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Projective change between arbitrary (α,β)-metric and Randers metric

By DAXIAO ZHENG (Shanghai) and QUN HE (Shanghai)

Abstract. In this paper, we study a special class of Finsler metrics: (α,β)-metric, and obtain some necessary and sufficient conditions for them to be projectively or Douglas related to Randers metric.

1. Introduction

No matter in Riemannian geometry or in Finsler geometry, geodesics are very important study objects. We say that a Finsler metric is projectively related to another metric if they have the same geodesics as oriented point sets. In Riemannian geometry, two Riemannian metrics α and $\tilde{\alpha}$ are projectively related if and only if their spray coefficients have the relation

$$G^i_\alpha = G^i_{\tilde{\alpha}} + \lambda_{x_k} y^k y^i,$$

where $\lambda = \lambda(x)$ is a scalar function on the base manifold, and (x^i, y^i) denote the local coordinates in the tangent bundle TM. Two Finsler metrics F and $\tilde{F}$ are projectively related if and only if their spray coefficients have the relation

$$G^i = \tilde{G}^i + P(y)y^i,$$

where $P(y)$ is a scalar function on $TM \setminus \{0\}$ and homogeneous of degree one in y. The change of a Finsler metric F to another Finsler metric $\tilde{F} := F + \tilde{\beta}$ is called a Randers change, where $\tilde{\beta}$ is a nonzero one form on the base manifold satisfying

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$\|\tilde{\beta}\| < 1$. It has been proved in [5] that F is projectively related to its Randers change $\tilde{F}$ if and only if $\tilde{\beta}$ is closed.

The projective change between two Finsler spaces has been studied by many geometers [1], [3], [12]. An interesting result concerned with the theory of projective change was given in Rapcsák's paper, and the necessary and sufficient condition for projective change was obtained. The authors Z. Shen and Civi Yildirim studied on a class of projectively flat metrics with constant flag curvature in [11]. In 2008, Y. Shen and Y. Yu studied the projective change between two Randers metrics. In 2009, Ningwei Cui and Yi-Bing Shen studied projective change between $F = \frac{(\alpha+\beta)^2}{\alpha}$ and a Randers metric and gave more detailed descriptions. In 2011, M. Zohrehvand and M. M. Rezaii studied the projective change between two special classes of (α, β)-metrics $F = \frac{\alpha^2}{\alpha-\beta}$ and Randers metric. In this paper, we will study the projective change between arbitrary (α, β)-metric and Randers metric. More precisely, we have the following result.

Theorem 1.1. *Let $F = \alpha\phi(s)$ be an (α, β)-metric and $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ be a Randers metric on a manifold M with dimension $n \geq 3$. Suppose $\phi \neq Ce^{\int_0^s \frac{k_1 t + k_2\sqrt{1+k_3 t^2}}{1+k_1 t^2 + k_2 t\sqrt{1+k_3 t^2}}dt}$, where C, k_1, k_2 and k_3 are constants. Then F is Douglas related to $\tilde{F}$ if and only if they are Douglas metrics.*

Theorem 1.2. *Let $F = \alpha\phi(s)$ and $\tilde{F} = \tilde{\alpha}\tilde{\phi}(s)$ be two (α, β)-metrics on M with dimension $n \geq 3$. Suppose they are Douglas metrics. Then F is projectively related to $\tilde{F}$ if and only if*

$$G^i_\alpha = G^i_{\tilde{\alpha}} + \theta y^i - \tau(k_1\alpha^2 + k_2\beta^2)b^i + \tilde{\tau}(k_1\tilde{\alpha}^2 + k_2\tilde{\beta}^2)\tilde{b^i},$$

where $\tau = \tau(x)$ and $\tilde{\tau} = \tilde{\tau}(x)$ are scalar functions and θ is a 1-form on M.

Theorem 1.3. *Let $F = \alpha\phi(s)$ be an (α, β)-metric and $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ be a Randers metric on a manifold M with dimension $n \geq 3$. Suppose $\phi \neq Ce^{\int_0^s \frac{k_1 t + k_2\sqrt{1+k_3 t^2}}{1+k_1 t^2 + k_2 t\sqrt{1+k_3 t^2}}dt}$, where C, k_1, k_2 and k_3 are constants and β is not parallel with respect to α. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:*

$$\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\}\phi'' = (k_1 + k_2 s^2)(\phi - s\phi'), \tag{1.1}$$

$$b_{i|j} = 2\tau\{(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\}, \tag{1.2}$$

$$d\tilde{\beta} = 0, \tag{1.3}$$

$$G^i_\alpha = G^i_{\tilde{\alpha}} + \theta y^i - \tau(k_1\alpha^2 + k_2\beta^2)b^i, \tag{1.4}$$

where $\tau=\tau(x)$ is a scalar function on M and k_1, k_2 and k_3 are constants with $(k_2,k_3)\neq(0,0)$, θ is a 1-form on M.

Theorem 1.4. *Let $F=\alpha\phi(s)$ be a non-Randers type (α,β)-metric and $\tilde{F}=\tilde{\alpha}+\tilde{\beta}$ be a Randers metric on M with dimension $n\geq 3$, and let $\phi=Ce^{\int_0^s\frac{k_1t+k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$, where C, k_1, k_2 and k_3 are constants with $k_2\neq 0$. Suppose that β is not parrel with respect to α. Then F is Douglas related to $\tilde{F}$ if and only if the following conditions hold:*

(a) *F has isotropic S-curvature,*

(b) *$\tilde{\alpha}=\sqrt{c_1}\sqrt{\alpha^2+k_3\beta^2}$,*

(c) *$d\tilde{\beta}=k_2\sqrt{c_1}d\beta$,*

where c_1 is a scalar function on M.

Theorem 1.5. *Let $F=\alpha\phi(s)$ be a non-Randers type (α,β)-metric and $\tilde{F}=\tilde{\alpha}+\tilde{\beta}$ be Randers metric on M with dimension $n\geq 3$, and let $\phi=Ce^{\int_0^s\frac{k_1t+k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$, where C, k_1, k_2 and k_3 are constants with $k_2\neq 0$. Suppose that β is not parrel with respect to α. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:*

(a) *F has isotropic S-curvature,*

(b) *$\tilde{\alpha}=\sqrt{c_1}\sqrt{\alpha^2+k_3\beta^2}$,*

(c) *$d\tilde{\beta}=k_2\sqrt{c_1}d\beta$,*

(d) *$G^i_\alpha=G^i_{\tilde{\alpha}}+k_1\beta s^i_0+\theta y^i$,*

where θ is a 1-form on M.

2. Preliminary

For a given Finsler metric $F=F(x,y)$, the geodesics of F satisfy the following ODEs:

$$\frac{d^2x^i}{dt^2}+2G^i\left(x,\frac{dx}{dt}\right)=0,$$

where $G^i=G^i(x,y)$ are the geodesic coefficients, given by

$$G^i=\frac{1}{4}g^{il}\{[F^2]_{x^my^l}y^m-[F^2]_{x^l}\}.$$

The equivalent condition that a Finsler metric F is projective to $\tilde{F}$ has been characterized by using spray coefficients.

Let $(M, \tilde{F})$ be a Finsler space. Another Finsler metric F on M is projective to $\tilde{F}$ if and only if there exists a scalar function $P(y)$ on $TM \setminus \{0\}$, i.e., homogeneous of degree one in y, such that

$$G^i = \tilde{G}^i + P(y)y^i, \tag{2.1}$$

where G^i and $\tilde{G}^i$ are spray coefficients of F and $\tilde{F}$ respectively. In what follows, we will explain what is a (regular) (α, β)-*metric*. Let $\phi = \phi(s)$, $|s| < b_0$, be a positive C^∞ function satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi'' > 0 \quad (|s| \leqslant b < b_0).$$

For a given Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and a 1-form $\beta = b_iy^i$ satisfying $\|\beta_x\|_\alpha < b_0$ for any $x \in M$, we call $F := \alpha\phi(s), s = \frac{\beta}{\alpha}$, an (regular) $(\alpha, \beta)-$metric. In this case, the fundamental form of the metric tensor induced by F is positive definite. Let $\nabla\beta = b_{i|j}dx^i \otimes dx^j$ be the covariant derivative of β with respect to α. Denote

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{i|j}).$$

Clearly, β is closed if and only if $s_{ij} = 0$. Let $s_j := b^i s_{ij}$, $s^i_j := a^{il}s_{lj}$, $s_0 := s_iy^j$, $s^i_0 := s^i_jy^j$ and $r_{00} := r_{ij}y^iy^j$. The geodesic coefficients G^i of F and the geodesic coefficients G^i_α of α are related by follows

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2\alpha Q s_0 + r_{00}\}\{\Psi b^i + \Theta\alpha^{-1}y^i\}, \tag{2.2}$$

where

$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \tag{2.3}$$

$$Q = \frac{\phi'}{\phi - s\phi'}, \tag{2.4}$$

$$\Psi = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \tag{2.5}$$

Definition 2.1. Let

$$D^i_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right), \tag{2.6}$$

where G^i are the spray coefficients of F. The tensor $D := D^i_{jkl}\partial i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

It is easily seen from (2.1) that the Douglas tensor is a projective invariant. Noting that the spray coefficients of a Riemannian metric are quadratic forms, one can see that the Douglas tensor vanishes from (2.6). It means that Douglas tensor is a non-Riemannian quantity. From [4], we get

$$D^i_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1} T^m_{y^m} y^i\right), \tag{2.7}$$

where

$$\begin{aligned} T^i &= \alpha Q s^i_0 + \Psi\{-2\alpha s_0 + r_{00}\} b^i, \\ T^m_{y^m} &= Q' s_0 + \Psi' \alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ &\quad + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]. \end{aligned} \tag{2.8}$$

A simple caculation shows

$$T^m_{y^m} = 2\Psi(r_0 + s_0) + \Psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0]. \tag{2.9}$$

Let F and $\tilde{F}$ be two (α,β)-metrics and assume that they have the same Douglas tensors, i.e. $D^i_{jkl} = \tilde{D}^i_{jkl}$. From (2.7) we have

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \tilde{T}^i - \frac{1}{n+1}(T^m_{y^m} - \tilde{T}^m_{y^m}) y^i\right) = 0.$$

Then there exists a class of scalar functions $H^i_{jk} := H^i_{jk}(x)$ on M such that

$$T^i - \tilde{T}^i - \frac{1}{n+1}(T^m_{y^m} - \tilde{T}^m_{y^m}) y^i = H^i_{00},$$

where $H^i_{00} := H^i_{jk} y^j y^k$, T^i and $T^m_{y^m}$ are given by (2.8) and (2.9) respectively. For a Randers metric, S. Bácsó and M. Matsumoto proved the following

Lemma 2.1 ([3]). *A Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is closed.*

Later on, B. Li and Z. Shen got more general consequence:

Lemma 2.2 ([2]). *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be an (α,β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space $R^n (n \geq 3)$, where $\alpha = \sqrt{\alpha_{ij}(x) y^i y^j}$ and $\beta = b_i(x) y^i$. Suppose that the following conditions:* (a) *β is not parallel with respect to α,* (b) *$\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ for any constants k_1, k_2 and k_3,* (c) *$db \neq 0$ everywhere or $b =$ constant on $\mathcal{U}$. Then F is a Douglas metric on $\mathcal{U}$ if and only if the function $\phi = \phi(s)$ satisfies*

$$\{1 + (k_1 + k_2 s^2) s^2 + k_3 s^2\}\phi'' = (k_1 + k_2 s^2)(\phi - s\phi'), \tag{2.10}$$

and β satisfies

$$b_{i|j} = 2\tau\{(1 + k_1 b^2) a_{ij} + (k_2 b^2 + k_3) b_i b_j\}, \tag{2.11}$$

where $\tau = \tau(x)$ is a scalar function on $\mathcal{U}$ and k_1, k_2 and k_3 are constants with $(k_2, k_3) \neq (0, 0)$.

We will prove that "$b = \text{const}$" in condition (c) contradicts (a) or (b) when F is a Douglas metric. In fact, condition (c) in Lemma 2.2 can be removed.

Lemma 2.3. *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be an (α, β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space $R^n (n \geq 3)$, where $\alpha = \sqrt{\alpha_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Suppose that the following conditions:* (a) *β is not parallel with respect to α,* (b) *$\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ for any constants k_1, k_2 and k_3. If* $b = \text{constant}$ *on $\mathcal{U}$, then F is not a Douglas metric.*

Proof. Suppose that F is a Douglas metric, from Lemma 2.2, we have

$$\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\}\phi'' = (k_1 + k_2 s^2)(\phi - s\phi'),$$
$$b_{i|j} = 2\tau\{(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\}.$$

Since $b = \text{const}$, from [6] we have

$$r_0 + s_0 = 0.$$

Since β is closed, we get

$$r_0 = b^i r_{ij} y^j = 2\tau\{(1 + k_1 b^2) + (k_2 b^2 + k_3)b^2\}\beta = 0.$$

Since β is not parallel with respect to α, $\tau \neq 0$, we get

$$(1 + k_1 b^2) + (k_2 b^2 + k_3)b^2 = 0.$$

Noting that

$$\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\}\phi'' = (k_1 + k_2 s^2)(\phi - s\phi'),$$

by setting $s = b$,

$$k_1 + k_2 b^2 = 0,$$

we have

$$(k_2 s^2 + k_3)\phi'' = k_2(\phi - s\phi').$$

If $k_2 = 0$, it is easy to check that F is a Randers metric. If $k_2 \neq 0$, we have

$$\Psi = \frac{1}{2}\frac{\phi''}{\phi - s\phi' + (b^2 - s^2)\phi''} = \frac{1}{2}\frac{1}{\frac{k_3}{k_2} + b^2}.$$

From [2], we see that F is of Randers type which contradicts the condition. $\square$

Lemma 2.4. *Let $F=\alpha\phi(s)$, $s=\frac{\beta}{\alpha}$, be an (α,β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space $R^n(n\geq 2)$, where $\alpha=\sqrt{\alpha_{ij}(x)y^iy^j}$ and $\beta=b_i(x)y^i$. Suppose that the following conditions:* (a) *β is not parallel with respect to α,* (b) *$\phi\neq k_1\sqrt{1+k_2s^2}+k_3s$ for any constants k_1, k_2 and k_3. Then F is a Douglas metric on $\mathcal{U}$ if and only if the function $\phi=\phi(s)$ satisfies*

$$\{1+(k_1+k_2s^2)s^2+k_3s^2\}\phi''=(k_1+k_2s^2)(\phi-s\phi'), \tag{2.12}$$

and β satisfies

$$b_{i|j}=2\tau\{(1+k_1b^2)a_{ij}+(k_2b^2+k_3)b_ib_j\}, \tag{2.13}$$

where $\tau=\tau(x)$ is a scalar function on $\mathcal{U}$ and k_1, k_2 and k_3 are constants with $(k_2,k_3)\neq(0,0)$.

Proof. $\Rightarrow$: Set $\mathcal{U}=U'\cup U''$, where $U'=\{x\in\mathcal{U}\mid db\neq 0\}$ and $U''=\{x\in\mathcal{U}\mid db=0\}$. From Lemma 2.3, we can assume $U''^{\circ}=\emptyset$, i.e., $\partial U'\cap\mathcal{U}=U''$. Since U' is an open set, from Lemma 2.2, (2.12) and (2.13) hold in U'. Since ϕ and β are smooth, (2.12) holds in $(-b(x),b(x))$ for any $x\in\mathcal{U}$. Next we consider β. For any $x_0\in U''$, we just need to prove τ continues at x_0. If

$$[(1+k_1b^2)+(k_2b^2+k_3)b^2]|_{x_0}\neq 0,$$

then

$$\tau(x_0)=\frac{b^ib^jb_{i|j}}{[(1+k_1b^2)+(k_2b^2+k_3)b^2]b^2}\Big|_{x_0}.$$

If $[(1+k_1b^2)+(k_2b^2+k_3)b^2]|_{x_0}=0$, similarly to Lemma 2.3 we can prove that F is of Randers type which contradicts the condition.

$\Leftarrow$: obvious. □

3. On Douglas related metric

It is known that for some special (α,β)-metrics, they are Douglas related to a Randers metric if and only if they are Douglas metrics, such as Matsumoto metric $F=\frac{(\alpha+\beta)^2}{\alpha}$. In what follows, we consider more general case.

Lemma 3.1. *Let $\Phi_1+\Phi_2\bar{\alpha}+\Phi_3\tilde{\alpha}=0$, where $\tilde{\alpha}=\sqrt{\Psi_1+\Psi_2\bar{\alpha}}$, $\bar{\alpha}=\sqrt{\sum_{i=1}^n y_i^2}$, $n\geq 3$, Φ_1, Ψ_1, Φ_2, Φ_3 and Ψ_2 are homogeneous polynomials. Then one of the following holds:*

(a) $\Phi_1=\Phi_2=\Phi_3=0$,

(b) $\tilde{\alpha} = k\bar{\alpha} + k_1$,
where $k = \frac{\Phi_2}{\Phi_3}$ *and* $k_1 = \frac{\Phi_1}{\Phi_3}$ *are homogeneous polynomials.*

Proof. If $\Phi_1 + \Phi_2\bar{\alpha} + \Phi_3\tilde{\alpha} = 0$, then we have

$$(\Phi_1 + \Phi_2\bar{\alpha})^2 = \Phi_3^2(\Psi_1 + \Psi_2\bar{\alpha}).$$

Since Φ_1, Ψ_1, Φ_2, Φ_3 and Ψ_2 are homogeneous polynomials, we have

$$2\Phi_1\Phi_2 = \Phi_3^2\Psi_2, \tag{3.1}$$

$$\Phi_1^2 + \Phi_2^2\bar{\alpha}^2 = \Phi_3^2\Psi_1. \tag{3.2}$$

The irreducible decomposition of Φ_3 is $\Phi_3 = {g_1}^{i_1}{g_2}^{i_2}\cdots{g_n}^{i_n}$, where each g_j is an irreducible polynomial. If Φ_3 is not a factor of Φ_2, then there exists a g_j which is not a factor of Φ_2. From (3.1), ${g_j}^{2i_j}$ has to be a factor of Φ_1 and therefore a factor of $\Phi_2^2\bar{\alpha}^2$ from (3.2). Since $\bar{\alpha}^2$ is irreducible, g_j must be a factor of Φ_2 which contradicts assumption. Thus either Φ_2 have the factor Φ_3 or $\Phi_1 = \Phi_2 = \Phi_3 = 0$. a simple argument shows that either Φ_1 have the factor Φ_3 or $\Phi_1 = \Phi_2 = \Phi_3 = 0$. This finishes the proof of the lemma. □

From Lemma 3.1, we have the following lemma.

Lemma 3.2. *Let* $\Phi_1 + \Phi_2\bar{\alpha} + \Phi_3\tilde{\alpha} + \Phi_4\bar{\alpha}\tilde{\alpha} = 0$, *where* $\tilde{\alpha} = \sqrt{\Psi_1 + \Psi_2\bar{\alpha}}$, $\bar{\alpha} = \sqrt{\sum_{i=1}^n y_i^2}$, $n \geq 2$, Φ_1 *and* Ψ_1 *are homogeneous polynomials of degree two,* Φ_2, Φ_3 *and* Ψ_2 *are homogeneous polynomials of degree one, and* Φ_4 *is a constant. Then one of the following holds:*
(a) $\Phi_1 = \Phi_2 = \Phi_3 = \Phi_4 = 0$,
(b) $\tilde{\alpha} = k\bar{\alpha} + \sum_{i=1}^n k_i y_i$,
where k *and* k_i *are constants.*

Proof. Multiplying $(\Phi_3 - \Phi_4\bar{\alpha})$ on both sides of $\Phi_1 + \Phi_2\bar{\alpha} + \Phi_3\tilde{\alpha} + \Phi_4\bar{\alpha}\tilde{\alpha} = 0$, yields

$$\Phi_1\Phi_3 - \Phi_2\Phi_4\bar{\alpha}^2 + (\Phi_2\Phi_3 - \Phi_1\Phi_4)\bar{\alpha} + ({\Phi_3}^2 - \Phi_4^2\bar{\alpha}^2)\tilde{\alpha} = 0.$$

From Lemma 3.1 we have case (i)

$$\Phi_1\Phi_3 - \Phi_2\Phi_4\bar{\alpha}^2 = \Phi_2\Phi_3 - \Phi_1\Phi_4 = {\Phi_3}^2 - \Phi_4^2\bar{\alpha}^2 = 0.$$

Since $\bar{\alpha}^2$ is irreducible, from

$${\Phi_3}^2 - \Phi_4^2\bar{\alpha}^2 = 0,$$

we get $\Phi_3 = \Phi_4 = 0$. Then the equation $\Phi_1 + \Phi_2\bar{\alpha} + \Phi_3\tilde{\alpha} + \Phi_4\bar{\alpha}\tilde{\alpha} = 0$ becomes:

$$\Phi_1 + \Phi_2\bar{\alpha} = 0,$$

which means $\Phi_1 = \Phi_2 = 0$.

Case (ii): $\tilde{\alpha} = k'\bar{\alpha} + k_1{}'$, where $k' = \frac{\Phi_2\Phi_3 - \Phi_1\Phi_4}{\Phi_3{}^2 - \Phi_4^2\bar{\alpha}^2}$ and $k_1{}' = \frac{\Phi_1\Phi_3 - \Phi_2\Phi_4\bar{\alpha}^2}{\Phi_3{}^2 - \Phi_4^2\bar{\alpha}^2}$ are homogeneous polynomials. Since Φ_1 is a homogeneous polynomial of degree two, Φ_2 and Φ_3 are homogeneous polynomials of degree one, Φ_4 is a constant, we obtain (b). □

Lemma 3.3. *Let $\tilde{\alpha} = \mu(x, \frac{\beta}{\alpha})\alpha$, where $\tilde{\alpha}$ and α are Riemannian metrics on M. Then $\mu(x, \frac{\beta}{\alpha}) = \sqrt{c_1(x) + c_2(x)\frac{\beta^2}{\alpha^2}}$, where c_1 and c_2 are scalar functions on M.*

Proof. From Theorem 2.1 of [14], we have

$$\tilde{\alpha} = \sqrt{c_1\alpha^2 + c_2\beta^2 + c_3\alpha\beta}.$$

Noting that $\tilde{\alpha}$ is Riemannian, we have $c_3 = 0$, hence $\mu\big(x, \frac{\beta}{\alpha}\big) = \sqrt{c_1(x) + c_2(x)\frac{\beta^2}{\alpha^2}}$. □

Remark 3.1. In fact, Theorem 2.1 of [14] told us: the Cartan tensor of (α, β)-metric is of the form $C_{ijk} = K_{ij}B_k + K_{jk}B_i + K_{ki}B_j$ if and only if $F^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where K_{ij} is the angular metric tensor of the Riemannian metric α and B_i are some tensor fields. From the proof of Theorem 2.1 of [14], one can see the consequence also holds for general (α, β)-*metric*.

Theorem 3.1. *Let $F = \alpha\phi(s)$ be an (α, β)-metric and $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ be a Randers metric on a manifold M with dimension $n \geq 3$. Suppose $\phi \neq Ce^{\int_0^s \frac{k_1 t + k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$, where C, k_1, k_2 and k_3 are constants. Then F is Douglas related to $\tilde{F}$ if and only if they are both Douglas metrics.*

Proof. Supposing F is Douglas related to $\tilde{F}$, then there exists a class of scalar functions $H^i_{jk} := H^i_{jk}(x)$ such that

$$T^i - \tilde{T}^i - \frac{1}{n+1}(T^m_{y^m} - \tilde{T}^m_{y^m})y^i = H^i_{00}.$$

Hence

$$H^i_{00} + \tilde{\alpha}\tilde{s}^i_0 = \alpha Q s^i_0 + \Phi\{-2\alpha Q s_0 + r_{00}\}b^i - \frac{1}{n+1}\{2\Phi(r_0 + s_0) + \Phi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0]\}y^i. \tag{3.3}$$

Choose a special coordinate system at a point as in [1]. Take a change of coordinates $(s, y^a) \to (y^i)$ by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \bar{\alpha} = \sqrt{\sum_{a=2}^{n} y_a^2},$$

and

$$\alpha = \sqrt{\delta_{ij} y^i y^j}, \quad \beta = by^1.$$

Set

$$s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_{i|j} := \frac{1}{2}(b_{i|j} + b_{j|i}).$$

Then we have

$$s_0 = \sum_{a=2}^{n} bs_{1a} y^a \quad s_0^i = s_j^i y^j = s_1^i \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha} + \sum_{a=2}^{n} s_a^i y^a,$$

$$r_0 = br_{11}\frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha} + b\sum_{a=2}^{n} r_{1a} y^a,$$

$$r_{00} = 2r_{1a}\frac{s}{\sqrt{b^2 - s^2}} y^a \bar{\alpha} + r_{11}\frac{s^2}{b^2 - s^2}\bar{\alpha}^2 + r_{ab} y^a y^b.$$

Case(i): $\tilde{\alpha} = \mu(x, s)\alpha$. Then we have

$$\tilde{s}_0^i = \tilde{s}_1^i \frac{s}{\sqrt{b^2 - s^2}}\mu(x, s)\bar{\alpha} + \sum_{a=2}^{n} s_a^i y^a.$$

From Lemma 3.3, we have

$$\mu(x, s) = \sqrt{c_1(x) + c_2(x)s^2}.$$

(3.3) can be written as

$$\Phi_1^i + \Phi_2^i \bar{\alpha} = 0.$$

Set $i = 1$, or $i = a$ in the above eqution, we have

$$\begin{aligned} \mu\tilde{s}_a^1 = &\left[Q - 2Q\Phi b^2 - \frac{2\Phi s}{n+1} + \frac{2Q\Phi' s(b^2 - s^2)}{n+1}\right] s_{1a} \\ &+ \left[\frac{2n\Phi s}{n+1} - \frac{2s^2\Phi'(b^2 - s^2)}{(n+1)b^2}\right] r_{1a} - 2H_{1a}^1 \frac{s}{b}, \end{aligned} \tag{3.4}$$

$$\left(\mu\tilde{s}^1_a\frac{sb}{b^2-s^2} - Qs_{1a}\frac{sb}{b^2-s^2} - H^a_{11}\frac{s^2}{b^2-s^2}\right)\bar{\alpha}^2 = H^a_{bc}y^by^c$$
$$+\frac{2y^ay^c}{n+1}\left[\Phi b(r_{1c}+s_{1c}) + \frac{1}{b}\Phi'(b^2-s^2)r_{1c} - 2bQ\Phi'(b^2-s^2)s_{1c}\right], \quad (3.5)$$

$$\left[2H^a_{1c}\frac{s}{b}y^c + \mu\tilde{s}^a_cy^c - s_{ac}Qy^c + \frac{2\Phi sr_{11}}{n+1}y^a + \frac{\Phi's^2(b^2-s^2)r_{11}y^a}{(n+1)b^2}\right]\frac{b^2}{b^2-s^2}\bar{\alpha}^2$$
$$= -\frac{\Phi'(b^2-s^2)r_{bc}y^by^c}{n+1}y^a. \quad (3.6)$$

From (3.5), we have

$$\mu(x,s)s\tilde{s}_{1a} = sQs_{1a} + H^a_{11}\frac{s^2}{b} + H^a_{bb}\frac{b^2-s^2}{b}, \quad b \neq a.$$

By taking $s = 0$, we get $H^a_{bb} = 0$. Then

$$\mu\tilde{s}^1_a = Qs_{1a} + H^a_{11}\frac{s}{b}. \quad (3.7)$$

From (3.6), we get

$$\mu\tilde{s}^a_c = s_{ac}Q - 2H^a_{1c}\frac{s}{b} \quad a \neq c. \quad (3.8)$$

If $s_{ij} = 0$, $\forall i,j$, then from (3.7) and (3.8), $\tilde{s}_{ij} = 0$,, i.e. β is closed, which shows that $\tilde{F}$ is Douglas metric. Otherwise,

$$Q = k_1s + k_2\sqrt{1+k_3s^2}$$

where k_1, k_2 and k_3 are constants. Then

$$\phi = Ce^{\int_0^s \frac{k_1t+k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$$

which contradicts the condition.

Case(ii): $\tilde{\alpha} \neq \mu(x,s)\alpha$. Then (3.3) can be written as

$$\Phi^i_1 + \Phi^i_2\alpha + \Phi^i_3\tilde{\alpha} + \Phi^i_4\alpha\tilde{\alpha} = 0,$$

where Φ^i_1, Φ^i_2 Φ^i_3, Φ^i_4 are polynomials of y^a. From Lemma 3.1 and Lemma 3.2, we have
(a) $\Phi^i_1 = \Phi^i_2 = \Phi^i_3 = 0$ and $\Phi^i_4 = 0$, or
(b) $\tilde{\alpha} = k(x,s)\alpha + \sum_{a=2}^n k_a(x,s)y^a$,
where $\Phi^i_3 = \tilde{s}^i_ay^a$ and $\Phi^i_4 = \tilde{s^i_1}\frac{s}{b}$.

If (a) holds, we have $\tilde{s}^i_j = 0$. From Lemma 2.2, we have $\tilde{D}^i_{jkl} = 0$.

If (b) holds, we will prove that $k_a(x,s)=0$. Since

$$\tilde{\alpha}=\sqrt{\tilde{a}_{11}y^1y^1+\tilde{a}_{ab}y^ay^b+2\tilde{a}_{1b}y^1y^b}$$
$$=\sqrt{\tilde{a}_{11}\frac{s^2}{b^2-s^2}\bar{\alpha}^2+\tilde{a}_{ab}y^ay^b+2\tilde{a}_{1b}\frac{s}{\sqrt{b^2-s^2}}\bar{\alpha}y^b},$$

from (b), we have

$$\tilde{a}_{11}\frac{s^2}{b^2-s^2}\bar{\alpha}^2+\tilde{a}_{ab}y^ay^b=k^2\bar{\alpha}^2+k_ak_by^ay^b, \tag{3.9}$$

$$\tilde{a}_{1b}\frac{s}{\sqrt{b^2-s^2}}y^b=kk_by^b. \tag{3.10}$$

From (3.9), we have

$$\tilde{a}_{ab}=k_ak_b, \quad b\neq a, \tag{3.11}$$

$$\tilde{a}_{11}\frac{s^2}{b^2-s^2}+\tilde{a}_{aa}=k^2+k_a^2. \tag{3.12}$$

From (3.10), we have

$$\tilde{a}_{1b}\frac{s}{\sqrt{b^2-s^2}}=k_bk. \tag{3.13}$$

Letting (3.12) $\times\, k_b^2$ and $s=0$, and using (3.11) and (3.13), we check that

$$\tilde{a}_{aa}k_b^2=\tilde{a}_{ab}^2 \quad a\neq b.$$

Similarly,

$$\tilde{a}_{bb}k_a^2=\tilde{a}_{ab}^2 \quad a\neq b.$$

Thus

$$\tilde{a}_{aa}\tilde{a}_{bb}\tilde{a}_{ab}^2=\tilde{a}_{ab}^4.$$

From the positive definiteness of $\tilde{\alpha}$, we have

$$\tilde{a}_{ab}=0 \quad a\neq b.$$

Thus from (3.11), there exists some a such that $k_a=0$. From (3.12), we have $\tilde{a}_{11}\frac{s^2}{b^2-s^2}+\tilde{a}_{aa}=k^2$ and $\tilde{a}_{11}\frac{s^2}{b^2-s^2}+\tilde{a}_{bb}=k^2+k_b^2$, for any $b\neq a$, which implies $k_b^2=\tilde{a}_{bb}-\tilde{a}_{aa}$. Thus k_b is independent of s. By taking $s=0$ in (3.10), from $k>0$, we have $k_b=0$. Then $\tilde{\alpha}=k(x,s)\alpha$, which contradicts the condition. This completes the proof of the theorem. □

From the proof above and Lemma 3.3, we have following proposition.

Proposition 3.1. *Let $F=\alpha\phi(s)$ be an (α,β)-metric and $\tilde{F}=\tilde{\alpha}+\tilde{\beta}$ be a Randers metric on M with dimension $n\geq 3$. Suppose $\tilde{\alpha}\neq\sqrt{c_1(x)+c_2(x)s^2}\alpha$. Then F is Douglas related to $\tilde{F}$ if and only if they are both Douglas metrics.*

For two Randers metrics, they had already been considered in [6]. For $\phi = Ce^{\int_0^s \frac{k_1t+k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$ and F is non-Randers and $k_2\neq 0$, we have

Theorem 3.2. *Let $F=\alpha\phi(s)$ is a non-Randers type (α,β)-metric and $\tilde{F}=\tilde{\alpha}+\tilde{\beta}$ be a Randers metric on M with dimension $n\geq 3$, and $\phi = Ce^{\int_0^s \frac{k_1t+k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$, where C, k_1, k_2 and k_3 are constants with $k_2\neq 0$. Suppose β is not parrel with respect to α. Then F is Douglas related to $\tilde{F}$ if and only if the following conditions hold:*

(a) *F has isotropic S-curvature,*

(b) $\tilde{\alpha}=\sqrt{c_1}\sqrt{\alpha^2+k_3\beta^2}$,

(c) $d\tilde{\beta}=k_2\sqrt{c_1}d\beta$.

where c_1 is a scalar function on M.

First we need to prove the following lemmas.

Lemma 3.4. *Let $F=\alpha\phi(s), s=\frac{\beta}{\alpha}$, be an (α,β)-metric and $\phi = Ce^{\int_0^s \frac{k_1t+k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$, where C, k_1, k_2 and k_3 are constants with $k_2\neq 0$. Then F is a Randers type metric if and only if $k_1=k_3$.*

Proof. If $k_1=k_3$, we have $\phi=C\sqrt{1+k_1s^2}+Ck_2s$, which means that F is of Randers type. On the contrary, if F is of Randers type, we have

$$\Psi=\frac{1}{2}\frac{k_1\sqrt{1+k_3s^2}+k_2k_3s}{(1+b^2k_1)\sqrt{1+k_3s^2}+k_2s(1+b^2k_3)}=\frac{1}{2}\frac{k_2'}{1+k_2'b^2},$$

for some constant k_2'. Since the right side of the above equation is independent of s, we have $k_1=k_3$. □

Lemma 3.5. *$F=\alpha\phi(s)$ is a non-Randers type (α,β)-metric on M with dimension $n\geq 3$ and $\phi = Ce^{\int_0^s \frac{k_1t+k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$, where C, k_1, k_2 and k_3 are any constants with $k_2\neq 0$. Then F has isotropic S-curvature if and only if β satisfies*

$$r_{00}=0,\quad s_0=0.$$

Proof. From Theorem 1 of [7], F has isotropic S-curvature if and only if one of the following conditions hold

(i) $$r_j + s_j = 0, \quad \Phi = 0,$$

(ii) $$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0,$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2},$$

(iii) $$r_{00} = 0, \quad s_0 = 0,$$

where
$$\Delta = 1 + sQ + (b^2 - s^2)Q',$$

and
$$\Phi = -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''.$$

From Lemma 3.4, we just need to prove that $k_1 = k_3$ when $\Phi = 0$ or $\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2-s^2}$. Since $\phi = Ce^{\int_0^s \frac{k_1 t + k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$, we have

$$\Delta = 1 + b^2 k_1 + \frac{k_2 s(1 + k_3 b^2)}{\sqrt{1 + k_3 s^2}},$$

and

$$\Phi = \frac{-k_2}{(1+k_3s^2)^{\frac{3}{2}}}\Big[(k_1 + nk_3 + (n+1)k_1k_3b^2)s^2 + n + 1 + (nk_1 + k_3)b^2 + (n+1)k_2(1 + k_3b^2)s\sqrt{1 + k_3s^2}\Big].$$

If $\Phi = 0$, we get $(k_1 + nk_3 + (n+1)k_1k_3b^2) = 0$, $n + 1 + (nk_1 + k_3)b^2 = 0$ and $1 + k_3b^2 = 0$. Thus $k_1 = k_3 = -\frac{1}{b^2}$.

If $\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2-s^2}$, we see that $\Delta^2 = 0$ when $s^2 = b^2$ for Φ is a continuous function.Thus $1 + b^2k_1 + k_2(\pm b)\sqrt{1 + k_3b^2} = 0$, i.e., $k_1 = k_3 = -\frac{1}{b^2}$. □

Proof of Theorem 3.2. ⇒: Supposing $\tilde{\alpha} \neq \sqrt{c_1 + c_2s^2}\alpha$, from Proposition 3.1, we see that F is a Douglas metric. From Lemma 2.4, we have

$$\{1 + (k_1' + k_2's^2)s^2 + k_3's^2\}\phi'' = (k_1' + k_2's^2)(\phi - s\phi').$$

Since $\phi = Ce^{\int_0^s \frac{k_1 t + k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$, we get $k_1 = k_3$. From Lemma 3.4, F is of Randers type which contradicts the condition. Hence $\tilde{\alpha} = \sqrt{c_1 + c_2 s^2}\alpha$, and F is not a Douglas metric.

Combining (3.4), (3.5) and (3.6) yields

$$\sqrt{c_1+c_2s^2}\tilde{s}^1_a = \left[Q - 2Q\Psi b^2 - \frac{2\Psi s}{n+1} + \frac{2Q\Psi' s(b^2-s^2)}{n+1}\right] s^1_a + \left[\frac{2n\Psi s}{n+1} - \frac{2\Psi' s^2(b^2-s^2)}{(n+1)b^2}\right] r_{1a} - 2H^1_{1a}\frac{s}{b}, \tag{3.14}$$

$$2H^a_{1a}\frac{s}{b} + \frac{1}{n+1}\left\{2\Psi r_{11}s + \Psi'(b^2-s^2)\left[r_{11}\frac{s^2}{b^2} + r_{aa}\left(\frac{b^2-s^2}{b^2}\right)\right]\right\} = 0. \tag{3.15}$$

$$\sqrt{c_1+c_2s^2}\tilde{s}^1_a = Qs^1_a + H^a_{11}\frac{s}{b}, \tag{3.16}$$

$$\sqrt{c_1+c_2s^2}\tilde{s}^a_b = Qs^a_b - 2H^a_{1b}\frac{s}{b},\ b \neq a, \tag{3.17}$$

$$r_{ab} = 0, \quad a \neq b \quad \text{and} \quad r_{aa} = r_{bb}. \tag{3.18}$$

Since F is not a Douglas metric, we have $s^j_i \neq 0$ for some i, j. From (3.16), (3.17) and $Q = k_1 s + k_2\sqrt{1+k_3s^2}$, we have

$$\sqrt{c_1}\tilde{s}^1_a = k_2 s^1_a, \qquad \sqrt{c_1}\tilde{s}^a_b = k_2 s^a_b, \quad c_2 = k_3 c_1. \tag{3.19}$$

Taking into account $\phi = Ce^{\int_0^s \frac{k_1 t + k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$, we have

$$\Psi = \frac{1}{2}\frac{k_1\sqrt{1+k_3s^2} + k_2k_3s}{(1+b^2k_1)\sqrt{1+k_3s^2} + k_2(1+b^2k_3)s},$$

and

$$\Psi' = \frac{1}{2}\frac{k_2(k_3-k_1)}{\sqrt{1+k_3s^2}[(1+b^2k_1)\sqrt{1+k_3s^2} + k_2(1+b^2k_3)s]^2}.$$

Since $k_1 \neq k_3$, from (3.14), considering the degree of s, we have $s^1_a = \tilde{s}^1_a = 0$, $r_{1a} = 0$, i.e., $r_0 = s_0 = 0$. By taking $s = 0$ in (3.15), we have $\Psi'(0)r_{aa} = 0$, hence $r_{aa} = 0$. Substituting it into (3.15), we have $r_{11} = 0$. From Lemma 3.5, we see that F has isotropic *S-curvature*. Since $s_0 = 0$, (3.19) is equivalence to $d\tilde{\beta} = k_2\sqrt{c_1}d\beta$.

$\Leftarrow$:

$$D^i_{jkl} - \tilde{D}^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}(\alpha Q s^i_0 - \tilde{\alpha}\tilde{s}^i_0) = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}(k_1\beta s^i_0) = 0. \qquad \square$$

4. Examples

Example 1. $F = \alpha\phi(s)$, $\phi = e^{\operatorname{arcsinh}(s)}$, $s = \frac{\beta}{\alpha}$ and $\tilde{F} = \sqrt{\alpha^2+\beta^2}+\beta$. F has isotropic $S-curvature$, then F is Douglas related to $\tilde{F}$. In this case, $\mu(x,s) = \sqrt{1+s^2}$.

Example 2. $F = \alpha\phi(s)$, $\phi = \sqrt{1+s+s^2}e^{\frac{1}{\sqrt{3}}\arctan(\frac{2}{\sqrt{3}}(s+\frac{1}{2}))}$, $s = \frac{\beta}{\alpha}$ and $\tilde{F} = \alpha+\beta$. F has isotropic $S-curvature$, then F is Douglas related to $\tilde{F}$. In this case, $\mu(x,s) = 1$.

Example 3. $F = \alpha+\beta$, and $\tilde{F} = \mu(x)\alpha+\mu(x)\beta$, then F is Douglas related to $\tilde{F}$. In this case, the condition of isotropic $S-curvature$ can be removed.

5. On projective related metric

Since projectively related metrics have to be Douglas related metrics, from Theorem 1.1 and Theorem 1.3, we only need to consider projectively related Douglas metrics.

Theorem 5.1. *Let $F = \alpha\phi(s)$ and $\tilde{F} = \tilde{\alpha}\tilde{\phi}(s)$ be two (α,β)-metrics on M with dimension $n \geq 3$. Suppose they are Douglas metrics. Then F is projectively related to $\tilde{F}$ if and only if*

$$G^i_\alpha = G^i_{\tilde{\alpha}} + \theta y^i - \tau(k_1\alpha^2+k_2\beta^2)b^i + \tilde{\tau}(k_1\tilde{\alpha}^2+k_2\tilde{\beta}^2)\tilde{b}^i,$$

where $\tau = \tau(x)$ and $\tilde{\tau} = \tilde{\tau}(x)$ are scalar functions and θ is a 1-form on M.

Proof. If one of the metrics is of Randers type or β is parallel with respect to α, it is easy to get the conclution. Thus, we can assume (a) β is not parallel with respect to α, (b) $\phi \neq \sqrt{1+k_2s^2}+k_3s$ for any constants k_1, k_2 and k_3. From Lemma 2.4 and (2.5), we have

$$\Psi = \frac{1}{2}\frac{k_1+k_2s^2}{1+k_1b^2+(k_2b^2+k_3)s^2},$$

and

$$r_{00} = 2\tau\alpha^2\{1+k_1b^2+(k_2b^2+k_3)s^2\}.$$

Then from equation (2.2), we have

$$G^i = G^i_\alpha + r_{00}(\Psi b^i + \Theta\alpha^{-1}y^i) = G^i_\alpha + \tau(k_1\alpha^2+k_2\beta^2)b^i + r_{00}\Theta\alpha^{-1}y^i,$$

and

$$\tilde{G^i} = G^i_{\tilde{\alpha}} + \tilde{\tau}(\tilde{k_1}\tilde{\alpha}^2 + \tilde{k_2}\tilde{\beta}^2)\tilde{b}^i + \tilde{r}_{00}\tilde{\Theta}\tilde{\alpha}^{-1}y^i.$$

Thus F is projectively related to $\tilde{F}$ if and only if

$$G^i_{\alpha} = G^i_{\tilde{\alpha}} + \theta y^i - \tau(k_1\alpha^2 + k_2\beta^2)b^i + \tilde{\tau}(k_1\tilde{\alpha}^2 + k_2\tilde{\beta}^2)\tilde{b^i}. \qquad \square$$

From Theorem 1.1 and Theorem 5.1, we have the following

Theorem 5.2. *Let $F = \alpha\phi(s)$ be an (α, β)-metric and $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ be a Randers metric on a manifold M with dimension $n \geq 3$. Suppose $\phi \neq Ce^{\int_0^s \frac{k_1 t + k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$, where C, k_1, k_2 and k_3 are constants and β is not parallel with respect to α. Then F is Projectively related to $\tilde{F}$ if and only if the following conditions hold:*

$$\{1 + (k_1 + k_2s^2)s^2 + k_3s^2\}\phi'' = (k_1 + k_2s^2)(\phi - s\phi'), \tag{5.1}$$

$$b_{i|j} = 2\tau\{(1 + k_1b^2)a_{ij} + (k_2b^2 + k_3)b_ib_j\}, \tag{5.2}$$

$$d\tilde{\beta} = 0, \tag{5.3}$$

$$G^i_{\alpha} = G^i_{\tilde{\alpha}} + \theta y^i - \tau(k_1\alpha^2 + k_2\beta^2)b^i, \tag{5.4}$$

where $\tau = \tau(x)$ is a scalar function on M and k_1, k_2 and k_3 are constants with $(k_2, k_3) \neq (0, 0)$, θ is a 1-form on M.

Similarly, for $\phi = Ce^{\int_0^s \frac{k_1 t + k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$ and F is of non-Randers type and $k_2 \neq 0$, we have

Theorem 5.3. *Let $F = \alpha\phi(s)$ be a (α, β)-metric of non-Randers type and $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ be a Randers metric on M with dimension $n \geq 3$, and $\phi = Ce^{\int_0^s \frac{k_1 t + k_2\sqrt{1+k_3t^2}}{1+k_1t^2+k_2t\sqrt{1+k_3t^2}}dt}$, where C, k_1, k_2 and k_3 are constants with $k_2 \neq 0$. Suppose β is not parrel with respect to α. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:*

(a) *F has isotropic S-curvature,*

(b) *$\tilde{\alpha} = \sqrt{c_1}\sqrt{\alpha^2 + k_3\beta^2}$,*

(c) *$d\tilde{\beta} = k_2\sqrt{c_1}d\beta$.*

(d) *$G^i_{\alpha} = G^i_{\tilde{\alpha}} + k_1\beta s^i_0 + \theta y^i$, where θ is a 1-form on M*

It is an immediate consequence of Theorem 1.4.

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DAXIAO ZHENG
DEPARTMENT OF MATHEMATICS
TONGJI UNIVERSITY
SHANGHAI 200092
P.R. CHINA

E-mail: 18917240898@163.com

CORRESPONDING AUTHOR:
QUN HE
DEPARTMENT OF MATHEMATICS
TONGJI UNIVERSITY
SHANGHAI 200092
P.R. CHINA

E-mail: hequn@tongji.edu.cn

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