := \|x - y\|$) is complete then the space $(X, \|\cdot\|)$ is called a generalized Banach space. Some examples of b-metric spaces are given by V. Berinde [2], S. Czerwik [9], J. Heinonen [11]. Here we give some examples of generalized b-metric spaces.

Notice that if $A, B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $A = [a_{ij}]$, $B = [b_{ij}]$, for $i, j \in \{1, 2, \ldots, m\}$ then by $A \leq B$ we mean $a_{ij} \leq b_{ij}$, for $i, j \in \{1, 2, \ldots, m\}$.

Example 2.2. Let X be a set with the cardinal $card(X) \geq 3$. Suppose that $X = X_1 \cup X_2$ is a partition of X such that $card(X_1) \geq 2$. Let $S = \left[\begin{smallmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{smallmatrix}\right] \geq \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ be a matrix of real numbers. Then, the functional $d : X \times X \to \mathbb{R}_+^2$ defined by:

$$d(x, y) := \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & x = y \\ 2 \begin{bmatrix} s_{11} \\ s_{22} \end{bmatrix}, & x, y \in X_1 \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & \text{otherwise} \end{cases}$$

is a generalized b-metric on X.

Example 2.3. The set $\ell^p(\mathbb{R})$ (with $0 < p < 1$), where $\ell^p(\mathbb{R}) := \{(x_n)_{n \in \mathbb{N}^*} \subset$

$\mathbb{R} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, together with the functional $d : (\ell^p(\mathbb{R}) \times \ell^q(\mathbb{R}))^2 \to \mathbb{R}^2_+$,

$$d(x,y) := \begin{bmatrix} \left(\sum\limits_{n=1}^{\infty} |x_{1n} - y_{1n}|^p\right)^{1/p} \\ \left(\sum\limits_{n=1}^{\infty} |x_{2n} - y_{2n}|^q\right)^{1/q} \end{bmatrix}$$

is a generalized b-metric space with $S = \begin{bmatrix} 2^{1/p} & s_{12} \\ s_{12} & 2^{1/q} \end{bmatrix} > \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Notice that the above example holds for the general case $\ell^p(X)$ with $0 < p < 1$, where X is a generalized Banach space.

Example 2.4. The space $L^p[0,1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, together with the functional

$$d(x,y) := \begin{bmatrix} \left(\int_0^1 |x_1(t) - y_1(t)|^p dt\right)^{1/p} \\ \left(\int_0^1 |x_2(t) - y_2(t)|^q dt\right)^{1/q} \end{bmatrix},$$

for each $(x_1,y_1),(x_2,y_2) \in L^p[0,1] \times L^q[0,1]$

is a generalized b-metric space with $S = \begin{bmatrix} 2^{1/p} & 0 \\ 0 & 2^{1/q} \end{bmatrix}$.

Notice that in a generalized b-metric space (X,d) the notions of convergent sequence, Cauchy sequence, completeness are similar to those for usual metric spaces. Since generalized b-metrics do not induce topologies, the notions of open set and closed set should be clearly established in this context.

We consider now the following families of subsets of a generalized b -metric space (X,d):

$\mathcal{P}(X) := \{Y \mid Y \subset X\}$; $\quad P(X) := \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}$;

$P_b(X) := \{Y \in P(X) \mid Y$ is bounded$\}$; $\quad P_{cp}(X) := \{Y \in P(X) \mid Y$ is compact$\}$;

$P_{cl}(X) := \{Y \in P(X) \mid Y$ is closed$\}$; $\quad P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$.

If (X,d) is a generalized b-metric space with $d(x,y) := [d_1(x,y) \dots d_m(x,y)]$, then we write:

$$D(A,B) = \begin{bmatrix} D_{d_1}(A,B) \\ \dots \\ D_{d_m}(A,B) \end{bmatrix},$$

where

$$D_{d_i} : P(X) \times P(X) \to [0,+\infty] , \qquad D_{d_i}(A,B) = \inf\{d_i(a,b) \mid a \in A,\ b \in B\}$$

represents the generalized gap functional generated by d_i, for $i \in \{1,\dots,m\}$;

$$\rho(A,B) = \begin{bmatrix} \rho_{d_1}(A,B) \\ \dots \\ \rho_{d_m}(A,B) \end{bmatrix},$$

where

$$\rho_{d_i} : P(X) \times P(X) \to [0,+\infty], \qquad \rho_{d_i}(A,B) = \sup\{D_{d_i}(a,B) \mid a \in A\}$$

resents the generalized excess functional generated by d_i, for $i \in \{1,\dots,m\}$;

$$H(A,B) = \begin{bmatrix} H_{d_1}(A,B) \\ \dots \\ H_{d_m}(A,B) \end{bmatrix},$$

where

$$H_{d_i} : P(X) \times P(X) \to [0,+\infty], \qquad H_{d_i}(A,B) = \max\{\rho_{d_i}(A,B), \rho_{d_i}(B,A)\}$$

represents the generalized Pompeiu–Hausdorff functional generated by d_i, for $i \in \{1,\dots,m\}$;

$$\delta(A,B) = \begin{bmatrix} \delta_{d_1}(A,B) \\ \dots \\ \delta_{d_m}(A,B) \end{bmatrix},$$

where

$$\delta_{d_i} : P(X) \times P(X) \to [0,+\infty], \qquad \delta_{d_i}(A,B) = \sup\{d_i(a,b) : a \in A, b \in B\}$$

represents the generalized delta functional generated by d_i, for $i \in \{1,\dots,m\}$. In particular, $\delta(A) := \delta(A,A)$ is the diameter of the set A.

Let (X,d) be a generalized b-metric space. If $F : X \to P(X)$ is a multivalued operator, then we denote by $\mathrm{Fix}(F)$ the fixed point set of F, i.e., $\mathrm{Fix}(F) := \{x \in X \mid x \in F(x)\}$ and by $\mathrm{SFix}(F)$ the strict fixed point set of F, i.e., $\mathrm{SFix}(F) := \{x \in X \mid \{x\} = F(x)\}$. The symbol $\mathrm{Graph}(F)$ denotes the graph of F, i.e., $\mathrm{Graph}(F) := \{(x,y) \in X \times X : y \in F(x)\}$.

By definition, a square matrix of real numbers is said to be convergent to zero if $A^n \longrightarrow 0$ as $n \to \infty$ (see R. S. Varga [21]). Some examples of matrices that are convergent to zero can be founded in R. Precup [18].

Lemma 2.5 ([18])**.** *Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. Then the following statements are equivalent:*

(i) *A is a matrix convergent to zero;*

(ii) *The eigenvalues of A are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$;*

(iii) *The matrix $I - A$ is non-singular and $(I - A)^{-1} = I + A + \cdots + A^n + \ldots$;*

(iv) *The matrix $I - A$ is non-singular and $(I - A)^{-1}$ has nonnegative elements;*

(v) *$A^n q \longrightarrow 0$ and $qA^n \longrightarrow 0$ as $n \to \infty$, for any $q \in \mathbb{R}^m$.*

3. Main results

The following results are useful for some of the proofs in the paper.

Lemma 3.1. *Let (X, d) be a generalized b-metric space and let $A, B \in P(X)$. We suppose that there exists $\eta \in \mathbb{R}^m_+, \eta > 0$ such that:*

(i) *for each $a \in A$ there is $b \in B$ such that $d(a, b) \leq \eta$;*

(ii) *for each $b \in B$ there is $a \in A$ such that $d(a, b) \leq \eta$.*

Then, $H(A, B) \leq \eta$.

Proof. It follows immediately from the definition of Pompeiu–Hausdorff generalized functional. □

Lemma 3.2. *Let (X, d) be a generalized b-metric space, $A \in P(X)$ and $x \in X$. Then $D(x, A) = 0$ if and only if $x \in \overline{A}$.*

Proof. We show that $\bar{A} = \{x \in X \mid D(x, A) = 0\}$.

Obviously, $D(x, A) = 0$ implies $x \in \overline{A}$. Now, let $x \in \overline{A}$, which means that for any $r \in \mathbb{R}^m_+$, $r > 0$ we have $A \cap B(x, r) \neq \emptyset$, i.e., for any $r \in \mathbb{R}^m_+$, $r > 0$, there exists $a \in A$ such that $d(x, a) < r$, i.e., $D(x, A) = 0$. □

Lemma 3.3. *Let (X, d) be a generalized b-metric space and let $(x_n)_{n \in \mathbb{N}} \subset X$. Let $S \in M_{m,m}(\mathbb{R})$, with $S \geq I$. Then:*

$$d(x_0, x_n) \leq Sd(x_0, x_1) + \cdots + S^{n-1} d(x_{n-2}, x_{n-1}) + S^{n-1} d(x_{n-1}, x_n).$$

Proof. We have

$$\begin{aligned} d(x_0, x_n) &\leq Sd(x_0, x_1) + Sd(x_1, x_n) \leq Sd(x_0, x_1) + S^2 d(x_1, x_2) + S^2 d(x_2, x_n) \\ &\leq Sd(x_0, x_1) + \cdots + S^{n-1} d(x_{n-2}, x_{n-1}) + S^{n-1} d(x_{n-1}, x_n), \end{aligned}$$

which completes the proof. □

Lemma 3.4. *Let (X,d) be a generalized b-metric space and let $S \in M_{m,m}(\mathbb{R})$, with $S \geq I$. Then for all $A,B,C \in P(X)$ we have:*

$$H(A,C) \leq S[H(A,B) + H(B,C)].$$

Proof. We have

$$d(a,c) \leq Sd(a,b) + Sd(b,c), \quad \text{for any } a \in A,\ b \in B,\ c \in C.$$

Taking $\inf_{c \in C}$ we have

$$D(a,C) \leq Sd(a,b) + SD(b,C), \quad \text{for any } a \in A,\ b \in B.$$

Thus,

$$D(a,C) \leq Sd(a,b) + SH(B,C), \quad \text{for any } a \in A,\ b \in B.$$

It follows that

$$\sup a \in AD(a,C) \leq SH(A,B) + SH(B,C)$$

and analogously,

$$\sup c \in CD(c,A) \leq SH(A,B) + SH(B,C).$$

Hence,

$$H(A,C) \leq S[H(A,B) + H(B,C)],$$

which completes the proof. □

Lemma 3.5. *Let (X,d) be a generalized b-metric space and let $A,B \in P_{cl}(X)$. Then for each $\alpha \in \mathbb{R}^m_+$, $\alpha > 0$ and for each $b \in B$, there exists $a \in A$ such that*

$$d(a,b) \leq H(A,B) + \alpha.$$

If, moreover, $A,B \in P_{cp}(X)$ and $S \in M_{m,m}(\mathbb{R})$, with $S \geq I$, then for each $b \in B$, there exists $a \in A$ such that

$$d(a,b) \leq SH(A,B).$$

Proof. The first statement follows immediately from the definition of Pompeiu–Hausdorff generalized functional. Now, let $\varepsilon_n = \left[\frac{1}{n} \dots \frac{1}{n}\right]$, $n \in \mathbb{N}^*$. Then for each $b \in B$, there exists $a_n \in A$ such that

$$d(a_n,b) \leq H(A,B) + \varepsilon_n, \quad n \in \mathbb{N}^*.$$

We may assume that $a_n \longrightarrow a \in A$. Therefore,

$$d(a,b) \leq Sd(a,a_n) + Sd(a_n,b) \leq Sd(a,a_n) + SH(A,B) + S\varepsilon_n, \quad n \in \mathbb{N}^*.$$

Letting $n \to \infty$, we get that

$$d(a,b) \leq SH(A,B),$$

which is the desired conclusion. □

Lemma 3.6. *Let (X,d) be a generalized b-metric space and let $A, B \in P_{cl}(X)$. For each $q > 1$ and for all $a \in A$, there exists $b \in B$ such that:*

$$d(a,b) \leq qH(A,B).$$

Proof. We may assume that $A \neq B$. Then $H_{d_i}(A,B) > 0$, for all $i \in \{1,\dots,m\}$. We suppose that there exists $q > 1$ and there exists $a \in A$ such that for all $b \in B$, we have $d(a,b) \nleq qH(A,B)$. That is, there exists $j \in \{1,\dots,m\}$ such that

$$d_j(a,b) > qH_{d_j}(A,B).$$

Taking $\inf b \in B$ we have

$$D_{d_j}(a,B) \geq qH_{d_j}(A,B).$$

Hence, we get the contradiction

$$H_{d_j}(A,B) \geq D_{d_j}(A,B) \geq qH_{d_j}(A,B) > H_{d_j}(A,B),$$

which completes the proof. □

Lemma 3.7. *Let (X,d) be a generalized b-metric space and let $A, B \in P_b(X)$. For each $q > 1$ and for all $a \in A$, there exists $b \in B$ such that:*

$$\delta(A,B) \leq qd(a,b).$$

Proof. We may assume that $A \neq B$. Then $\delta_{d_i}(A,B) > 0$, for all $i \in \{1,\dots,m\}$. We suppose that there exists $q > 1$ and there exists $a \in A$ such that for all $b \in B$, we have $\delta(A,B) \nleq qd(a,b)$. That is, there exists $j \in \{1,\dots,m\}$ such that

$$\delta_{d_j}(A,B) > qd_j(a,b).$$

Taking $\sup_{b \in B}$ we have

$$\delta_{d_j}(A,B) \geq q\delta_{d_j}(a,B).$$

Hence, we get the contradiction

$$\delta_{d_j}(A,B) \geq q\delta_{d_j}(A,B) > \delta_{d_j}(A,B),$$

which completes the proof. □

Lemma 3.8. *Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ be a matrix convergent to zero. Then, there exists $Q > 1$ such that for any $q \in (1, Q)$ we have that qA is convergent to* 0.

PROOF. Since A is convergent to zero, we have that the spectral radius $\rho(A) < 1$. Next, since $q\rho(A) = \rho(qA) < 1$, we can choose $Q := \frac{1}{\rho(A)} > 1$ and hence, the conclusion follows. □

Definition 3.9. Let (X, d) be a generalized b-metric space and let $f : X \to X$ be a singlevalued operator. Then, f is called a left A -contraction if there exists a matrix $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ convergent to zero such that

$$d[f(x), f(y)] \leq Ad(x, y), \quad \text{for any } x, y \in X.$$

Definition 3.10. Let (X, d) be a generalized b-metric space. Then $f : X \to X$ is a Picard operator (briefly PO), if we have that:

(i) $\text{Fix}(f) = \{x^*\}$ for some x^* in X;

(ii) for each $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ (where $x_n = f^n(x_0)$), converges to x^*.

Definition 3.11. Let (X, d) be a generalized b-metric space and let $f : X \to X$ be a PO. Then f is a M-Picard operator (briefly MPO) if $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and there exists the operator $f^\infty : X \to X$, $f^\infty(x) = \lim_{n\to\infty} f^n(x_0)$ such that $d(x_0, f^\infty(x_0)) \leq Md(x_0, f(x_0))$, for each $x_0 \in X$.

Now we present some fixed point theorems in generalized b-metric spaces for singlevalued operators.

Theorem 3.12. *Let (X, d) be a complete generalized b-metric space with $S \in \mathcal{M}_{m,m}(\mathbb{R}_+), S \geq I$ and let $f : X \to X$ be a left A-contraction such that $AS = SA$ and $SA < I$. Then f is a $(I - SA)^{-1} S$-Picard operator.*

PROOF. Let $x_0 \in X$. Inductively, for any $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we have

$$\begin{aligned}
&d(x_n, x_{n+p}) \\
&\quad \leq Sd(x_n, x_{n+1}) + \cdots + S^{p-1}d(x_{n+p-2}, x_{n+p-1}) + S^{p-1}d(x_{n+p-1}, x_{n+p}) \\
&\quad \leq SA^n d(x_0, x_1) + \cdots + S^{p-1}A^{n+p-2}d(x_0, x_1) + S^{p-1}A^{n+p-1}d(x_0, x_1) \\
&\quad \leq SA^n \left(I + SA + \cdots + S^{p-2}A^{p-2} + S^{p-2}A^{p-1}\right) d(x_0, x_1) \\
&\quad \leq SA^n \left(I + SA + \cdots + S^{p-2}A^{p-2} + S^{p-1}A^{p-1} + \dots\right) d(x_0, x_1) \\
&\quad \leq SA^n (I - SA)^{-1} d(x_0, x_1).
\end{aligned}$$

Letting $n \to \infty$, we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X. By completeness of X, it follows that there exists $x^* \in X$ such that for any $x_0 \in X$, the sequence $(x_n) \longrightarrow x^*$ when $n \to \infty$. We have

$$d[x^*, f(x^*)] \leq Sd(x^*, x_{n+1}) + Sd[x_{n+1}, f(x^*)] \leq Sd(x^*, x_{n+1}) + SAd(x_n, x^*)$$

and thus, x^* is a fixed point of f in X.

For the uniqueness, we suppose that $y^* \in X$ is another fixed point of f with $y^* \neq x^*$. Then

$$d(y^*, x^*) = d[f(y^*), f(x^*)] \leq Ad(y^*, x^*).$$

It follows that

$$(I - A)\, d(y^*, x^*) \leq 0$$

Since $(I - A) \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $(I - A) \neq 0$, we have the only one possibility $d(y^*, x^*) = 0$ and thus, $y^* = x^*$.

Since in a generalized b-metric space d is not continuous in general, we will use the following error estimate for the fixed point

$$d(x_n, x^*) = d[f^n(x_0), f^n(x^*)] \leq A^n d(x_0, x^*), \quad \text{for any } n \in \mathbb{N}.$$

We have

$$d(x_0, x^*) \leq Sd(x_0, x_1) + Sd(x_1, x^*) \leq Sd(x_0, x_1) + SAd(x_0, x^*)$$

and thus,

$$d(x_0, x^*) \leq (I - SA)^{-1} Sd(x_0, x_1).$$

Since $SA \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $SA < I$ it follows that SA is a matrix convergent to zero and since $S \geq I$, it follows that $(I - SA)^{-1}S$ has nonnegative elements.

Hence, f is a $(I - SA)^{-1}S$-Picard operator. □

Our Theorem 3.12 can be used, for example, to establish the existence and the uniqueness of the solution for a system of integral equations. In this respect, let us consider the case of two Volterra-type equations system (see the following result).

Theorem 3.13. *Let $I = [0, a]$ (with $a > 0$) be an interval of the real axis and consider the following system of integral equations in $C(I, X_1) \times C(I, X_2)$:*

$$\begin{cases} x_1(t) = \lambda_1 \int_0^t k_1(t, s, x_1(s), x_2(s))\, ds \\ x_2(t) = \lambda_2 \int_0^t k_2(t, s, x_1(s), x_2(s))\, ds \end{cases} \tag{3.1}$$

for $t \in I$, where $\lambda_i \in \mathbb{R}$, for $i \in \{1, 2\}$.

We assume that:

i) $k_1 \in C\left(I^2 \times X_1 \times X_2, X_1\right), k_2 \in C\left(I^2 \times X_1 \times X_2, X_2\right)$;

ii) *there exist the matrices* $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$, $Q = \left[\begin{smallmatrix} q & 0 \\ 0 & q \end{smallmatrix}\right] \in \mathcal{M}_{2,2}(\mathbb{R}_+)$ *with* $q > 1$ *such that*

$$\|k_i(t,s,u_1,u_2) - k_i(t,s,v_1,v_2)\|_{X_i} \le q(a_{i1}\|u_1 - v_1\|_{X_1} + a_{i2}\|u_2 - v_2\|_{X_2}),$$
$$\textit{for each } (t,s,u_1,u_2),\ (t,s,v_1,v_2) \in I^2 \times X_1 \times X_2,\ i \in \{1,2\}.$$

Then, the integral equations system (3.1) *has a unique solution* $x^* := \left[\begin{smallmatrix} x_1^* \\ x_2^* \end{smallmatrix}\right]$ *in* $C(I, X_1) \times C(I, X_2)$.

Proof. For $i \in \{1,2\}$ and $x := \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in C(I,X_1) \times C(I,X_2)$, we define

$$f_i : C(I,X_1) \times C(I,X_2) \to C(I,X_i),$$
$$x \longmapsto f_i x$$
$$f_i x(t) := \lambda_i \int_0^t k_i(t,s,x_1(s),x_2(s))ds, \quad \text{for any } t \in I.$$

By i), the operators f_1, f_2 are well defined. Moreover, the system (3.1) can be re-written as a fixed point equation in the following form

$$x = f(x),$$

where $f := \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$. Obviously, $x^* := \left[\begin{smallmatrix} x_1^* \\ x_2^* \end{smallmatrix}\right]$ is a solution for (3.1) if and only if x^* is a fixed point for the operator f.

We show that f is a left M contraction. Let $x := \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$, $y := \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in C(I,X_1) \times C(I,X_2)$. For $i \in \{1,2\}$, we have

$$\begin{aligned}
&\|f_i(x)(t) - f_i(y)(t)\|_{X_i} \\
&\quad \le |\lambda_i| \int_0^t \|k_i(t,s,x_1(s),x_2(s)) - k_i(t,s,y_1(s),y_2(s))\|_{X_i} ds \\
&\quad \le |\lambda_i| \int_0^t q\left(a_{i1}\|x_1(s) - y_1(s)\|_{X_1} + a_{i2}\|x_2(s) - y_2(s)\|_{X_2}\right) ds \\
&\quad = |\lambda_i|\, q \left(a_{i1}\|x_1 - y_1\|_{B_1} \int_0^t e^{\tau s} ds + a_{i2}\|x_2 - y_2\|_{B_2} \int_0^t e^{\tau s} ds\right) \\
&\quad \le \frac{|\lambda_i|}{\tau} e^{\tau t} q\left(a_{i1}\|x_1 - y_1\|_{B_1} + a_{i2}\|x_2 - y_2\|_{B_2}\right),
\end{aligned}$$

where $\|u\|_B := \begin{bmatrix} \|u_1\|_{B_1} \\ \|u_2\|_{B_2} \end{bmatrix} = \begin{bmatrix} \sup_{t\in[0,a]} e^{-\tau t}\|u_1(t)\|_{X_1} \\ \sup_{t\in[0,a]} e^{-\tau t}\|u_2(t)\|_{X_2} \end{bmatrix}$, $\tau > 0$ denotes the Bielecki-type norm on the generalized Banach space $C(I,X_1) \times C(I,X_2)$.

Thus, we obtain that

$$\|f_i(x) - f_i(y)\|_{B_i} \leq \frac{|\lambda_i|}{\tau} q(a_{i1}\|x_1 - y_1\|_{B_1} + a_{i2}\|x_2 - y_2\|_{B_2}), \quad \text{for } i \in \{1, 2\}.$$

These inequalities can be written in the vector form

$$\|f(x) - f(y)\|_B \leq M\|x - y\|_B,$$

where

$$M = \left[\frac{|\lambda_i|\, q a_{ij}}{\tau}\right]_{i,j\in\{1,2\}}.$$

Taking $\tau > \max i, j \in \{1, 2\}|\lambda_i|q^2 a_{ij}$, we have that that the matrix M is convergent to zero and thus, f is a left M-contraction. Moreover, $MQ = QM$ and $QM < I$. By Theorem 3.12 it follows that there exists a unique fixed point $x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ in $C(I, X_1) \times C(I, X_2)$ for $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$. □

Definition 3.14. Let (X, d) be a generalized b-metric space and let $f : X \to X$ be a singlevalued operator. Then, f is called a left (A, B, C)-contraction if there exist the matrices $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, where A is convergent to zero with $A + B + C < I$ such that

$$d[f(x), f(y)] \leq Ad(x, y) + Bd[x, f(x)] + Cd[y, f(y)], \quad \text{for any } x, y \in X.$$

Theorem 3.15. *Let (X, d) be a complete generalized b-metric space with $S \in \mathcal{M}_{m,m}(\mathbb{R}_+), S \geq I$ and let $f : X \to X$ be a left (A, B, C)-contraction such that $KS = SK$, where $K := (I - C)^{-1}(A + B)$ and $SA < I$. Then f is a $(I - SA)^{-1}S(I - B)$-Picard operator.*

Proof. Let $x_0 \in X$. We have

$$\begin{aligned} d(x_n, x_{n+1}) &= d[f(x_{n-1}), f(x_n)] \leq Ad(x_{n-1}, x_n) + Bd[x_{n-1}, f(x_{n-1})] \\ &\quad + Cd[x_n, f(x_n)] = (A + B)\, d(x_{n-1}, x_n) + Cd(x_n, x_{n+1}) \end{aligned}$$

and inductively

$$d(x_n, x_{n+1}) \leq (I-C)^{-1}(A+B)d(x_{n-1}, x_n) \leq \cdots \leq \left[(I-C)^{-1}(A+B)\right]^n d(x_0, x_1).$$

Since $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $A + B + C < I$, we get that $K \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $K < I$. Thus, K is convergent to zero. For any $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we have

$$\begin{aligned} &d(x_n, x_{n+p}) \\ &\quad\leq Sd(x_n, x_{n+1}) + \cdots + S^{p-1}d(x_{n+p-2}, x_{n+p-1}) + S^{p-1}d(x_{n+p-1}, x_{n+p}) \\ &\quad\leq SK^n d(x_0, x_1) + \cdots + S^{p-1}K^{n+p-2}d(x_0, x_1) + S^{p-1}K^{n+p-1}d(x_0, x_1) \\ &\quad\leq SK^n\left(I + SK + \cdots + S^{p-2}K^{p-2} + S^{p-2}K^{p-1}\right) d(x_0, x_1) \end{aligned}$$

$$\leq SK^n \left(I + SK + \cdots + S^{p-2}K^{p-2} + S^{p-1}K^{p-1} + \dots\right) d(x_0, x_1)$$
$$\leq SK^n (I - SK)^{-1} d(x_0, x_1).$$

Letting $n \to \infty$, we obtain that the sequence (x_n) is Cauchy in X. By completeness of X, it follows that there exists $x^* \in X$ such that for any $x_0 \in X$, the sequence $(x_n) \longrightarrow x^*$ when $n \to \infty$. We have

$$\begin{aligned} d[x^*, f(x^*)] &\leq Sd(x^*, x_{n+1}) + Sd[x_{n+1}, f(x^*)] \\ &\leq Sd(x^*, x_{n+1}) + SAd(x_n, x^*) + SBd(x_n, x_{n+1}) + SCd[x^*, f(x^*)] \\ &\leq Sd(x^*, x_{n+1}) + SAd(x_n, x^*) + SBK^n d(x_0, x_1) + SCd[x^*, f(x^*)] \end{aligned}$$

and thus,

$$\begin{aligned} d[x^*, f(x^*)] &\leq (I - SC)^{-1} Sd(x^*, x_{n+1}) + (I - SC)^{-1} SAd(x_n, x^*) \\ &\quad + (I - SC)^{-1} SBK^n d(x_0, x_1). \end{aligned}$$

Letting $n \to \infty$, we get that x^* is a fixed point of f in X.

For the uniqueness, we suppose that $y^* \in X$ is another fixed point of f with $y^* \neq x^*$. Then

$$d(y^*, x^*) = d[f(y^*), f(x^*)] \leq Ad(y^*, x^*) + Bd[y^*, f(y^*)] + Cd[x^*, f(x^*)].$$

It follows that

$$(I - A) d(y^*, x^*) \leq 0.$$

Since $(I - A) \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $(I - A) \neq 0$, we have the only one possibility $d(y^*, x^*) = 0$ and thus, $y^* = x^*$.

Since in a generalized b-metric space d is not continuous in general, we will use the following error estimate for the fixed point. For any $n \in \mathbb{N}^*$, we have

$$\begin{aligned} d(x_n, x^*) = d[f(x_{n-1}), f(x^*)] &\leq Ad(x_{n-1}, x^*) + Bd[x_{n-1}, x_n] + Cd[x^*, f(x^*)] \\ &\leq Ad(x_{n-1}, x^*) + BK^{n-1} d(x_0, x_1) \\ &\leq A[Ad(x_{n-2}, x^*) + Bd(x_{n-2}, x_{n-1})] + BK^{n-1} d(x_0, x_1) \\ &\leq A^2 d(x_{n-2}, x^*) + ABK^{n-2} d(x_0, x_1) + BK^{n-1} d(x_0, x_1) \\ &\leq \cdots \leq A^n d(x_0, x^*) + \sum_{i=0}^{n-1} A^i BK^{n-i-1} d(x_0, x_1). \end{aligned}$$

Then

$$d(x_0, x^*) \leq Sd(x_0, x_1) + Sd(x_1, x^*) \leq Sd(x_0, x_1) + SAd(x_0, x^*) + SBd(x_0, x_1)$$

and thus,

$$d(x_0, x^*) \leq (I - SA)^{-1} S(I - B) d(x_0, x_1).$$

Since $SA \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $SA < I$ it follows that SA is a matrix convergent to zero and since $S \geq I, 0 \leq B < I$, it follows that $(I - SA)^{-1}S(I - B)$ has nonnegative elements.

Hence, f is a $(I - SA)^{-1}S(I - B)$-Picard operator. $\square$

It is known (see Czerwik [9]) that if (X, d) is a generalized b-metric space, then the functional $H : P_{b,cl}(X) \times P_{b,cl}(X) \to [0, +\infty]^m$ is a generalized b-metric in $P_{b,cl}(X)$. Also, if (X, d) is a complete generalized b-metric space, we have that $(P_{b,cl}(X), H)$ is a complete generalized b-metric space. Notice that a generalized Pompeiu–Hausdorff functional $H : P_{b,cl}(X) \times P_{b,cl}(X) \to [0, +\infty]^m$ can be introduced in the setting of generalized b-metric spaces (H_i is the vector-valued Pompeiu–Hausdorff metric on $P_{b,cl}(X)$ generated by d_i, where $i \in \{1, \dots, m\}$) and thus, the concept of a multivalued left A-contraction in Nadler's sense can be formulated.

Definition 3.16. Let $Y \subset X$ be a nonempty set and let $F : Y \to P_{cl}(X)$ be a multivalued operator. Then, F is called a multivalued left A-contraction in Nadler's sense if $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and

$$H[F(x), F(y)] \leq Ad(x, y), \quad \text{for any } x, y \in Y.$$

Definition 3.17. Let (X, d) be a generalized b-metric space. Then $F : X \to P(X)$ is a multivalued weak Picard operator (briefly MWP operator), if for each $x \in X$ and $y \in F(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(i) $x_0 = x, x_1 = y$;

(ii) $x_{n+1} \in F(x_n)$;

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to a fixed point of F.

Definition 3.18. Let (X, d) be a generalized b-metric space and let $F : X \to P(X)$ be a MWP operator. Then we define the multivalued operator $F^\infty : \mathrm{Graph}(F) \to P(\mathrm{Fix}(F))$ by the formula $\{F^\infty(x, y) = z \in \mathrm{Fix}(F)$: there exists a sequence of successive approximations of F starting from (x, y) that converges to $z\}$.

Definition 3.19. Let X, Y be two nonempty sets and let $F : X \to P(Y)$ be a multivalued operator. Then a singlevalued operator $f : X \to Y$ is a selection for F if $f(x) \in F(x)$, for any $x \in X$.

Definition 3.20. Let (X,d) be a generalized b-metric space and let $F : X \to P(X)$ be a MWP operator. Then F is a M -multivalued weak Picard operator (briefly M-MWP operator) if $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and there exists a selection f^∞ of F^∞ such that $d(x, f^\infty(x,y)) \le Md(x,y)$, for all $(x,y) \in \mathrm{Graph}(F)$.

Now we present some fixed point theorems in generalized b-metric spaces for multivalued operators.

Theorem 3.21. *Let (X,d) be a complete generalized b-metric space with $S \in \mathcal{M}_{m,m}(\mathbb{R}_+), S \ge I$ and let $F : X \to P_{cl}(X)$ be a multivalued left A-contraction in Nadler's sense such that $AS = SA$ and $SA < I$. Then F is a $(I - SA)^{-1} S$-multivalued weak Picard operator.*

Proof. Let $x_0 \in X$ such that $x_1 \in F(x_0)$. Let $q \in \left(1, \frac{1}{\rho(A)}\right)$. For $F(x_0), F(x_1)$ and for $x_1 \in F(x_0)$, by Lemma 3.6, it follows that there exists $x_2 \in F(x_1)$ such that

$$d(x_1, x_2) \le qH[F(x_0), F(x_1)] \le qAd(x_0, x_1).$$

For $F(x_1), F(x_2)$ and for $x_2 \in F(x_1)$, there exists $x_3 \in F(x_2)$ such that

$$d(x_2, x_3) \le qH[F(x_1), F(x_2)] \le qAd(x_1, x_2) \le (qA)^2 d(x_0, x_1).$$

Inductively, there exists the sequence $(x_n) \in X$ such that $x_{n+1} \in F(x_n)$ and

$$d(x_n, x_{n+1}) \le (qA)^n d(x_0, x_1), \quad \text{for any } n \in \mathbb{N}^*.$$

For any $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we have

$$\begin{aligned}
&d(x_n, x_{n+p}) \\
&\quad \le Sd(x_n, x_{n+1}) + \dots + S^{p-1} d(x_{n+p-2}, x_{n+p-1}) + S^{p-1} d(x_{n+p-1}, x_{n+p}) \\
&\quad \le S(qA)^n \left[I + \dots + S^{p-2}(qA)^{p-2} + S^{p-2}(qA)^{p-1}\right] d(x_0, x_1) \\
&\quad \le S(qA)^n \left(I + \dots + q^{p-2}S^{p-2}A^{p-2} + q^{p-1}S^{p-1}A^{p-1}\right) d(x_0, x_1) \\
&\quad \le S(qA)^n \left(I + \dots + q^{p-2}S^{p-2}A^{p-2} + q^{p-1}S^{p-1}A^{p-1} + \dots\right) d(x_0, x_1) \\
&\quad \le S(qA)^n (I - qSA)^{-1} d(x_0, x_1).
\end{aligned}$$

Letting $n \to \infty$ and using Lemma 3.8, we obtain that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy in X. By completeness of X, it follows that there exists $x^* \in X$ such that for any $x_0 \in X$, the sequence $(x_n) \longrightarrow x^*$ when $n \to \infty$.

We have

$$\begin{aligned} D\left[x^*, F\left(x^*\right)\right] &\leq Sd\left(x^*, x_{n+1}\right) + SD\left[x_{n+1}, F\left(x^*\right)\right] \\ &\leq Sd\left(x^*, x_{n+1}\right) + SH\left[F\left(x_n\right), F\left(x^*\right)\right] \leq Sd\left(x^*, x_{n+1}\right) + SAd\left(x_n, x^*\right) \end{aligned}$$

and letting $n \to \infty$, we get that $D\left[x^*, F\left(x^*\right)\right] = 0$. By Lemma 3.2, it follows that $x^* \in \overline{F\left(x^*\right)}$. Hence, $x^* \in F\left(x^*\right)$.

Since in a generalized b-metric space d is not continuous in general, we will use the following error estimate for the fixed point. For any $n \in \mathbb{N}^*$, we have

$$d\left(x_n, x^*\right) = qH\left[F\left(x_{n-1}\right), F\left(x^*\right)\right] \leq qAd\left(x_{n-1}, x^*\right) \leq \cdots \leq (qA)^n\, d\left(x_0, x^*\right).$$

Then

$$d\left(x_0, x^*\right) \leq Sd\left(x_0, x_1\right) + Sd\left(x_1, x^*\right) \leq Sd\left(x_0, x_1\right) + qSAd\left(x_0, x^*\right)$$

and thus,

$$d\left(x_0, x^*\right) \leq (I - qSA)^{-1}\, Sd\left(x_0, x_1\right).$$

Letting $q \searrow 1$, we get that

$$d\left(x_0, x^*\right) \leq (I - SA)^{-1}\, Sd\left(x_0, x_1\right).$$

Since $SA \in \mathcal{M}_{m,m}\left(\mathbb{R}_+\right)$ and $SA < I$ it follows that SA is a matrix convergent to zero and since $S \geq I$, it follows that $(I - SA)^{-1} S$ has nonnegative elements.

Hence, F is a $(I - SA)^{-1} S$-multivalued weak Picard operator. $\square$

Remark 3.22. In a similar manner with the proof of Theorem 3.13 (using Theorem 3.21) can be obtained existence results for the following integral inclusion system in $C\left(I, X_1\right) \times C\left(I, X_2\right)$:

$$\begin{cases} x_1(t) \in \lambda_1 \int_0^t K_1\left(t, s, x_1(s), x_2(s)\right) ds \\ x_2(t) \in \lambda_2 \int_0^t K_2\left(t, s, x_1(s), x_2(s)\right) ds \end{cases} \tag{3.2}$$

for $t \in I := [0, a]$ (where $\lambda_i \in \mathbb{R}$, $i \in \{1, 2\}$).

Definition 3.23. Let $Y \subset X$ be a nonempty set and let $F : Y \to P_{cl}(X)$ be a multivalued operator. Then, F is called a multivalued left (A, B, C)-contraction if there exist the matrices $A, B, C \in \mathcal{M}_{m,m}\left(\mathbb{R}_+\right)$, where A is convergent to zero with $A + B + C < I$ such that

$$H\left[F(x), F(y)\right] \leq Ad(x, y) + BD\left[x, F(x)\right] + CD\left[y, F(y)\right], \quad \text{for any } x, y \in Y.$$

Theorem 3.24. *Let (X,d) be a complete generalized b-metric space with $S\in\mathcal{M}_{m,m}(\mathbb{R}_+)$, $S\geq I$ and let $F:X\to P_{cl}(X)$ be a multivalued left (A,B,C)-contraction such that $KS=SK$, where $K:=(I-qC)^{-1}(A+B)$, $q\in\left(1,\frac{1}{\rho(A+B+C)}\right)$ and $SA<I$. Then F is a $(I-SA)^{-1}S(I-B)$-multivalued weak Picard operator.*

Proof. Let $x_0\in X$ such that $x_1\in F(x_0)$. For $F(x_0),F(x_1)$ and for $x_1\in F(x_0)$, by Lemma 3.6, it follows that there exists $x_2\in F(x_1)$ such that

$$d(x_1,x_2)\leq qH\left[F(x_0),F(x_1)\right]\leq qAd(x_0,x_1)$$
$$+qBD\left[x_0,F(x_0)\right]+qCD\left[x_1,F(x_1)\right]\leq q(A+B)\,d(x_0,x_1)+qCd(x_1,x_2).$$

Thus,

$$d(x_1,x_2)\leq q\,(I-qC)^{-1}\,(A+B)\,d(x_0,x_1).$$

For $F(x_1),F(x_2)$ and for $x_2\in F(x_1)$, there exists $x_3\in F(x_2)$ such that

$$d(x_2,x_3)\leq qH\left[F(x_1),F(x_2)\right]\leq qAd(x_1,x_2)$$
$$+qBD\left[x_1,F(x_1)\right]+qCD\left[x_2,F(x_2)\right]\leq q(A+B)\,d(x_1,x_2)+qCd(x_2,x_3).$$

Thus,

$$d(x_2,x_3)\leq q(I-qC)^{-1}(A+B)d(x_1,x_2)\leq\left[q(I-qC)^{-1}(A+B)\right]^2d(x_0,x_1).$$

Inductively, there exists the sequence $(x_n)\in X$ such that $x_{n+1}\in F(x_n)$ and

$$d(x_n,x_{n+1})\leq\left[q\,(I-qC)^{-1}\,(A+B)\right]^n d(x_0,x_1),\quad\text{for any } n\in\mathbb{N}^*.$$

For any $n\in\mathbb{N}$ and $p\in\mathbb{N}^*$, we have

$$\begin{aligned}
&d(x_n,x_{n+p})\\
&\quad\leq Sd(x_n,x_{n+1})+\cdots+S^{p-1}d(x_{n+p-2},x_{n+p-1})+S^{p-1}d(x_{n+p-1},x_{n+p})\\
&\quad\leq S\,(qK)^n\left[I+\cdots+S^{p-2}\,(qK)^{p-2}+S^{p-2}\,(qK)^{p-1}\right]d(x_0,x_1)\\
&\quad\leq S\,(qK)^n\left(I+\cdots+q^{p-2}S^{p-2}K^{p-2}+q^{p-1}S^{p-1}K^{p-1}\right)d(x_0,x_1)\\
&\quad\leq S\,(qK)^n\left(I+\cdots+q^{p-2}S^{p-2}K^{p-2}+q^{p-1}S^{p-1}K^{p-1}+\ldots\right)d(x_0,x_1)\\
&\quad\leq S\,(qK)^n\,(I-qSK)^{-1}\,d(x_0,x_1). \qquad (*)
\end{aligned}$$

We show that K is convergent to zero and $\frac{1}{\rho(A+B+C)}\leq\frac{1}{\rho(K)}$.

Since $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $A+B+C<I$, we have that $(A+B+C)$ is convergent to zero. It follows that $q(A+B+C)$ is convergent to zero and thus, $q(A+B+C)<I$. Then

$$A+B+qC \leq q(A+B+C) < I \tag{3.3}$$

and

$$0 < I - q(A+B+C) \leq I - qC \tag{3.4}$$

By (3.3) it follows that $K<I$ and by (3.4) it follows that $K \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. Thus, K is convergent to zero.

We observe that

$$0 \leq C\left[I - q(A+B+C)\right].$$

It follows that

$$A+B \leq A+B+C - qC(A+B+C)$$

and thus,

$$(I-qC)^{-1}(A+B) \leq A+B+C.$$

By the properties of spectral radius, we get that $\rho(K) \leq \rho(A+B+C)$ and thus, $\frac{1}{\rho(A+B+C)} \leq \frac{1}{\rho(K)}$.

Now, letting $n \to \infty$ in (*) and using Lemma 3.8, we obtain that the sequence (x_n) is Cauchy in X. By completeness of X, it follows that there exists $x^* \in X$ such that for any $x_0 \in X$, $x_n \longrightarrow x^*$ when $n \to \infty$.

We have

$$\begin{aligned} D[x^*, F(x^*)] &\leq Sd(x^*, x_{n+1}) + SD[x_{n+1}, F(x^*)] \\ &\leq Sd(x^*, x_{n+1}) + SH[F(x_n), F(x^*)] \\ &\leq Sd(x^*, x_{n+1}) + SAd(x_n, x^*) + SBD[x_n, F(x_n)] + SCD[x^*, F(x^*)]. \end{aligned}$$

Thus,

$$0 \leq D\left[x^*, F(x^*)\right] \leq (I-SC)^{-1} S\left[d(x^*, x_{n+1}) + Ad(x_n, x^*) + Bd(x_n, x_{n+1})\right]$$

and letting $n \to \infty$, we get that $D\left[x^*, F(x^*)\right] = 0$. By Lemma 3.2, it follows that $x^* \in \overline{F(x^*)}$. Hence, $x^* \in F(x^*)$.

Since in a generalized b-metric space d is not continuous in general, we will use the following error estimate for the fixed point. For any $n \in \mathbb{N}^*$, we have

$$\begin{aligned} d(x_n, x^*) &= qH\left[F(x_{n-1}), F(x^*)\right] \leq qAd(x_{n-1}, x^*) + qBd(x_{n-1}, x_n) \\ &\leq qAd(x_{n-1}, x^*) + qBK^{n-1}d(x_0, x_1) \\ &\leq qA\left[qAd(x_{n-2}, x^*) + qBd(x_{n-2}, x_{n-1})\right] + qBK^{n-1}d(x_0, x_1) \end{aligned}$$

$$\leq (qA)^2 d(x_{n-2}, x^*) + q^2 ABK^{n-2} d(x_0, x_1) + qBK^{n-1} d(x_0, x_1)$$
$$\leq \cdots \leq (qA)^n d(x_0, x^*) + \sum_{i=0}^{n-1} q^{i+1} A^i BK^{n-i-1} d(x_0, x_1).$$

Then

$$d(x_0, x^*) \leq Sd(x_0, x_1) + Sd(x_1, x^*) \leq Sd(x_0, x_1) + qSAd(x_0, x^*) + qSBd(x_0, x_1)$$

and thus,

$$d(x_0, x^*) \leq (I - qSA)^{-1} S(I - B) d(x_0, x_1).$$

Letting $q \searrow 1$, we get that

$$d(x_0, x^*) \leq (I - SA)^{-1} S(I - B) d(x_0, x_1).$$

Since $SA \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $SA < I$ it follows that SA is a matrix convergent to zero and since $S \geq I, 0 \leq B < I$, it follows that $(I - SA)^{-1} S(I - B)$ has nonnegative elements.

Hence, F is a $(I - SA)^{-1} S(I - B)$-multivalued weak Picard operator. □

We give some addition results for the strict fixed point set of F.

Theorem 3.25. *If all the assumption of Theorem 3.24 holds and* SFix(F) *is nonempty, then:*

$$\mathrm{Fix}(F) = \mathrm{SFix}(F) = \{x^*\}.$$

Proof. By Theorem 3.24, it follows that $x^* \in \mathrm{Fix}(F)$. We suppose that there exists $y^* \in \mathrm{Fix}(F)$ such that $y^* \neq x^*$. Then

$$d(y^*, x^*) = D[y^*, F(x^*)] \leq H[F(y^*), F(x^*)]$$
$$\leq Ad(y^*, x^*) + BD[y^*, F(y^*)] + CD[x^*, F(x^*)] = Ad(y^*, x^*).$$

It follows that

$$(I - A) d(y^*, x^*) \leq 0.$$

Since $(I - A) \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $(I - A) \neq 0$, we have the only one possibility $d(y^*, x^*) = 0$ and thus, $y^* = x^*$. Hence, $\mathrm{Fix}(F) = \{x^*\}$. On the other hand, since $\mathrm{SFix}(F)$ is nonempty and $\mathrm{SFix}(F) \subset \mathrm{Fix}(F) = \{x^*\}$, we conclude that $\mathrm{Fix}(F) = \mathrm{SFix}(F) = \{x^*\}$. □

Theorem 3.26. *Let (X,d) be a complete generalized b-metric space with $S \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $S \geq I$ and let $F : X \to P_b(X)$ be such that $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, where A is convergent to zero with $A+B+C < I$, $KS = SK$, where $K := (I-C)^{-1}(A+B)$, $SA < I$ and*

$$\delta\left[F(x),F(y)\right] \leq Ad(x,y) + B\delta\left[x,F(x)\right] + C\delta\left[y,F(y)\right], \quad \text{for any } x,y \in X.$$

Then $\operatorname{SFix}(F) = \{x^*\}$.

Proof. Let $q \in \left(1, \frac{1}{\rho(A+B+C)}\right)$. For $\{x\}, F(x)$ and for $x \in X$ it follows that there exists a selection $f : X \to X, f(x) \in F(x)$ such that

$$\delta\left[x, F(x)\right] \leq qd\left[x, f(x)\right].$$

We have

$$d\left[f(x), f(y)\right] \leq \delta\left[F(x), F(y)\right] \leq Ad(x,y)$$
$$+ B\delta\left[x,F(x)\right] + C\delta\left[y,F(y)\right] \leq Ad(x,y) + qBd\left[x,f(x)\right] + qCd\left[y,f(y)\right].$$

Since $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $A+B+C < I$, we have that $(A+B+C)$ is convergent to zero. It follows that $q(A+B+C)$ is convergent to zero and thus, $q(A+B+C) < I$. Then

$$A + qB + qC \leq q(A+B+C) < I.$$

By Theorem 3.15, it follows that there exists a unique $x^* \in X$ such that $x^* = f(x^*) \in F(x^*)$, i.e., $x^* \in \operatorname{Fix}(F)$.

We show that $x^* \in \operatorname{SFix}(F)$. We have

$$0 \leq \delta\left[x^*, F(x^*)\right] \leq \delta\left[F(x^*), F(x^*)\right] \leq Ad(x^*, x^*)$$
$$+ B\delta\left[x^*, F(x^*)\right] + C\delta\left[x^*, F(x^*)\right] = (B+C)\,\delta\left[x^*, F(x^*)\right].$$

It follows that

$$0 \leq (I - B - C)\,\delta\left[x^*, F(x^*)\right] \leq 0.$$

Since $(I-B-C) \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $(I-B-C) \neq 0$, we have the only one possibility $\delta\left[x^*, F(x^*)\right] = 0$ and thus, we obtain that $F(x^*) = \{x^*\}$.

For the uniqueness, we suppose that there exists $y^* \in \operatorname{SFix}(F)$ such that $y^* \neq x^*$. Then

$$d(x^*, y^*) = \delta\left[F(x^*), F(y^*)\right]$$
$$\leq Ad(x^*, y^*) + B\delta\left[x^*, F(x^*)\right] + C\delta\left[y^*, F(y^*)\right] = Ad(x^*, y^*).$$

It follows that

$$(I - A)\,d(x^*, y^*) \leq 0.$$

Since $(I-A) \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $(I-A) \neq 0$, we have the only one possibility $d(y^*, x^*) = 0$ and thus, $y^* = x^*$. Hence, $\operatorname{SFix}(F) = \{x^*\}$. □

Remark 3.27. If we choose $B = C = 0$ in Theorem 3.26 implies that $\delta\,[F(x), F(x)] = 0$, for any $x \in X$ which yields that F is a singlevalued operator. Therefore the statement of Theorem 3.26 is nontrivial if $B + C > 0$.

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IOAN-RADU PETRE
DEPARTMENT OF APPLIED MATHEMATICS
BABES-BOLYAI UNIVERSITY
1, KOGALNICEANU STR.
400084, CLUJ-NAPOCA
ROMANIA

E-mail: ioan.petre@ubbcluj.ro

MONICA BOTA
DEPARTMENT OF APPLIED MATHEMATICS
BABES-BOLYAI UNIVERSITY
1, KOGALNICEANU STR.
400084, CLUJ-NAPOCA
ROMANIA

E-mail: bmonica@math.ubbcluj.ro

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