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On some classes of almost pseudo Ricci symmetric manifolds

By UDAY CHAND DE (Kolkata) and PRAJJWAL PAL (Nadia)

Abstract. The object of the present paper is to study almost pseudo Ricci symmetric manifolds. Some geometric properties have been studied. Among others we obtain a sufficient condition for an almost pseudo Ricci symmetric manifold to be a quasi Einstein manifold. Next we consider conformally flat almost pseudo Ricci symmetric manifolds. Some global properties have been studied. Finally we give an example to verify the sufficient condition for an almost pseudo Ricci symmetric manifold to be a quasi Einstein manifold.

1. Introduction

As it is well known, symmetric spaces play an important role in differential geometry. The study of locally symmetric Riemannian spaces was initiated in the late twenties by E. Cartan [1], who, in particular, obtained a classification of those spaces. Let $(M^n, g), (n = \dim M)$ be a Riemannian manifold, i.e., a manifold M with the Riemannian metric g, and let ∇ be the Levi–Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric [1] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [2]. The class of locally symmetric Riemannian manifolds is very natural generalization of the class of manifolds of constant curvature. During the last five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to different extent such as conformally symmetric manifolds by M. C. Chaki and B. Gupta [3], recurrent

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manifolds introduced by A. G. Walker [4], conformally recurrent manifolds by T. Adati and T. Miyazawa [5] , pseudo symmetric manifolds by M. C. Chaki [6], weakly symmetric manifolds by L. Tamássy and T. Q. Binh [7] etc. A non-flat Riemannian manifold (M^n, g), $(n > 2)$ is said to be a pseudo symmetric manifold [6] if its curvature tensor satisfies the condition

$$(\nabla_X R)(Y,Z)W = 2A(X)R(Y,Z)W + A(Y)R(X,Z)W \\ + A(Z)R(Y,X)W + A(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho,$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X,\rho) = A(X)$$

for all X and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-form A is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a locally symmetric manifold in the sense of E. Cartan. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$. In a recent paper U. C. De and A. K. Gazi [8] introduced the notion of almost pseudo symmetric manifolds. A Riemannian manifold (M^n, g), $(n > 2)$ is said to be almost pseudo symmetric if its curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the condition:

$$(\nabla_U \tilde{R})(X,Y,Z,W) = [A(U) + B(U)]\tilde{R}(X,Y,Z,W) + A(X)\tilde{R}(U,Y,Z,W) \\ + A(Y)\tilde{R}(X,U,Z,W) + A(Z)\tilde{R}(X,Y,U,W) + A(W)\tilde{R}(X,Y,Z,U),$$

where A, B are non-zero 1-forms defined by $g(X,P) = A(X)$, and $g(X,Q) = B(X)$, for all vector fields X, ∇ denotes the operator of covariant differentiation with respect to the metric g, $\tilde{R}$ is defined by $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type $(1,3)$. The 1-forms A and B are called the associated 1-forms of the manifold. Such a manifold is denoted by $(APS)_n$. Here the vector fields P and Q are called the basic vector fields of the manifold corresponding to the associated 1-forms A and B respectively. If in the above equation $B = A$, then the $(APS)_n$ reduces to a $(PS)_n$. The notion of weakly symmetric manifold was introduced by L. Tamássy and T. Q. Binh [7]. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetric if the curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(Y)R(X,Z)W \\ + C(Z)R(Y,X)W + D(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho, \quad (1.1)$$

where ∇ denotes the Levi–Civita connection on (M^n, g), and A, B, C, D and ρ are the 1-forms and a vector field, respectively which are non-zero simultaneously. Such a manifold is denoted by $(WS)_n$. It was proved in [9] that the 1-forms and the vector field must be related as follows

$$B(X) = C(X) = D(X), \quad g(X, \rho) = D(X), \quad \text{for all } X.$$

That is, the weakly symmetric manifold is characterized by the condition

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + D(Y)R(X, Z)W \\ + D(Z)R(Y, X)W + D(W)R(Y, Z)X + g(R(Y, Z)W, X)\rho. \quad (1.2)$$

The 1-forms A and D are called the associated 1-forms, and the vector field ρ is called the associated vector field of the manifold. If $A = 2D$, the (M^n, g) reduces to a pseudo symmetric manifold in the sense of M. C. Chaki [6]. Again if $A = D = 0$, the manifold reduces to a symmetric manifold in the sense of E. Cartan. The existence of a $(WS)_n$ was proved by M. Prvanović [10] and a concrete example is given by U. C. De and S. Bandyopadhyay ([9], [11]). This justifies the name weakly symmetric manifold defined by (1.1). In 1993 L. Tamássy and T. Q. Binh [12] introduced the notion of weakly Ricci symmetric manifold. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + C(Z)S(Y, X), \quad (1.3)$$

where A, B, C are three non-zero 1-forms, and ∇ denotes the operator of covariant differentiation with respect to the metric g. Such an n-dimensional manifold is denoted by $(WRS)_n$. If in (1.3) the 1-form A is replaced by $2A$; B and C are replaced by A, then the manifold is called a pseudo Ricci symmetric manifold introduced by M. C. Chaki [13]. This implies that pseudo Ricci symmetric manifold is a particular case of a weakly Ricci symmetric manifold defined by (1.3).

The notion of an almost pseudo Ricci symmetric manifold was introduced by M. C. Chaki and T. Kawaguchi [14]. It was a generalization of the notion of pseudo Ricci symmetric manifold and was defined as follows:

A non-flat Riemannian manifold (M^n, g) is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies

$$(\nabla_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X), \quad (1.4)$$

where A and B are two 1-forms and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. In such a case A and B are called the associated 1-forms and an n-dimensional manifold of this kind is denoted by $A(PRS)_n$. If $B = A$, then (1.4) takes the following form:

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X),$$

which is called a pseudo Ricci symmetric manifold introduced by Chaki [13]. Let

$$g(X,P) = A(X) \text{ and } g(X,Q) = B(X), \quad \text{for all } X \tag{1.5}$$

Then P, Q are called the basic vector fields of the manifold corresponding to the associated 1-forms A and B, respectively. A. Gray [15] introduced the notion of cyclic parallel Ricci tensor and Codazzi type Ricci tensor. The Ricci tensor S of type $(0,2)$ is said to be of cyclic parallel, if it is non-zero, and

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{1.6}$$

Again a Riemannian manifold is said to be of Codazzi type if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

In a recent paper U. C. De and A. K. Gazi [8] studied almost pseudo symmetric manifolds. In subsequent papers ([16], [17]) U. C. De and A. K. Gazi studied almost pseudo conformally symmetric manifolds and conformally flat almost pseudo Ricci symmetric manifolds.

It may be mentioned that any pseudo Ricci symmetric manifold is a particular case of an $A(PRS)_n$, but a $(WRS)_n$ is not a particular case of an $A(PRS)_n$. In the above said paper [17] U. C. De and A. K. Gazi gave two examples of $A(PRS)_n$. Among these two examples one is a conformally flat $A(PRS)_n$ and the other one is a non-conformally flat $A(PRS)_n$. Also in a recent paper A. De, C. Üzgür and U. C. De [18] studied conformally flat almost pseudo Ricci symmetric space-time. Motivated by these works we further study $A(PRS)_n$ $(n > 2)$.

We also have a very useful lemma as follows:

Walker's Lemma [4]. *If a_{ij}, b_i are numbers satisfying $a_{ij} = a_{ji}$, and $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$, for $i,j,k = 1,2,\dots,n$, then either all $a_{ij} = 0$ or, all b_i are zero.*

The paper is organized as follows: After preliminaries in Section 2, we study $A(PRS)_n(n > 2)$ with cyclic parallel Ricci tensor. In Section 4, we consider $A(PRS)_n$ admitting a parallel vector field. Next we obtain a sufficient condition for an $A(PRS)_n$ to be a quasi Einstein manifold. Section 6 is devoted to study conformally flat $A(PRS)_n(n > 3)$. Section 7 and Section 8 deal with some global properties of $A(PRS)_n$ having Codazzi type Ricci tensor. Finally we give an example of an $A(PRS)_4$ to illustrate the result already obtained in Section 5.

2. Preliminaries

Let S and r denote the Ricci tensor of type $(0, 2)$ and the scalar curvature respectively. L denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is,

$$g(LX, Y) = S(X, Y), \tag{2.1}$$

for any vector fields X, Y. Let $\bar{A}$ and $\bar{B}$ are two 1-forms defined by

$$A(LX) = \bar{A}(X), \quad B(LX) = \bar{B}(X). \tag{2.2}$$

Then $\bar{A}$ and $\bar{B}$ are called auxiliary 1-forms corresponding to the 1-forms A and B respectively. Putting $Y = Z = e_i$ in (1.4), where $\{e_i\}$, $i = 1, 2, \ldots, n$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, $1 \leq i \leq n$, we get

$$dr(X) = [A(X) + B(X)]r + 2\bar{A}(X). \tag{2.3}$$

We know [17] that a conformally flat $A(PRS)_n$ is a quasi Einstein manifold of the form

$$S(X, Y) = ag(X, Y) + bT(X)T(Y), \tag{2.4}$$

where $a = \frac{r-t}{n-1}$, $b = \frac{nt-r}{n-1}$ are scalars, and $T(X) = \frac{B(X)}{\sqrt{B(Q}}$, $t = \frac{\bar{B}(Q)}{B(Q)}$. Also $T(X) = g(X, \rho)$, so that $\rho = \frac{Q}{\sqrt{B(Q)}}$ is a unit vector field.

Also it is known ([19], [20]) that for a conformally flat Riemannian manifold $(n > 3)$

$$(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) = \frac{1}{2(n-1)}[g(Y, Z)dr(X) - g(X, Y)dr(Z)]. \tag{2.5}$$

Now contracting (2.4) over Y, Z we have

$$r = an + b. \tag{2.6}$$

Now let us suppose that a is constant. Then from (2.6) it follows that

$$dr(X) = db(X). \tag{2.7}$$

Hence from (2.4) we get

$$\begin{aligned}(\nabla_Z S)(X,Y) = da(Z)g(X,Y) + db(Z)T(X)T(Y) \\ + b[(\nabla_Z T)(X)T(Y) + T(X)(\nabla_Z T)(Y)].\end{aligned} \tag{2.8}$$

Now if we consider an $A(PRS)_n (n > 3)$ whose Ricci tensor is of Codazzi type, then we have

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = 0. \tag{2.9}$$

Using (1.4) in (2.9) we get

$$B(X)S(Y,Z) = B(Y)S(X,Z). \tag{2.10}$$

Contracting over Y, Z we get

$$B(X)r = \bar{B}(X). \tag{2.11}$$

Again putting $Y = Q$ in (2.10) we get

$$B(X)S(Q,Z) = B(Q)S(X,Z)$$

or

$$S(X,Z) = \frac{B(X)\bar{B}(Z)}{B(Q)}. \tag{2.12}$$

Using (2.11) in (2.12) we get

$$S(X,Z) = rT(X)T(Z), \tag{2.13}$$

where $T(X) = \frac{B(X)}{\sqrt{B(Q)}}$. Let $T(X) = g(X,\rho)$. Then $g(X,\rho) = g\big(X, \frac{Q}{\sqrt{B(Q)}}\big)$. Hence $\rho = \frac{Q}{\sqrt{B(Q)}}$. So $g(\rho,\rho) = \frac{B(Q)}{B(Q)} = 1$. Hence ρ is a unit vector field. If $r = 0$, then from (2.13) we get $S = 0$ which is not admissible. Hence in a $A(PRS)_n$, $r \neq 0$. From (2.13) we have

$$S(X,X) = rT(X)T(X), \quad \text{for all } X. \tag{2.14}$$

that is,

$$S(X,X) = r[g(X,\rho)]^2, \quad \text{for all } X. \tag{2.15}$$

Hence $S(\rho,\rho) = r$, since ρ is a unit vector field. Let θ be the angle between ρ and an arbitrary vector field X, then $\cos\theta = \frac{g(X,\rho)}{\sqrt{g(\rho,\rho)g(X,X)}} = \frac{g(X,\rho)}{\sqrt{g(X,X)}}$. Hence $[g(X,\rho)]^2 \le g(X,X) = |X|^2$. If $r > 0$, then $r|X|^2 \ge r[g(X,\rho)]^2$. Thus from (2.15) we have

$$S(X,X) \le r|X|^2. \tag{2.16}$$

3. $A(PRS)_n(n > 3)$ with cyclic parallel Ricci tensor

In (1.4) if we replace Y, Z by X we get

$$(\nabla_X S)(X,X) = [A(X) + B(X)]S(X,X) + A(X)S(X,X) + A(X)S(X,X).$$

or

$$(\nabla_X S)(X,X) = [3A(X) + B(X)]S(X,X). \tag{3.1}$$

Now if the Ricci tensor is non-zero, then from (3.1) it follows that $(\nabla_X S)(X,X)=0$ if and only if $3A(X) + B(X) = 0$. Hence we have the following theorem:

Theorem 3.1. *In an $A(PRS)_n$ the Ricci curvature $S(X,X)$ is covariantly constant in the direction of X if and only if $3A(X) + B(X) = 0$.*

Again interchanging X, Y, Z in (1.4) and then summing them we get

$$\begin{aligned}(\nabla_X S)(Y,Z) &+ (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ &= F(X)S(Y,Z) + F(Y)S(X,Z) + F(Z)S(X,Y),\end{aligned}$$

$$F(X) = 3A(X) + B(X) \tag{3.2}$$

Now if the Ricci tensor of the manifold is cyclic parallel, then from (3.2) and (1.6) we have

$$F(X)S(Y,Z) + F(Y)S(X,Z) + F(Z)S(X,Y) = 0. \tag{3.3}$$

Then by Walker's lemma we can see that either, $F = 0$ or, $S = 0$. But since $S \neq 0$, we have $T = 0$, which implies that

$$3A(X) + B(X) = 0. \tag{3.4}$$

Conversely, if $3A(X) + B(X) = 0$, then from (3.2) we obtain

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0,$$

which implies that the Ricci tensor is cyclic parallel. Thus we can state the following theorem:

Theorem 3.2. *An $A(PRS)_n(n > 2)$ admits cyclic parallel Ricci tensor if and only if the associated 1-forms A and B satisfy the relation* (3.4).

4. $A(PRS)_n$ admitting a parallel vector field

In this section we suppose that an $A(PRS)_n$ admits a parallel vector field V ([21], [22]) which is not orthogonal to the associated vector field P. Since in the defining condition of $A(PRS)_n$, the 1-form A is non-zero, therefore the metrically equivalent vector field P corresponding to the 1-form A does not vanish at least at a point $x \in A(PRS)_n$. Then

$$\nabla_X V = 0, \tag{4.1}$$

for all $X \in A(PRS)_n$. Applying Ricci identity to (4.1) we get

$$\tilde{R}(X, Y, Z, V) = 0. \tag{4.2}$$

Contracting Y and Z in (4.2) we get

$$S(X, V) = 0.. \tag{4.3}$$

Now from (4.1) and (4.3) it follows that

$$(\nabla_X S)(Y, V) = 0. \tag{4.4}$$

From (1.4) we get by (4.2), (4.3) and (4.4)

$$(\nabla_X S)(Y, V) = [A(X) + B(X)]S(Y, V) + A(Y)S(X, V) + A(V)S(Y, X),$$

or

$$A(V)S(X, Y) = 0. \tag{4.5}$$

Now since by assumption $A(V) \neq 0$, so from (4.5) we have $S(X, Y) = 0$ for all vector fields X, Y. Hence

$$C(X, Y, Z) = R(X, Y, Z),$$

where C is the Weyl conformal curvature tensor. Then $C = 0$ implies $R = 0$, that is, the manifold is flat, which is inadmissible by definition of $A(PRS)_n$. Thus we have the following theorem:

Theorem 4.1. *If an $A(PRS)_n$ admits a parallel vector field which is not orthogonal to the associated vector field P, then the manifold can not be conformally flat.*

5. Sufficient condition for an $A(PRS)_n$ to be a quasi Einstein manifold

In an $A(PRS)_n$ the Ricci tensor S satisfies

$$(\nabla_X S)(Y,Z) = [A(X)+B(X)]S(Y,Z)+B(Y)S(X,Z)+B(Z)S(X,Y), \quad (5.1)$$

In a Riemannian manifold a vector field P defined by $g(X,P) = A(X)$ for all vector field X is said to be a concircular vector field [23] if

$$(\nabla_X A)(Y) = \alpha g(X,Y) + \omega(X)A(Y), \quad (5.2)$$

where α is a non-zero scalar and ω is a closed 1-form. If P is a unit vector field, then the equation (5.2) can be written as

$$(\nabla_X A)(Y) = \alpha[g(X,Y) - A(X)A(Y)]. \quad (5.3)$$

We suppose that in an $A(PRS)_n$ the vector field P is a unit concircular vector field defined by (5.3) where α is a non-zero scalar. Applying Ricci identity to (5.3) we obtain

$$A(R(X,Y)Z) = \alpha^2[g(X,Z)A(Y) - g(Y,Z)A(X)]. \quad (5.4)$$

Putting $Y = Z = e_i$ in (5.4), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$A(LX) = (n-1)\alpha^2 A(X),$$

where L is the Ricci operator defined by $g(LX,Y) = S(X,Y)$, which implies that

$$S(X,P) = (n-1)\alpha^2 A(X). \quad (5.5)$$

Now,

$$(\nabla_Y S)(X,P) = \nabla_Y S(X,P) - S(\nabla_Y X, P). \quad (5.6)$$

Applying (5.5) and (5.3) in (5.6) we get

$$(\nabla_Y S)(X,P) = (n-1)\alpha^3[g(X,Y) - A(X)A(Y)] - S(X, \nabla_Y P). \quad (5.7)$$

Since $(\nabla_X g)(Y,P) = 0$ we have

$$(\nabla_Y A)(X) = \nabla_Y A(X) - A(\nabla_Y X)$$

or

$$(\nabla_Y A)(X) = \nabla_Y g(X, P) - g(\nabla_Y X, P)$$

or

$$(\nabla_Y A)(X) = g(X, \nabla_Y P). \tag{5.8}$$

Using (5.3) in (5.8) yields

$$\alpha[g(X, Y) - A(X)A(Y)] = g(X, \nabla_Y P),$$

that is,

$$g(\alpha Y, X) - g(\alpha A(Y)P, X) = g(X, \nabla_Y P),$$

which implies

$$\nabla_Y P = \alpha Y - \alpha A(Y)P = \alpha(Y - A(Y)P).$$

Therefore,

$$S(X, \nabla_Y P) = S(X, \alpha Y) - S(X, \alpha A(Y)P).$$

Hence

$$S(X, \nabla_Y P) = \alpha[S(X, Y) - A(Y)S(X, P)]. \tag{5.9}$$

Applying (5.9) in (5.7) we get

$$\begin{aligned}(\nabla_Y S)(X, P) = (n-1)\alpha^3[g(X, Y) - A(X)A(Y)] \\ - \alpha S(X, Y) + \alpha A(Y)S(X, P).\end{aligned} \tag{5.10}$$

Again using (5.5) in (5.10) we get

$$(\nabla_Y S)(X, P) = (n-1)\alpha^3 g(X, Y) - \alpha S(X, Y). \tag{5.11}$$

Putting $Z = P$ and using (5.5) and (5.11) in (5.1) we get

$$\begin{aligned}a(n-1)\alpha^3 g(X, Y) - \alpha S(X, Y) = [A(X) + B(X)](n-1)\alpha^2 A(Y) \\ + (n-1)\alpha^2 A(X)B(Y) + B(P)S(X, Y),\end{aligned}$$

which implies

$$\begin{aligned}[\alpha + B(P)]S(X, Y) = (n-1)\alpha^3 g(X, Y) \\ - [A(X) + B(X)](n-1)\alpha^2 A(Y) - (n-1)\alpha^2 A(X)B(Y).\end{aligned} \tag{5.12}$$

Putting $Y = P$ in (5.12) and using (5.5) we get

$$[\alpha + B(P)](n-1)\alpha^2 A(X) = (n-1)\alpha^3 A(X) \\ - [A(X) + B(X)](n-1)\alpha^2 - (n-1)\alpha^2 A(X)B(P). \quad (5.13)$$

From (5.13) it follows that

$$B(X) = [2B(P) + 1]A(X). \quad (5.14)$$

Let us suppose

$$\alpha + B(P) \neq 0. \quad (5.15)$$

Using (5.14) in (5.12) we have

$$S(X,Y) = \frac{(n-1)\alpha^3}{\alpha + B(P)} g(X,Y) - [A(X) - (2B(P)+1)A(X)] \left[\frac{(n-1)\alpha^2 A(Y)}{\alpha + B(P)}\right] \\ - \frac{(n-1)\alpha^2 A(X)}{\alpha + B(P)}[-(2B(P)+1)A(Y)]$$

or

$$S(X,Y) = \frac{(n-1)\alpha^3}{\alpha + B(P)} g(X,Y) + \frac{(n-1)\alpha^2}{\alpha + B(P)}[4B(P)+1]A(X)A(Y). \quad (5.16)$$

or

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \quad (5.17)$$

where $a = \frac{(n-1)\alpha^3}{\alpha+B(P)}$ and $b = \frac{(n-1)\alpha^2[4B(P)+1]}{\alpha+B(P)}$ as $\alpha \neq 0$ and $B \neq 0$. Thus we have the following:

Theorem 5.1. *If in an $A(PRS)_n$, the basic vector field P is a unit concircular vector field, then the manifold is a quasi Einstein manifold provided $\alpha + B(P) \neq 0$.*

6. Conformally flat $A(PRS)_n$

In this section we consider conformally flat $A(PRS)_n$. Using (2.7) in (2.8) we get

$$(\nabla_Z S)(X,Y) = dr(Z)T(X)T(Y) + b[(\nabla_Z T)(X)T(Y) + T(X)(\nabla_Z T)(Y)]. \quad (6.1)$$

Substituting (6.1) in (2.5) we get

$$\begin{aligned} &dr(X)T(Y)T(Z) + b[(\nabla_X T)(Y)T(Z) + T(Y)(\nabla_X T)(Z)] \\ &\quad - dr(Z)T(Y)T(X) - b[(\nabla_Z T)(Y)T(X) + T(Y)(\nabla_Z T)(X)] \\ &= \alpha[dr(X)g(Y,Z) - dr(Z)g(Y,X)], \end{aligned} \tag{6.2}$$

where $\alpha = \frac{1}{2(n-1)}$. Putting $X = Y = e_i$ in (6.2), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, $1 \le i \le n$, we get

$$\begin{aligned} &dr(e_i)T(e_i)T(Z) + b[(\nabla_{e_i} T)(e_i)T(Z) + T(e_i)(\nabla_{e_i} T)(Z)] \\ &\quad - dr(Z)T(e_i)T(e_i) - b[(\nabla_Z T)(e_i)T(e_i) + T(e_i)(\nabla_Z T)(e_i)] \\ &= \alpha[dr(e_i)g(e_i,Z) - dr(Z)g(e_i,e_i)]. \end{aligned}$$

or

$$\alpha(1-n)dr(Z) = dr(\rho)T(Z) + b(\nabla_\rho T)(Z) + bT(Z)(\delta T) - dr(Z), \tag{6.3}$$

where we put $\delta T = \Sigma_{i=1}^n (\nabla_{e_i} T)(e_i)$.

Again putting $X = Y = \rho$ in (6.2) we get

$$\begin{aligned} &dr(\rho)T(\rho)T(Z) + b[(\nabla_\rho T)(\rho)T(Z) + T(\rho)(\nabla_\rho T)(Z)] - dr(Z)T(\rho)T(\rho) \\ &\quad - b[(\nabla_Z T)(\rho)T(\rho) + T(\rho)(\nabla_Z T)(\rho)] = \alpha[dr(\rho)g(\rho,Z) - dr(Z)g(\rho,\rho)] \end{aligned}$$

or

$$b(\nabla_\rho T)(Z) = (\alpha - 1)[dr(\rho)T(Z) - dr(Z)]. \tag{6.4}$$

Substituting (6.4) in (6.3) we get

$$\alpha(1-n)dr(Z) = dr(\rho)T(Z) + (\alpha-1)[dr(\rho)T(Z) - dr(Z)] + bT(Z)(\delta T) - dr(Z)$$

or

$$[\alpha(1-n) + (\alpha-1) + 1]dr(Z) - [(\alpha-1)+1]dr(\rho)T(Z) - bT(Z)(\delta T) = 0$$

or

$$\alpha(n-2)dr(Z) + \alpha dr(\rho)T(Z) + bT(Z)(\delta T) = 0. \tag{6.5}$$

Now putting $Z = \rho$ in (6.5), it yields

$$\alpha(n-2)dr(\rho) + \alpha dr(\rho) + b(\delta T) = 0$$

or

$$b(\delta T) = -\alpha(n-1)dr(\rho). \tag{6.6}$$

Using (6.6) in (6.5) we get

$$\alpha(n-2)dr(Z) + \alpha dr(\rho)T(Z) + \{-\alpha(n-1)dr(\rho)\}T(Z) = 0$$

or

$$\alpha(n-2)dr(Z) = \alpha(n-2)dr(\rho)T(Z).$$

Since $\alpha \neq 0$ *and* $n > 3$ we have

$$dr(Z) = dr(\rho)T(Z). \tag{6.7}$$

Putting $Y = \rho$ in (6.2) we get

$$\begin{aligned} dr(X)T(Z) + b[(\nabla_X T)(\rho)T(Z) + (\nabla_X T)(Z)] - dr(Z)T(X) \\ - b[(\nabla_Z T)(\rho)T(X) + (\nabla_Z T)(X)] = \alpha[dr(X)T(Z) - dr(Z)T(X)] \end{aligned}$$

or

$$b[(\nabla_X T)Z - (\nabla_Z T)(X)] = (\alpha - 1)[dr(X)T(Z) - dr(Z)T(X)]. \tag{6.8}$$

Using (6.7) in (6.8) we get

$$b[(\nabla_X T)Z - (\nabla_Z T)X] = (\alpha - 1)[dr(\rho)T(X)T(Z) - dr(\rho)T(Z)T(X)]$$

or

$$b[(\nabla_X T)Z - (\nabla_Z T)X] = 0.$$

Since $b \neq 0$ we have

$$(\nabla_X T)Z - (\nabla_Z T)X = 0. \tag{6.9}$$

This means that the 1-form T defined by $g(X, \rho) = T(X)$ is closed, that is, $dT(X, Y) = 0$. Hence it follows that

$$g(\nabla_X \rho, Y) = g(\nabla_Y \rho, X), \tag{6.10}$$

for all X, Y.

Now putting $Y = \rho$ in (6.10), we get

$$g(\nabla_X \rho, \rho) = g(\nabla_\rho \rho, X). \tag{6.11}$$

Since $g(\nabla_X \rho, \rho) = 0$, from (6.11) it follows that $g(\nabla_\rho \rho, X) = 0$ for all X. Hence $\nabla_\rho \rho = 0$. This means that the integral curves of the vector field ρ are geodesic. Therefore we can state the following theorem:

Theorem 6.1. *In a conformally flat $A(PRS)_n$ with non-constant scalar curvature, the integral curves of the vector field ρ are geodesics, provided $a =$* constant.

7. Killing vector field in a compact, orientable $A(PRS)_n (n > 3)$ without boundary

In this section we consider a compact, orientable $A(PRS)_n(n > 3)$ manifold M without boundary and with Codazzi type Ricci tensor. It is known ([24], [25]) that in such a manifold M the following relation holds

$$\int_M [S(X,X) - |\nabla X|^2 - (\operatorname{div} X)^2]dv \le 0 \quad \text{for all } X. \tag{7.1}$$

If X is a Killing vector field, then $\operatorname{div} X = 0$ [25]. Hence (7.1) takes the form

$$\int_M [S(X,X) - |\nabla X|^2]dv = 0. \tag{7.2}$$

We have $r \neq 0$ in a $A(PRS)_n$. Hence either $r > 0$ or $r < 0$. First we suppose that $r > 0$. Then by (2.16) we get

$$r|X|^2 \ge S(X,X).$$

Therefore

$$r|X|^2 - |\nabla X|^2 \ge S(X,X) - |\nabla X|^2.$$

Consequently,

$$\int_M [r|X|^2 - |\nabla X|^2]dv \ge \int_M [S(X,X) - |\nabla X|^2]dv,$$

and by (7.2)

$$\int_M [r|X|^2 - |\nabla X|^2]dv \ge 0. \tag{7.3}$$

Next we suppose that $r < 0$. Then

$$\int_M [r|X|^2 - |\nabla X|^2]dv \le 0. \tag{7.4}$$

Hence from (7.3) and (7.4) it follows that $X = 0$. This leads to the following theorem:

Theorem 7.1. *In a compact, orientable $A(PRS)_n(n > 3)$ without boundary with Codazzi type Ricci tensor, there does not exist any non-zero Killing vector field.*

8. Harmonic vector fields in a compact, orientable $A(PRS)_n(n > 3)$ without boundary

A vector field V in a Riemannian manifold M is said to be harmonic if ([26], [25])

$$d\omega = 0 \quad \text{and} \quad \delta\omega = 0, \tag{8.1}$$

where $\omega(X) = g(X, V)$ for all X. It is known ([26], [25]) that in a compact, orientable Riemannian manifold M the following relation holds for any vector field X.

$$\int_M [S(X,X) - \frac{1}{2}|d\omega|^2 + |\nabla X|^2 - (\delta\omega)^2]dv = 0, \tag{8.2}$$

where dv denotes the volume element of M. Now considering a compact, orientable $A(PRS)_n(n > 3)$ manifold without boundary but with a Codazzi type Ricci tensor, it follows from (8.2) and (8.1) that for any harmonic vector field

$$\int_M [S(X,X) + |\nabla X|^2] = 0. \tag{8.3}$$

Using (2.15) in (8.3) we have

$$\int_M [r\{g(X,\rho)\}^2 + |\nabla X|^2]dv = 0. \tag{8.4}$$

Since $r > 0$, from (8.4) we get

$$g(X,\rho) = 0 \quad \text{and} \quad \nabla X = 0. \tag{8.5}$$

From the first part of (8.5) it follows that X is orthogonal to ρ, and from the second part it follows that the vector field X is parallel. Hence we have the following:

Theorem 8.1. *In a compact, orientable $A(PRS)_n(n > 3)$ without boundary and with a Codazzi type Ricci tensor, any harmonic vector field in the $A(PRS)_n$ is parallel and orthogonal to the vector field ρ.*

9. Example of a quasi Einstein $A(PRS)_n$

We show in case of a concrete $A(PRS)_4$ that the above found conditions are sufficient in order that the $A(PRS)_4$ be quasi Einstein.

Example 9.1. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian manifold endowed with the Riemannian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{9.1}$$

where $(i, j = 1, 2, 3, 4)$, $q = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant. Here the only non-vanishing components of the Christoffel symbols and the curvature tensors are respectively:

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{q}{1+2q}, \quad \Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{q}{1+2q},$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{q}{1+2q}, \quad R_{2332} = R_{2442} = R_{3443} = \frac{q^2}{1+2q}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors and their covariant derivatives are:

$$R_{11} = \frac{3q}{(1+2q)^2}, \quad R_{22} = R_{33} = R_{44} = \frac{q}{1+2q},$$

$$R_{11,1} = \frac{3q(1-2q)}{(1+2q)^3}, \quad R_{22,1} = R_{33,1} = R_{44,1} = \frac{q}{(1+2q)^2}.$$

It can be easily shown that the scalar curvature r of this $(\mathbb{R}^4, g)$ is $\frac{6q(1+q)}{(1+2q)^3}$, which is non-vanishing and non-constant.

Let us choose the associated 1-forms as follows:

$$A_i(x) = \begin{cases} -\dfrac{q}{1+2q} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{9.2}$$

$$B_i(x) = \begin{cases} \dfrac{1+q}{1+2q} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{9.3}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.4) reduces to the equations

$$R_{11,1} = [A_1 + B_1]R_{11} + A_1 R_{11} + A_1 R_{11}, \tag{9.4}$$

$$R_{22,1} = [A_1 + B_1]R_{22} + A_2 R_{12} + A_2 R_{12}, \tag{9.5}$$

$$R_{33,1} = [A_1 + B_1]R_{33} + A_3 R_{13} + A_3 R_{13}, \tag{9.6}$$

$$R_{44,1} = [A_1 + B_1]R_{44} + A_4R_{14} + A_4R_{14}, \tag{9.7}$$

since for the other cases (1.4) holds trivially. By (9.2) and (9.3) we get the following relation for the right hand side(R.H.S.) and the left hand side(L.H.S.) of (9.4)

$$\begin{aligned}\text{R.H.S. of (9.4)} &= [A_1 + B_1]R_{11} + A_1R_{11} + A_1R_{11} = 3A_1R_{11} + B_1R_{11}\\ &= 3\left(-\frac{q}{1+2q}\right)\left(\frac{3q}{(1+2q)^2}\right) + \left(\frac{1+q}{1+2q}\right)\frac{3q}{(1+2q)^2}\\ &= \frac{3q(1-2q)}{(1+2q)^2} = R_{11,1} = \text{L.H.S. of (9.4)}.\end{aligned}$$

By similar argument it can be shown that (9.5), (9.6), (9.7) are true. So, $\mathbb{R}^4$ is an $A(PRS)_n$ whose scalar curvature is non-zero and non-constant.

Now we shall show that the vector A_i is a unit concircular vector in $(\mathbb{R}^4, g)$ which satisfies the condition (5.15). To prove that A_i is a unit concircular vector field we have to prove that

$$A_{i,j} = \alpha[g_{i,j} - A_iA_j], \tag{9.8}$$

where ',' denotes the covariant differentiation with respect to the metric.

So (9.8) reduces to

$$A_{1,1} = \alpha[g_{11} - A_1A_1], \tag{9.9}$$

since for the other cases (9.8) holds trivially. Again $A_{1,1} = -\frac{q}{(1+2q)^2}$. So with the help of (9.2) and (9.3) after straightforward calculation we see that (9.9) is true. Hence A_i is a unit concircular vector field with $\alpha = -\frac{q}{(1+2q)^3-q^2}$.

Again, $\alpha + B(P) = \alpha - 2A_1B_1 - A_1 = \frac{(2q+18q^2+53q^3+68q^4+32q^5)}{(1+2q)^2\{(1+2q)^3-q^2\}} \neq 0$, hence (5.15) is satisfied. We shall now show that this $(\mathbb{R}^4, g)$ is a quasi Einstein manifold. Let us choose the scalar functions a and b (the associated scalars) and the 1-form as follows:

$$a = \frac{q}{(1+2q)^2}, \quad b = \frac{2q(1-q)}{(1+2q)^3}$$

$$E_i(x) = \begin{cases} \sqrt(1+2q) & \text{for } \ i = 1 \\ 0 & \text{otherwise,} \end{cases}.$$

at any point $x \in \mathbb{R}^4$. We can easily check that $(\mathbb{R}^4, g)$ is a quasi Einstein manifold, which justifies Theorem 5.1.

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UDAY CHAND DE
DEPARTMENT OF PURE MATHEMATICS
CALCUTTA UNIVERSITY
35, BALLYGUNGE CIRCULAR ROAD
KOL-700019, W. B.
INDIA

E-mail: uc_de@yahoo.com

PRAJJWAL PAL
CHAKDAHA CO-OPERATIVE
COLONY VIDYAYATAN(H.S)
P.O.- CHAKDAHA
DIST- NADIA, WEST BENGAL
INDIA

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