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Balancing numbers which are products of consecutive integers

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Abstract. In 1999 A. Behera and G. K. Panda defined balancing numbers as follows. A positive integer n is called a balancing number if $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+k)$ for some $k \in \mathbb{N}$. The sequence of balancing numbers is denoted by B_m for $m \in \mathbb{N}$. In this paper we show that the Diophantine equation $B_m = x(x+1)(x+2)(x+3)(x+4)$ has no solution with $m \geq 0$ and $x \in \mathbb{Z}$. We follow the ideas described in [13], that is we combine Baker's method and the so-called Mordell–Weil sieve to obtain all solutions.

1. Introduction

A positive integer n is called a balancing number if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+k)$$

for some $k \in \mathbb{N}$. The sequence of balancing numbers is denoted by B_m for $m \in \mathbb{N}$. We note that usually the initial values $B_0 = 0$, $B_1 = 1$ of the balancing sequence are used. Behera and Panda [3] proved many interesting results related to the sequence B_m. They showed that the balancing numbers fulfill the following recurrence relation

$$B_{m+1} = 6B_m - B_{m-1} \quad (m \geq 1)$$

where $B_0 = 0$ and $B_1 = 1$. Later several authors investigated balancing numbers and their various generalizations. In [21] Liptai proved that there are no Fibonacci balancing numbers and in [22] he showed that there are no Lucas balancing

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numbers. He used a method by BAKER and DAVENPORT [2]. SZALAY [29] obtained the same results by using different techniques. In [25] PANDA introduced the sequence balancing numbers. Let $\{s_m\}_{m=1}^{\infty}$ be a sequence such that $s_m \in \mathbb{R}$. An element s_m of this sequence is called a sequence balancing number if

$$s_1 + s_2 + \cdots + s_{m-1} = s_{m+1} + s_{m+2} + \cdots + s_{m+k}$$

for some $k \in \mathbb{N}$. Further generalization in this direction is due to BÉRCZES, LIPTAI and PINK [4]. Now let a, b two non-negative coprime integers and recall the following definition of [19]. A positive integer $an + b$ is called (a, b)-type balancing number if

$$(a + b) + (2a + b) + \cdots + (a(n - 1) + b) = (a(n + 1) + b) + \cdots + (a(n + k) + b)$$

for some $k \in \mathbb{N}$. Denote by $B_m^{(a,b)}$ the m-th positive integer $an + b$ among the (a, b)-type balancing numbers. KOVÁCS, LIPTAI and OLAJOS [19] proved some general finiteness results concerning the equation

$$B_m^{(a,b)} = f(x),$$

where f is a monic polynomial with integral coefficients. They also resolved some related Diophantine equations. LIPTAI, LUCA, PINTÉR and SZALAY [23] introduced the concept of (k, l)-power numerical center as follows. Let y, k, l be fixed positive integers with $y \geq 2$. A positive integer x with $x \leq y - 2$ is called a (k, l)-power numerical center for y if

$$1^k + \cdots + (x - 1)^k = (x + 1)^l + \cdots + (y - 1)^l.$$

The authors of [23] obtained certain effective and ineffective finiteness results for (k, l)-power numerical centers.

For positive integers k, x let

$$\Pi_k(x) = x(x + 1) \cdots (x + k - 1).$$

That is, $\Pi_k(x)$ is a polynomial in x of degree k. In [19] it was proved that the equation

$$B_m = \Pi_k(x)$$

for fixed $k \geq 2$ has only finitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. We note that in [19] the "small" solutions of the above equation with $k \in \{6, 8\}$ were also computed.

In this paper we deal with the case $k = 5$. That is we consider the equation

$$B_m = x(x + 1)(x + 2)(x + 3)(x + 4).$$

We prove the following theorem.

Theorem 1. *The Diophantine equation*

$$B_m = x(x+1)(x+2)(x+3)(x+4) \quad m \geq 1,\ x \in \mathbb{Z}$$

has no solution.

2. Auxiliary results

Consider the hyperelliptic curve

$$\mathcal{C}: \quad y^2 = F(x) := x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0, \tag{1}$$

where $b_i \in \mathbb{Z}$. Let α be a root of F and $J(\mathbb{Q})$ be the Jacobian of the curve $\mathcal{C}$. We have that

$$x - \alpha = \kappa\xi^2$$

where κ, $\xi \in K = \mathbb{Q}(\alpha)$ and κ comes from a finite set. By knowing the Mordell–Weil group of the curve $\mathcal{C}$ it is possible to provide a method to compute such a finite set. To each coset representative $\sum_{i=1}^{m}(P_i - \infty)$ of $J(\mathbb{Q})/2J(\mathbb{Q})$ we associate

$$\kappa = \prod_{i=1}^{m}\left(\gamma_i - \alpha d_i^2\right),$$

where the set $\{P_1, \ldots, P_m\}$ is stable under Galois action, all $y(P_i)$ are non-zero and $x(P_i) = \gamma_i/d_i^2$ where γ_i is an algebraic integer and $d_i \in \mathbb{Z}_{\geq 1}$. If P_i, P_j are conjugate, then we may suppose that $d_i = d_j$ and so γ_i, γ_j are conjugate. We have the following lemma (Lemma 3.1 in [13]).

Lemma 1. *Let $\mathcal{K}$ be a set of κ values associated as above to a complete set of coset representatives of $J(\mathbb{Q})/2J(\mathbb{Q})$. Then $\mathcal{K}$ is a finite subset of $\mathcal{O}_K$ and if (x, y) is an integral point on the curve* (1) *then $x - \alpha = \kappa\xi^2$ for some $\kappa \in \mathcal{K}$ and $\xi \in K$.*

As an application of his theory of lower bounds for linear forms in logarithms, Baker [1] gave an explicit upper bound for the size of integral solutions of hyperelliptic curves. This result has been improved by many authors (see e.g. [5], [7], [10], [28] and [30]).

In [13] an improved completely explicit upper bound was proved by combining ideas from [10], [11], [12], [20], [24], [26], [30], [31]. Now we will state the theorem which gives the improved bound. We introduce some notation. Let K be a number field of degree d and let r_K be its unit rank further R_K its regulator. For $\alpha \in K$ we denote by $\mathrm{h}(\alpha)$ the logarithmic height of the element α. Let

$$\partial_K = \begin{cases} \dfrac{\log 2}{d} & \text{if } d = 1,\, 2, \\ \dfrac{1}{4}\left(\dfrac{\log\log d}{\log d}\right)^3 & \text{if } d \geq 3 \end{cases}$$

and

$$\partial'_K = \left(1 + \frac{\pi^2}{\partial_K^2}\right)^{1/2}.$$

Define the constants

$$c_1(K) = \frac{(r_K\,!)^2}{2^{r_K - 1} d^{r_K}}, \qquad c_2(K) = c_1(K)\left(\frac{d}{\partial_K}\right)^{r_K - 1},$$

$$c_3(K) = c_1(K)\frac{d^{r_K}}{\partial_K}, \qquad c_4(K) = r_K d c_3(K),$$

$$c_5(K) = \frac{r_K^{r_K+1}}{2\partial_K^{r_K-1}}.$$

Let

$$\partial_{L/K} = \max\left\{[L:\mathbb{Q}], [K:\mathbb{Q}]\partial'_K, \frac{0.16[K:\mathbb{Q}]}{\partial_K}\right\},$$

where $K \subseteq L$ are number fields. Define

$$C(K,n) := 3 \cdot 30^{n+4} \cdot (n+1)^{5.5}\, d^2\, (1 + \log d).$$

The following theorem will be used to get an upper bound for the size of the integral solutions of our equation. It is Theorem 3 in [13].

Theorem 2. *Let α be an algebraic integer of degree at least 3 and κ be an integer belonging to K. Denote by α_1, α_2, α_3 distinct conjugates of α and by κ_1, κ_2, κ_3 the corresponding conjugates of κ. Let*

$$K_1 = \mathbb{Q}(\alpha_1, \alpha_2, \sqrt{\kappa_1\kappa_2}\,), \quad K_2 = \mathbb{Q}(\alpha_1, \alpha_3, \sqrt{\kappa_1\kappa_3}\,), \quad K_3 = \mathbb{Q}(\alpha_2, \alpha_3, \sqrt{\kappa_2\kappa_3}\,),$$

and

$$L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \sqrt{\kappa_1\kappa_2}, \sqrt{\kappa_1\kappa_3}\,).$$

In what follows R stands for an upper bound for the regulators of K_1, K_2 and K_3 and r denotes the maximum of the unit ranks of K_1, K_2, K_3. Let

$$c_j^* = \max_{1\leq i\leq 3} c_j(K_i), \qquad N = \max_{1\leq i,j\leq 3}\left|\mathrm{Norm}_{\mathbb{Q}(\alpha_i,\alpha_j)/\mathbb{Q}}(\kappa_i(\alpha_i - \alpha_j))\right|^2,$$

and

$$H^* = c_5^* R + \frac{\log N}{\min_{1\le i\le 3}[K_i : \mathbb{Q}]} + \mathrm{h}(\kappa).$$

Define

$$A_1^* = 2H^* \cdot C(L, 2r+1) \cdot (c_1^*)^2 \partial_{L/L} \cdot \left(\max_{1\le i\le 3} \partial_{L/K_i}\right)^{2r} \cdot R^2,$$

and

$$A_2^* = 2H^* + A_1^* + A_1^* \log\{(2r+1)\cdot \max\{c_4^*, 1\}\}.$$

If $x \in \mathbb{Z}\backslash\{0\}$ *satisfies* $x - \alpha = \kappa\xi^2$ *for some* $\xi \in K$ *then*

$$\log|x| \le 8A_1^* \log(4A_1^*) + 8A_2^* + H^* + 20\log 2 + 13\,\mathrm{h}(\kappa) + 19\,\mathrm{h}(\alpha).$$

To obtain a lower bound for the possible unknown integer solutions we are going to use the so-called Mordell–Weil sieve. The Mordell–Weil sieve has been successfully applied to prove the non-existence of rational points on curves (see e.g. [8], [9], [17] and [27]).

Let $C/\mathbb{Q}$ be a smooth projective curve (in our case a hyperelliptic curve) of genus $g \ge 2$. Let J be its Jacobian. We assume the knowledge of some rational points on C, so let D be a fixed rational point on C and let $\jmath$ be the corresponding Abel–Jacobi map:

$$\jmath : C \to J, \qquad P \mapsto [P - D].$$

Let W be the image in J of the known rational points on C and $D_1, \dots, D_r$ generators for the free part of $J(\mathbb{Q})$. By using the Mordell–Weil sieve we are going to obtain a very large and smooth integer B such that

$$\jmath(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q}).$$

Let

$$\phi : \mathbb{Z}^r \to J(\mathbb{Q}), \qquad \phi(a_1, \dots, a_r) = \sum a_i D_i,$$

so that the image of ϕ is the free part of $J(\mathbb{Q})$. The variant of the Mordell–Weil sieve explained in [13] provides a method to obtain a very long decreasing sequence of lattices in $\mathbb{Z}^r$

$$B\mathbb{Z}^r = L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \cdots \supsetneq L_k$$

such that

$$\jmath(C(\mathbb{Q})) \subset W + \phi(L_j)$$

for $j = 1, \dots, k$.

The next lemma [13, Lemma 12.1] gives a lower bound for the size of rational points whose image are not in the set W. Let h be the logarithmic height on J and $\hat{h}$ be the canonical height on J.

Lemma 2. *Let W be a finite subset of $J(\mathbb{Q})$ and L be a sublattice of $\mathbb{Z}^r$. Suppose that $\jmath(C(\mathbb{Q})) \subset W + \phi(L)$. Let μ_1 be a lower bound for $h - \hat{h}$ and*

$$\mu_2 = \max\left\{\sqrt{\hat{h}(w)} : w \in W\right\}.$$

Denote by M the height-pairing matrix for the Mordell–Weil basis $D_1, \ldots, D_r$ and let $\lambda_1, \ldots, \lambda_r$ be its eigenvalues. Let

$$\mu_3 = \min\left\{\sqrt{\lambda_j} : j = 1, \ldots, r\right\}$$

and $m(L)$ the Euclidean norm of the shortest non-zero vector of L. Then, for any $P \in C(\mathbb{Q})$, either $\jmath(P) \in W$ or

$$h(\jmath(P)) \geq (\mu_3 m(L) - \mu_2)^2 + \mu_1.$$

3. Proof of Theorem 1

It was shown by Liptai that the integers B_m satisfy the following equation

$$z^2 - 8y^2 = 1$$

for some integer z. So one has to determine all solution of the equation

$$z^2 = 8(x(x+1)(x+2)(x+3)(x+4))^2 + 1.$$

Rewrite the latter equation as follows

$$z^2 = 8(x^2 + 4x)^2(x^2 + 4x + 3)^2(x^2 + 4x + 4) + 1.$$

Let $X = 2x^2 + 8x$. We obtain that

$$\mathcal{C}: \quad Y^2 = X^2(X+6)^2(X+8) + 4, \tag{2}$$

where $Y = 2z$. It remains to find all integral points on $\mathcal{C}$. The rank of the Jacobian of $\mathcal{C}$ is 3, so classical Chabauty's method [14], [15], [16] cannot be applied. In this paper we combine Baker's method and the so-called Mordell–Weil sieve to obtain all integral solutions of equation (2).

Lemma 3. *The only integral solutions to the equation* (2) *are*

$$(0, \pm 2), (-6, \pm 2), (-8, \pm 2).$$

Proof. Let $J(\mathbb{Q})$ be the Jacobian of the genus two curve (2). Using magma [6] we determine a Mordell–Weil basis which is given by

$$D_1 = (0,2) - \infty,$$
$$D_2 = (-6,2) - \infty,$$
$$D_3 = (\omega, -\omega - 10) + (\overline{\omega}, -\overline{\omega} - 10) - 2\infty,$$

where ω is a root of the polynomial $x^2 + 7x + 4$. Let $f = x^2(x+6)^2(x+8) + 4$ and α be a root of f. We will choose for coset representatives of $J(\mathbb{Q})/2J(\mathbb{Q})$ the linear combinations $\sum_{i=1}^{3} n_i D_i$, where $n_i \in \{0,1\}$. We have

$$x - \alpha = \kappa \xi^2,$$

where κ belongs to a finite set (having 8 elements). This set can be constructed as described in Lemma 1. We apply Theorem 2 to get a large upper bound for $\log|x|$. A magma code was written by Gallegos–Ruiz [18] to obtain bounds for such equations. We used the above magma functions to compute an upper bound for $\log|x|$, the results are summarized in the following table

κ	bound for $\log\|x\|$
1	$4.17 \cdot 10^{204}$
$-\alpha$	$1.59 \cdot 10^{411}$
$-6-\alpha$	$3.11 \cdot 10^{430}$
$4+7\alpha+\alpha^2$	$1.59 \cdot 10^{411}$
$-8+6\alpha+\alpha^2$	$3.11 \cdot 10^{430}$
$13+9\alpha+\alpha^2$	$1.59 \cdot 10^{411}$
$6\alpha+\alpha^2$	$3.11 \cdot 10^{430}$
$-10+5\alpha+\alpha^2$	$3.11 \cdot 10^{430}$

The set of known rational points on the curve (2) is $\{\infty,(0,\pm 2),(-6,\pm 2),(-8,\pm 2)\}$. Let W be the image of this set in $J(\mathbb{Q})$. Applying the Mordell–Weil sieve implemented by Bruin and Stoll we obtain that

$$\jmath(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q}),$$

where

$$B = 2^6 \cdot 3^4 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19^2 \cdot 29 \cdot 31 \cdot 41 \cdot 43 \cdot 47 \cdot 61 \cdot 67 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97 \cdot 107 \cdot 109 \cdot 113,$$

that is

$$B = 46247720065121846143591520774334300410472000.$$

Now we use an extension of the Mordell–Weil sieve due to Samir Siksek to obtain a very long decreasing sequence of lattices in $\mathbb{Z}^3$. After that we apply Lemma 2 to obtain a lower bound for possible unknown rational points. We get that if (x, y) is an unknown integral point, then

$$\log|x| \geq 1.03 \times 10^{580}.$$

This contradicts the bound for $\log|x|$ we obtained by Baker's method. □

The statement of the Theorem now easily follows. It is enough to find the values of $X = 2x^2 + 8x$. Afterwards the values for x and m are recovered immediately.

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