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The difference quotient operator

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Introduction

In this paper we discuss the spectral properties of a linear operator defined on a Hilbert space. Using the results of [1], we determine the essential spectrum of our operator. This, together with the Berger–Shaw theorem enables us to assert that the operator is question, which we will call the difference quotient operator, and the adjoint of the multiplication operator are both essentially normal.

Our main result is an application of the Brown–Douglas–Fillmore theorem: we will show that the difference quotient operator is a compact perturbation of the adjoint of the multiplication operator, defined on an appropriately chosen space.

Preliminaries

We will define the difference quotient operator. We will be working in the Hilbert space $H^2(G,\mu)$, consisting of the L^2 closure of the polynomials in the independent complex variable z with respect to the usual two-dimensional Lebesgue area measure μ, defined on an open, simply connected, bounded subset of the complex plane, G, containing the origin, such that $\bar{G}$ is also connected. To avoid some pathological boundaries, we will further assume that $\partial(G) = \partial(\bar{G})$. The difference quotient operator is defined as follows:

$$T : H^2(G,\mu) \to H^2(G,\mu), \quad (Tf)(z) = \frac{f(z) - f(0)}{z}.$$

Proposition 1. *T is a bounded linear operator.*

Proof. Linearity is obvious, so we will establish an upper bound for the norm of T. Let $R > 0$ be the radius of the largest open disc around the origin that is inside G; call this disc D_R. By definition

$$\|T\| = \sup \|Tf\|_2 = \sup \left[\int_G \left| \frac{f(z) - f(0)}{z} \right|^2 d\mu \right]^{1/2} .$$

Then

$$\|T\|^2 = \sup \left[\int_{D_R} \left| \frac{f(z) - f(0)}{z} \right|^2 d\mu + \int_{G \setminus D_R} \left| \frac{f(z) - f(0)}{z} \right|^2 d\mu \right],$$

where the supremums are taken over the unit ball of $H^2(G, \mu)$. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \text{then} \quad \frac{f(z) - f(0)}{z} = \sum_{k=1}^{\infty} a_k\, z^{k-1}.$$

We obtain

$$\int_{D_R} \left| \frac{f(z) - f(0)}{z} \right|^2 d\mu = \int_0^R \int_0^{2\pi} \left(\sum_{k=1}^{\infty} |a_k|^2\, r^{2k-2} \right) r\, dr d\theta,$$

where we made use of the standard $z = re^{i\theta}$ change of variables. Carrying on the computation we obtain

$$\begin{aligned} \int_{D_R} \left| \frac{f(z) - f(0)}{z} \right|^2 d\mu &= \sum_{k=1}^{\infty} \left(\int_0^R \int_0^{2\pi} |a_k|^2\, r^{2k-1}\, dr\, d\theta \right) \\ &= \pi \left(\sum_{k=1}^{\infty} \frac{|a_k|^2\, R^{2k}}{k} \right). \end{aligned}$$

However,

$$\int_{D_R} |f(z)|^2\, d\mu = \pi \left(\sum_{k=0}^{\infty} \frac{|a_k|^2\, R^{2k+2}}{k+1} \right).$$

and we notice that

$$(*) \qquad \int_{D_R} \left| \frac{f(z) - f(0)}{z} \right|^2 d\mu \le \frac{2}{R^2} \int_{D_R} |f(z)|^2\, d\mu.$$

For $z \in G \backslash D_R$, $|z| \geq R$, so

$$\int_{G \backslash D_R} \left| \frac{f(z) - f(0)}{z} \right|^2 d\mu \leq \frac{1}{R^2} \int_{G \backslash D_R} |f(z) - f(0)|^2 \, d\mu.$$

We make use of the inequality

$$|a - b|^2 \leq 2(|a|^2 + |b|^2),$$

to obtain
(**)

$$\int_G \left| \frac{f(z) - f(0)}{z} \right|^2 d\mu \leq \frac{2}{R^2} \left[\int_{G \backslash D_R} |f(z)|^2 \, d\mu + |f(0)|^2 \int_{G \backslash D_R} 1 \, d\mu \right].$$

But

$$|f(0)| \leq \frac{1}{R\sqrt{\pi}} \|f\|_2 \qquad [4,\text{p. } 5],$$

so adding $(*)$ and $(**)$ yields

$$\int_{G \backslash D_R} \left| \frac{f(z) - f(0)}{z} \right|^2 d\mu$$

$$\leq \frac{2}{R^2} \left[\int_{G \backslash D_R} |f(z)|^2 \, d\mu + \int_{D_R} |f(z)|^2 \, d\mu + \frac{\mu(G) - \pi R^2}{\pi R^2} \right] = \frac{2}{R^4} \frac{\mu(G)}{\pi}.$$

Thus $\|T\| \leq \frac{1}{R^2}\sqrt{\frac{2\mu(G)}{\pi}}$, proving that T is bounded.

The spectrum

Proposition 2. $\sigma(T) = \{z : \frac{1}{z} \notin G\}$.

As a reminder, the spectrum of an operator T is the set of complex numbers λ for which $(T - \lambda)$ is not invertible. We will therefore try to determine those λ for which $(T - \lambda)$ is either not one-to-one or not onto. We have $(T - \lambda)f(z) = \frac{f(z)(1-\lambda z) - f(0)}{z}$, so the only possible candidate for $\ker(T - \lambda)$ is $\frac{C}{1-\lambda z}$. This function may or may not be in $H^2(G, \mu)$ depending on λ(C denotes an arbitrary constant).

Case 1. $\frac{1}{\lambda} \notin \bar{G}$. In this case $\frac{C}{1-\lambda z}$ is analytic in G, and also square integrable because in this case $|1 - \lambda z| > d > 0$ for some constant d, so

$$\int_G \frac{1}{|1 - \lambda z|^2} \, d\mu \leq \frac{1}{d^2} \int_G 1 \, d\mu < \infty.$$

Since $\frac{1}{\lambda} \notin \bar{G}$, $\frac{C}{1-\lambda z}$ is analytic on an open set containing $\bar{G}$ as well, thus by Runge's theorem it is the uniform limit of polynomials on $\bar{G}$. Thus $\frac{C}{1-\lambda z}$ is indeed in $H^2(G,\mu)$. We conclude that for $\frac{1}{\lambda} \notin \bar{G}$, $(T-\lambda)$ is not one-to-one.

Case 2. $\frac{1}{\lambda} \in G$. In this case $\frac{C}{1-\lambda z}$ is neither analytic, nor square integrable in G, so $\ker(T-\lambda) = 0$, so $(T-\lambda)$ is one-to-one if $\frac{1}{\lambda} \in G$.

Case 3. $\frac{1}{\lambda} \in \partial G$. As it turns out, the spectrum of T can be determined even without analyzing this case.

We will determine now for which λ is $(T-\lambda)$ onto.

Case 1. $\frac{1}{\lambda} \notin \bar{G}$. In this case it was shown that $(T-\lambda)$ is not one-to-one, so a given $g(z) \in H^2(G,\mu)$ may have several inverse images under $(T-\lambda)$.

Noticing that $(T-\lambda)\left(\frac{zg(z)+C}{1-\lambda z}\right) = g(z)$, we conclude that for $\frac{1}{\lambda} \notin \bar{G}, (T-\lambda)$ is onto, since for any constant C, $\frac{zg(z)+C}{1-\lambda z}$ us square integrable and analytic in G. $\frac{zg(z)+C}{1-\lambda z}$ is a uniform limit of a bounded sequence of polynomials on G because $\frac{1}{1-\lambda z}$ is. It follows immediately that $\frac{zg(z)+C}{1-\lambda z}$ belongs to $H^2(G,\mu)$.

Case 2. $\frac{1}{\lambda} \in G$. In this case $(T-\lambda)$ in one-to-one. Given $f(z) \in H^2(G,\mu)$, we show that $h(z) = \frac{zf(z)-\frac{1}{\lambda}f(\frac{1}{\lambda})}{1-\lambda z}$ is the inverse image of $f(z)$. Since $\frac{1}{\lambda} \in G$, $f(\frac{1}{\lambda})$ is defined. If $f \in H^2(G,\mu)$ then $zf(z) \in H^2(G,\mu)$. As for $a=0$ in Proposition 1, it can be shown that

$$(zf(z)-af(a))(z-a)^{-1} \in H^2(G,\mu),$$

whenever $a \in G$. Hence with $a = \lambda^{-1} \in G$, we obtain that

$$\begin{aligned}\lambda^{-1}(zf(z)-\lambda^{-1}f(\lambda^{-1}))(z-\lambda^{-1})^{-1} &= (zf(z)-\lambda^{-1}f(\lambda^{-1})(1-\lambda z)^{-1} \\ &\in H^2(G,\mu).\end{aligned}$$

We conclude that for $\frac{1}{\lambda} \in G, (T-\lambda)$ is onto.

Gathering the information obtained so far we conclude tha $(T-\lambda)$ is invertible if $\frac{1}{\lambda} \in G$ and it's not invertible if $\frac{1}{\lambda} \notin \bar{G}$. If $\sigma(T)$ denotes the spectrum of T, this means, that

$$\left\{z : \frac{1}{z} \notin \bar{G}\right\} \subset \sigma(T) \quad \text{and} \quad \left\{z : \frac{1}{z} \in G\right\} \cap \sigma(T) = \emptyset.$$

The three sets $\{z : \frac{1}{z} \notin \bar{G}\}, \{z : \frac{1}{z} \in \partial G\}$ and $\{z : \frac{1}{z} \in G\}$ partition the complex plane. Let's call these three sets A, B and C respectively. We know that the spectrum of any operator is a compact set, thus $\sigma(T)$ is closed. Since $A \subset \sigma(T)$, we have $\bar{A} \subset \sigma(T)$ as well. (In general, for two sets, $X \subset Y$ implies $\bar{X} \subset \bar{Y}$ because by definition $\bar{X}$ is the intersection of all closed sets containing X.) But it is easy to see that $\bar{A} = A \cup B = \{z : \frac{1}{z} \notin G\}$. Thus, since $C \cap \sigma(T) = \emptyset$, we obtain Proposition 2.

We notice that, since G and $\bar{G}$ are simply connected, $H^2(G, \mu) = R^2(G, \mu)$ as a consequence of Runge's theorem. Here we denote the L^2 closure of rational functions by $R^2(G, \mu)$. This means, that considering T to be defined on $R^2(G, \mu)$, the spectrum of T remains $\{z : \frac{1}{z} \notin G\}$.

The essential spectrum

We will now determine the essential spectrum of T.

Definition. The essential spectrum of an operator L, denoted $\sigma_e(L)$, is defined to be the set of all complex numbers λ such that $L - \lambda$ is not Fredholm. The left-essential spectrum of an operator L (respectively right essential spectrum) denoted $\sigma_{le}(L)$ respectively $\sigma_{re}(L)$ is defined to be the set of all numbers λ such that $L - \lambda$ is not left (respectively right) Fredholm.

The following theorem may be found in [4, p. 355, 373].

Theorem.

(1) $\sigma_{le}(L) \cup \sigma_{re}(L) = \sigma_e(L) \subset \sigma(L)$.
(2) *If $\lambda \in \partial\sigma(L)$ then either λ is an isolated point of $\sigma(L)$ or $\lambda \in \sigma_{le}(L) \cap \sigma_{re}(L)$.*

Lemma 3. *Let R be as in Proposition 1, $M = \sup\{|z| : z \in \bar{G}\}$. Then $\sigma(T)$ contains an open disc around the origin of radius $\frac{1}{M}$ and, denoting the spectral radius of T by $r(T)$, we have $\frac{1}{M} \leq r(T) \leq \frac{1}{R}$. Also, $\sigma(T)$ has no isolated points.*

Proof. Let $0 < |z| < \frac{1}{M}$. Then $|\frac{1}{z}| > M$, so $\frac{1}{z} \notin G$, thus $z \in \sigma(T)$. Clearly $z = 0$ is an element of $\sigma(T)$. Since $r(T) = \sup\{|z| : z \in \sigma(T)\}$, if we suppose $r(T) > \frac{1}{R}$, this would mean the existence of a point $z \in \sigma(T)$ such that $|\frac{1}{z}| < R$. But, by definition of R, this implies that $\frac{1}{z} \in G$, so $z \notin \sigma(T)$, contradiction. Finally $\sigma(T)$ has no isolated points because it is the closure of an open set.

We are now in position to determine the essential spectrum of T.

Proposition 4. $\sigma_e(T) = \partial\sigma(T)$.

PROOF. Denoting the resolvent set of T, i.e., the complement of the spectrum of T by $\rho(T)$, we partition the complex plane into the union of the three sets:

$$\mathbf{C} = \rho(T) \cup \text{ int } (\sigma(T)) \cup \partial\sigma(T).$$

By the first part of the theorem quoted above, $\sigma_e(T) \subset \sigma(T)$, so clearly the essential spectrum of T and the resolvent of T are disjoint. If $\lambda \in \text{int}(\sigma(T))$ then we saw earlier that $\ker(T-\lambda) = \frac{C}{1-\lambda z}$, so $\dim\ker(T-\lambda) = 1$, and that $(T-\lambda)$ is Fredholm and $\text{ind}(T-\lambda) = 1$. This means that λ is not an element of the essential spectrum of T. Therefore we must have $\sigma_e(T) \subset \partial\sigma(T)$.

On the other hand, the second part of the theorem and Lemma 3 imply that $\partial\sigma(T) \subset \sigma_{le}(T) \cup \sigma_{re}(T) \subset \sigma_e(T)$. Thus the desired conclusion follows.

To proceed further, we have to introduce some definitions.

Definition. The set of trace-class operators, $B_1(H)$ consists of those elements A of $B(H)$ for which the sum

$$\sum_{i\in\Lambda} |\langle Ae_i, e_i\rangle| \text{ is finite for all orthonormal bases } \{e_i\}_{i\in\Lambda}.$$

The following theorem is from [6, p. 16].

Theorem. *Trace-class operators are compact.*

For convenience we will state the BERGER–SHAW Theorem [2, p. 1193].

Theorem. *If A is an n-multicyclic hyponormal operator, then $[A^*, A]$ is trace class.*

We are now ready to prove

Lemma 5. $[T^*, T] = T^*T - TT^*$ *is trace class.*

PROOF. Consider the multiplication operator S on $H^2(G,\mu)$ defined by $(Sf)(z) = zf(z)$. S is subnormal, having $M_z : L^2(G,\mu) \to L^2(G,\mu)$ as its normal extension. $S^n(1) = z^n$, so $\{S^k(1)\}_{k=0}^{\infty}$ is a total set in $H^2(G,\mu)$, which means that $\{1\}$ is a cyclic vector for S. S being subnormal, it is also hyponormal, so the powerful Berger-Shaw theorem applies, and we conclude that $[S^*, S]$ is trace-class.

It is easy to verify that $\begin{cases} TS = I \\ ST = I + F \end{cases}$, where F is a rank one operator.

Therefore $[T^*, T] = TT^*[S^*, S]T^*T - TF^*ST^*T + TT^*F$. (We made use of the relations $S^*T^* = I, T^*S^* = I + F^*$).

Since $B_1(H)$ is a two-sided ideal, and finite rank operators are trace class, we conclude that $[T^*, T] \in B_1(H)$.

Definition. A is an essentially normal operator if $[A^*, A]$ is compact.

Thus the difference quotient operator is essentially normal, and so is the multiplication operator and its adjoint.

We will determine the essential spectrum of the operator $S : H^2(G, \mu) \to H^2(G, \mu)$ defined by $(Sf)(z) = zf(z)$. This is actually done in [1, p. 472]. Since G is simply connected and $\partial(G)$ has no isolated points by assumption, $\sigma_e(S) = \partial(G)$.

Proposition 6.

$$\operatorname{ran}(S - \lambda) = \begin{cases} \{f : f(z) \in H^2(G, \mu), f(\lambda) = 0\}, & \textit{if } \lambda \in G \\ H^2(G, \mu), & \textit{if } \lambda \notin \bar{G} \end{cases}.$$

Proof. Denote $M_\lambda = \{f \in H^2(G, \mu) : f(\lambda) = 0\}, \lambda \in G$. Since $(S-\lambda)g(z) = (z-\lambda)g(z)$, $f(z) = (z-\lambda)g(z) \in M_\lambda$ for all $g(z)$ in $H^2(G, \mu)$, so $\operatorname{ran}(S - \lambda) \subset M_\lambda$.

Conversely, let $f(z) \in M_\lambda$, then $f(z) = (z - \lambda)g(z)$ for some $g(z)$. We will show that $g(z) \in H^2(G, \mu)$. Choose $R > 0$ so that $D_R = \{z : |z - \lambda| \leq R\}$ is contained in G.

$$\|g(z)\|_2^2 = \int_{D_R} |g(z)|^2 \, d\mu + \int_{G \setminus D_R} |g(z)|^2 \, d\mu.$$

Now $g(z) = \frac{f(z)}{z-\lambda}$ has a removable singularity at λ, and D_R is closed, so $|g(z)|$ is bounded on D_R, thus $\int_{D_R} |g(z)|^2 \, d\mu < \infty$. On the other hand $\int_{G \setminus D_R} |g(z)|^2 \, d\mu < \frac{1}{R^2} \int_G |f(z)|^2 \, d\mu$.

This shows that $g(z) \in H^2(G, \mu)$, proving $M_\lambda \subset \operatorname{ran}(S - \lambda)$. We conclude $M_\lambda = \operatorname{ran}(S - \lambda)$ for $\lambda \in G$. We will show that M_λ is closed in $H^2(G, \mu)$. Define the linear functional

$$L_\lambda : H^2(G, \mu) \to \mathbf{C} \quad \text{by} \quad L_\lambda(f) = f(\lambda).$$

L_λ is bounded, for

$$\|L_\lambda\| = \sup \frac{|L_\lambda(f)|}{\|f\|_2} = \sup \frac{|f(\lambda)|}{\|f\|_2} \leq \frac{1}{R\sqrt{\pi}}.$$

The supremums are taken over $H^2(G,\mu)$.

M_λ is closed, because it is the kernel of L_λ. Now if $\lambda \notin \bar{G}$, given $g(z) \in H^2(G,\mu), f(z) = \frac{g(z)}{z-\lambda}$ is a uniform limit of a bounded sequence of polynomials on G because $(1-\lambda z)^{-1}$ is, therefore $f(z) \in H^2(G,\mu)$, showing that $(S-\lambda)$ is onto in this case.

It is a trivial observation that $\ker(S-\lambda) = \{0\}$ for all $\lambda \in \mathbf{C}$.

We may write $f(z) = (f(z) - f(\lambda)) + f(\lambda)$ for all $f(z) \in H^2(G,\mu)$. Here $(f(z) - f(\lambda))$ belongs to $\operatorname{ran}(S-\lambda)$ if $\lambda \in G$; $f(\lambda)$ being a constant function belongs to $H^2(G,\mu)$. Thus $H^2(G,\mu) = \operatorname{ran}(S-\lambda) + \mathbf{C}$. Therefore $\dim(\operatorname{ran}(S-\lambda))^\perp = 1$.

Summing up we coclude that $(S-\lambda)$ is a Fredholm operator for $\lambda \notin \partial G$, and

$$\operatorname{ind}(S-\lambda) = \begin{cases} -1 & \text{for } \lambda \in G, \\ 0 & \text{for } \lambda \notin \bar{G}. \end{cases}$$

We are now ready to use the Brown–Douglas–Fillmore theorem.

Theorem. *If T_1 and T_2 are essentially normal operators on a Hilbert space H, then a necessary and sufficient condition that T_1 be unitarily equivalent to some compact perturbation of T_2 is that T_1 and T_2 have the same essential spectrum Λ, and ind$(T_1 - \lambda I) =$ ind$(T_2 - \lambda I)$ for all $\lambda \notin \Lambda$. We refer the reader to* [3, p. 58-117] *for a proof.*

Let's define now our operator S on the space $H^2((\frac{\bar{1}}{G^*})^c, \mu)$, where $\frac{1}{G^*} = \{\bar{z} : \frac{1}{z} \in G\}$, $\frac{\bar{1}}{G^*}$ means the closure of $\frac{1}{G^*}$. Let $(\frac{\bar{1}}{G^*})^c = \Omega$. Then $\sigma_e(S^*) = (\sigma_e(S))^* = (\partial(\Omega))^* = \partial(\{\frac{1}{z} : z \in \bar{G}\}^c) = \{z : \frac{1}{z} \in \partial G\} = \sigma_e(T)$. We have used the fact that for any set $A, \partial A = \bar{A} \cap \overline{A^c} = \partial(A^c)$.

Therefore $T : H^2(G,\mu) \to H^2(G,\mu)$ and $S^* : H^2(\Omega,\mu) \to H^2(\Omega,\mu)$ have the same essential spectrum,they are both essentially normal, and for $\lambda \notin \sigma_e(T) = \sigma_e(S^*)$,

$$\operatorname{ind}(S^* - \lambda) = \begin{cases} 1, & \text{if } \lambda \in (\Omega)^*, \\ 0, & \text{if } \lambda \notin \overline{(\Omega)}^*. \end{cases}$$

If $\lambda \in (\Omega)^*$, we have

$$\lambda \in \left(\left\{ \overline{\frac{1}{z} : z \in G} \right\}^c \right)^* = \left\{ \frac{1}{z} : z \notin \bar{G} \right\}^* = \left\{ \frac{1}{z} : z \notin \bar{G} \right\} = \operatorname{int} \sigma(T).$$

If $\lambda \notin (\overline{(\Omega)})^*$, then

$$\lambda \notin \overline{\left(\overline{\left\{\frac{1}{z} : z \in G\right\}}^{c}\right)}^{*} = \overline{\left\{\frac{1}{z} : z \notin \bar{G}\right\}}^{*} = \left\{\frac{1}{z} : z \notin G\right\}^{*}$$
$$= \left\{\frac{1}{z} : z \notin G\right\} = \sigma(T).$$

So we conclude

$$\operatorname{ind}(S^* - \lambda) = \begin{cases} 1, & \text{if } \lambda \in \operatorname{int}\sigma(T) \\ 0, & \text{if } \lambda \notin \sigma(T) \end{cases} = \operatorname{ind}(T - \lambda) = \begin{cases} 1, & \text{if } \lambda \in \operatorname{int}\sigma(T) \\ 0, & \text{if } \lambda \notin \sigma(T) \end{cases}.$$

Finally we are able to assert:

Theorem. *$T : H^2(G, \mu) \to H^2(G, \mu)$ is unitarily equivalent to some compact perturbation of $S^* : H^2(\Omega, \mu) \to H^2(\Omega, \mu)$.*

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