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A duality property of Delaunay faces for line arrangements in $\mathbb{H}^3$

By ANDREW PRZEWORSKI (Charleston)

Abstract. For an arrangement of lines in $\mathbb{H}^3$, each face of the Delaunay cells is determined by three lines. We prove that the face associated with three lines is the same as the face associated with the three pairwise common perpendiculars to those lines, except in some degenerate circumstances.

1. Introduction

In [Prz12], we introduced the notion of Delaunay cells associated with an arrangement of flats in hyperbolic space. Although that paper dealt with flats of arbitrary dimension in hyperbolic space of arbitrary dimension, in this paper, we restrict attention to one-dimensional flats (i.e. lines) in $\mathbb{H}^3$.

Definition 1.1. [Prz12] Given a line ℓ in $\mathbb{H}^3$, define the projection function $\pi : \mathbb{H}^3 \to \ell$ as $\pi(x)$ is the point on ℓ which is closest to x. We then define the open Delaunay cell associated with the distinct lines $\ell_1, \cdots, \ell_n$ (for $n \leq 4$) to be the set of points $x \in \mathbb{H}^3$ for which $\pi_1(x), \cdots, \pi_n(x)$ are in general position and x lies in the relative interior of their convex hull.

We will often allow some or all of the ℓ_i to be points on $\partial\mathbb{H}^3$ rather than lines. In this case, the projection function is a constant function, equal to that particular point on $\partial\mathbb{H}^3$. When points on $\partial\mathbb{H}^3$ are allowed, the requirement that the ℓ_i be distinct will be replaced with the requirement that none of the ℓ_i are contained in any other of the ℓ_i.

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In [Prz12], we required that the ℓ_i be disjoint lines rather than merely distinct, and points on $\partial\mathbb{H}^3$ weren't allowed. While the primary motivation of this paper is to study the Delaunay cells from [Prz12], the main result of this paper can be proved in some situations which weren't relevant in [Prz12].

Typically, one would expect the open Delaunay cell associated with n lines to be $(n-1)$-dimensional, although there are degenerate cases in which it's empty. Assuming the open Delaunay cell associated with the disjoint lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4 is three-dimensional, its boundary will be two-dimensional. One portion of the boundary will be the open Delaunay cell associated with three of the lines. We refer to such a portion of the boundary as a face.

Any point x in the face associated with ℓ_1, ℓ_2, and ℓ_3 satisfies the criterion that $\pi_1(x)$, $\pi_2(x)$, and $\pi_3(x)$ are in general position and x lies in the relative interior of their convex hull. This face can be extended to a surface, which we call the coplanar surface.

Definition 1.2. Let each of ℓ_1, ℓ_2, and ℓ_3 be a line in $\mathbb{H}^3$ or point on $\partial\mathbb{H}^3$, with the requirement that none of the ℓ_i is contained in any other of the ℓ_i. We define the coplanar surface of ℓ_1, ℓ_2, and ℓ_3 to be the set

$$\{x \in \mathbb{H}^3 \mid x, \pi_1(x), \pi_2(x), \text{ and } \pi_3(x) \text{ are coplanar}\}$$

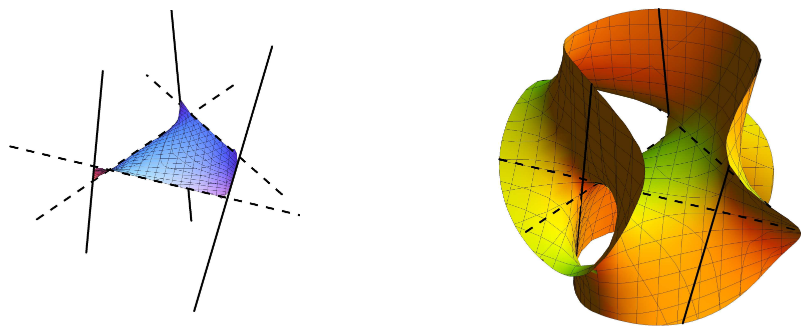

Figure 1. A face (left) and a coplanar surface (right). ℓ_1, ℓ_2, and ℓ_3 are solid lines. Pairwise common perpendiculars are dashed lines

One can easily see that ℓ_1, ℓ_2, and ℓ_3 all lie on the coplanar surface (or if they're points on $\partial\mathbb{H}^3$, the extension of the coplanar surface to $\overline{\mathbb{H}^3}$). Let $\ell_{ij}^{\perp}$ be the common perpendicular to ℓ_i and ℓ_j. We regard a point on $\partial\mathbb{H}^3$ as being perpendicular to any line that contains it. For example, if two distinct lines share a common endpoint, then the common endpoint is their common perpendicular. The common perpendicular to two distinct points on $\partial\mathbb{H}^3$ is the line which connects them. Then the restriction that none of the ℓ_i are contained in any of the other ℓ_i serves the purpose of guaranteeing that the pairwise common perpendiculars exist.

If they are lines, $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$ also lie on the coplanar surface (since if x is on $\ell_{ij}^{\perp}$, then x, $\pi_i(x)$, and $\pi_j(x)$ are collinear). In the case that any of the $\ell_{ij}^{\perp}$ is a point on $\partial\mathbb{H}^3$, it won't lie on the coplanar surface, but it will lie on the extension of the coplanar surface to $\overline{\mathbb{H}^3}$.

Except in degenerate cases, ℓ_1, ℓ_2, ℓ_3, $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{23}^{\perp}$ are six distinct lines which form a right-angled (nonplanar) hyperbolic hexagon, which happens to be the boundary of the face associated with the lines ℓ_1, ℓ_2, and ℓ_3. Degenerate cases include: some of the "lines" being points on $\partial\mathbb{H}^3$, the face being empty, some of the ℓ_i intersecting each other, or some of the $\ell_{ij}^{\perp}$ intersecting each other.

If we were given only the right-angled hyperbolic hexagon, it would be impossible to tell which of the three sides were the original three lines and which of the three sides were the pairwise common perpendiculars. In particular, the pairwise common perpendiculars to $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$ are (in some order) ℓ_1, ℓ_2, and ℓ_3. Then the coplanar surface of $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$ would contain the same six lines. Thus, it's natural to ask whether the coplanar surface of $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$ is the same as the coplanar surface of ℓ_1, ℓ_2, and ℓ_3.

Theorem 3.2. *Let each of ℓ_1, ℓ_2, and ℓ_3 be a line in $\mathbb{H}^3$ or a point on $\partial\mathbb{H}^3$, none of which contains any of the others. Then their coplanar surface is contained in the coplanar surface of their pairwise common perpendiculars. If none of the pairwise common perpendiculars contain any other of the pairwise common perpendiculars, then the two coplanar surfaces are identical.*

Theorem 3.6. *Let each of ℓ_1, ℓ_2, and ℓ_3 be a line in $\mathbb{H}^3$ or a point on $\partial\mathbb{H}^3$, none of which contains any of the others. Let $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$ be their pairwise common perpendiculars. If none of the $\ell_{ij}^{\perp}$ contain any other of the $\ell_{ij}^{\perp}$, then the face associated with ℓ_1, ℓ_2, and ℓ_3 is the same as the face associated with $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$.*

One could ask whether the same theorems hold for flats in higher dimensional hyperbolic space, but the general answer seems to be "no". In three-dimensional Euclidean geometry, pairs of lines don't always have a unique common perpendicular line. Avoiding these exceptional cases, it's very simple to prove that analogous theorems hold. Another natural question to ask is "since ℓ_1, ℓ_2, and ℓ_3 determine the same coplanar surface as $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$, are there any other triples of lines which determine that coplanar surface"? We expect to address this in a future paper.

The motivation is to study degenerate circumstances that can arise in Delaunay decompositions, such as multiple Delaunay faces having two-dimensional overlap. We also suspect that the Delaunay faces can provide the 2-handles of a Mom structure [GMM10], [GMM09], [Mil09], although this would likely require ruling out various exceptional circumstances.

In Section 2, we compute (up to a scalar multiple) the projection of the origin onto the common perpendicular to two lines. In Section 3, we prove the main results of the paper.

2. Projection onto the common perpendicular

Throughout the paper, we represent $\mathbb{H}^3$ in the Klein model. The Klein model is $D^3 = \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| < 1\}$. Lines and planes are accurately represented as lines and planes, but angles and distances are distorted. Rotation about the origin is a hyperbolic isometry. Other hyperbolic isometries are generally not as simple to describe.

A plane has a unit normal vector $\mathbf{n}$ (in a Euclidean sense, and chosen to point away from the origin). Then the equation of the plane can be written as $\mathbf{x} \cdot \mathbf{n} = C$ for some constant C. The pole of the plane is the point $\frac{\mathbf{n}}{C}$, which lies outside of D^3. A line is perpendicular to a plane (in a hyperbolic sense) if and only if it passes through the plane's pole. In the case that $C = 0$, the plane has no pole. In that case, the plane and line are perpendicular in a hyperbolic sense if and only if they are perpendicular in a Euclidean sense.

Definition 2.1. We will represent a line $\ell \in D^3$ as a vector $\mathbf{c} \in D^3$ and a unit vector $\mathbf{d} \in \mathbb{R}^3$. The vector $\mathbf{c}$ is the point on ℓ which is closest to $\mathbf{0}$, so is $\pi(\mathbf{0})$. The vector $\mathbf{d}$ is a direction vector for the line ℓ. Then $\mathbf{c} \cdot \mathbf{d} = 0$ and $\mathbf{d} \cdot \mathbf{d} = 1$.

We will also represent a point $\mathbf{c} \in \partial D^3$ as the vector $\mathbf{c}$ and a unit vector $\mathbf{d}$. We choose the unit vector $\mathbf{d}$ so $\mathbf{c} \cdot \mathbf{d} = 0$, but otherwise the choice of $\mathbf{d}$ is arbitrary. We include $\mathbf{d}$ merely for notational consistency between lines in D^3 and points on ∂D^3.

As a plane has a pole, so does a line. The pole of a line is another line, passing through the point $\frac{\mathbf{c}}{|\mathbf{c}|^2}$ with direction $\mathbf{c} \times \mathbf{d}$. A plane is perpendicular to a line if and only if the plane contains the line's pole. If $\mathbf{c} = \mathbf{0}$, then the line has no pole. In this case, a line and plane are perpendicular in a hyperbolic sense if and only if they are perpendicular in a Euclidean sense.

Proposition 2.2. *Let each of ℓ_1 and ℓ_2 be a line in D^3 or a point on ∂D^3 (described by vectors $\mathbf{c}_i$ and $\mathbf{d}_i$ as in Definition 2.1). If ℓ_1 and ℓ_2 have a common perpendicular line $\ell_{12}^{\perp}$ which lies in the x-y plane and is parallel (in a Euclidean sense) to the y-axis, then we may write $\mathbf{d}_i$ and $\mathbf{c}_i$ as*

$$\mathbf{d}_i = \frac{1}{\sqrt{1-(1-t_i^2)\cos^2\alpha\cos^2\beta_i}}(\sin\alpha\cos\beta_i, -t_i\cos\alpha\cos\beta_i, \sin\beta_i)$$

$$\mathbf{c}_i = \mathbf{p}_i - (\mathbf{p}_i\cdot\mathbf{d}_i)\mathbf{d}_i$$

where

$$\mathbf{p}_i = (\cos\alpha, t_i\sin\alpha, 0)$$

for $i \in \{1,2\}$, and some $|t_i| \leq 1$, $0 < \alpha < \pi$, and $0 \leq \beta_i < \pi$. Assuming $\ell_1 \neq \ell_2$, the ordered pairs (t_1, β_1) and (t_2, β_2) are not equal.

Proof. If the common perpendicular $\ell_{12}^{\perp}$ to ℓ_1 and ℓ_2 lies in the x-y plane and is parallel to the y-axis, then it crosses the x-axis somewhere within D^3. Let that point be $(\cos\alpha, 0, 0)$ with $0 < \alpha < \pi$. Then $\ell_{12}^{\perp}$ is the line $x = \cos\alpha$, $z = 0$. Each of ℓ_1 and ℓ_2 will intersect $\ell_{12}^{\perp}$ at a point $\mathbf{p}_i$ with y-coordinate between $-\sin\alpha$ and $\sin\alpha$. Let the y-coordinate of $\mathbf{p}_i$ be $t_i\sin\alpha$ for $i \in \{1,2\}$ and $|t_i| \leq 1$. Then ℓ_i lies in a plane which is (hyperbolically) perpendicular to $\ell_{12}^{\perp}$ at the point $\mathbf{p}_i$. Any plane perpendicular to $\ell_{12}^{\perp}$ must contain the pole of the line $\ell_{12}^{\perp}$, the line $x = \sec\alpha$, $y = 0$ (unless $\alpha = \frac{\pi}{2}$). The equation of such a plane is $(\sec\alpha - \cos\alpha)y = -(x - \sec\alpha)t_i\sin\alpha$, which simplifies to $y\sin\alpha = t_i(1 - x\cos\alpha)$. Note that this final form for the plane is correct even if $\alpha = \frac{\pi}{2}$, since in that case $\ell_{12}^{\perp}$ is the y-axis and the plane must be perpendicular to $\ell_{12}^{\perp}$ in even a Euclidean sense.

The direction vector $\mathbf{d}_i$ must be orthogonal to the normal vector of the plane. The normal vector to the plane is $(t_i\cos\alpha, \sin\alpha, 0)$ so $\mathbf{d}_i$ is a scalar multiple of $(\sin\alpha\cos\beta_i, -t_i\cos\alpha\cos\beta_i, \sin\beta_i)$ for some $\beta_i \in [0, 2\pi)$. Since the sign of $\mathbf{d}_i$ is irrelevant, we may choose $\beta_i \in [0, \pi)$. Scaling so $\mathbf{d}_i$ is a unit vector, we see that

$$\mathbf{d}_i = \frac{1}{\sqrt{1-(1-t_i^2)\cos^2\alpha\cos^2\beta_i}}(\sin\alpha\cos\beta_i, -t_i\cos\alpha\cos\beta_i, \sin\beta_i)$$

Line ℓ_i passes through the point $\mathbf{p}_i$ with direction $\mathbf{d}_i$. On ℓ_i, the closest point to the origin is $\mathbf{c}_i = \mathbf{p}_i - (\mathbf{p}_i\cdot\mathbf{d}_i)\mathbf{d}_i$.

If the ordered pairs (t_1, β_1) and (t_2, β_2) are equal, then ℓ_1 and ℓ_2 both pass through the same point $\mathbf{p}_1 = \mathbf{p}_2$ and have the same direction vectors, so aren't distinct. □

Remark 1. By rotating D^3 about the origin, most pairs of distinct lines can be transformed into lines of the form in the proposition. However, if ℓ_1 and ℓ_2

share an endpoint on ∂D^3, then they lack a common perpendicular line. The proposition does not produce any such pairs of lines, even if $\alpha = 0$.

Lemma 2.3. *With $\mathbf{d}_i$ and $\mathbf{p}_i$ as in the previous proposition, and i and j in $\{1,2\}$,*

(1) $\mathbf{p}_j \cdot \mathbf{d}_i = \frac{(1-t_it_j)\cos\alpha\sin\alpha\cos\beta_i}{\sqrt{1-(1-t_i^2)\cos^2\alpha\cos^2\beta_i}}$

(2) $1 - |\mathbf{p}_i|^2 = (1-t_i^2)\sin^2\alpha$

(3) *The y-coordinate of $(\mathbf{p}_i \cdot \mathbf{d}_i)\mathbf{p}_i + (1-|\mathbf{p}_i|^2)\mathbf{d}_i$ is zero.*

(4) $\mathbf{p}_j \cdot (\mathbf{p}_i - \mathbf{p}_j) = (t_it_j - t_j^2)\sin^2\alpha$

(5) $\mathbf{d}_j \cdot (\mathbf{p}_j - \mathbf{p}_i) = \frac{(t_it_j-t_j^2)\cos\alpha\sin\alpha\cos\beta_j}{\sqrt{1-(1-t_j^2)\cos^2\alpha\cos^2\beta_j}}$

Proof. Each of these claims can be verified through a short computation. □

Definition 2.4. If each of ℓ_1 and ℓ_2 is a line in D^3 or a point on ∂D^3 (represented as vectors $\mathbf{c}_i$ and $\mathbf{d}_i$), define

$$\mathbf{a} = (|\mathbf{c}_2|^2 - \mathbf{c}_1\cdot\mathbf{c}_2)\mathbf{c}_1 + (|\mathbf{c}_1|^2 - \mathbf{c}_1\cdot\mathbf{c}_2)\mathbf{c}_2 + (1-|\mathbf{c}_1|^2)(\mathbf{c}_2\cdot\mathbf{d}_1)\mathbf{d}_1 + (1-|\mathbf{c}_2|^2)(\mathbf{c}_1\cdot\mathbf{d}_2)\mathbf{d}_2$$

We will prove that the projection of $\mathbf{0}$ onto $\ell_{12}^{\perp}$ is a positive scalar multiple of $\mathbf{a}$. First, we need some technical results.

Proposition 2.5. *With $\mathbf{c}_i$ and $\mathbf{d}_i$ as in Proposition 2.2, the y-coordinate of $\mathbf{a}$ is 0.*

Proof. Rather than deal with the entire expression for $\mathbf{a}$, we start by reorganizing the terms in $(|\mathbf{c}_2|^2 - \mathbf{c}_1\cdot\mathbf{c}_2)\mathbf{c}_1 + (1-|\mathbf{c}_1|^2)(\mathbf{c}_2\cdot\mathbf{d}_1)\mathbf{d}_1$.

$$\begin{aligned}
&(|\mathbf{c}_2|^2 - \mathbf{c}_1\cdot\mathbf{c}_2)\mathbf{c}_1 + (1-|\mathbf{c}_1|^2)(\mathbf{c}_2\cdot\mathbf{d}_1)\mathbf{d}_1\\
&\quad= (\mathbf{c}_2\cdot(\mathbf{c}_2-\mathbf{c}_1))\mathbf{c}_1 + (1-|\mathbf{c}_1|^2)(\mathbf{c}_2\cdot\mathbf{d}_1)\mathbf{d}_1\\
&\quad= (\mathbf{c}_2\cdot(\mathbf{p}_2-\mathbf{p}_1+(\mathbf{p}_1\cdot\mathbf{d}_1)\mathbf{d}_1-(\mathbf{p}_2\cdot\mathbf{d}_2)\mathbf{d}_2))\,\mathbf{c}_1 + (1-|\mathbf{c}_1|^2)(\mathbf{c}_2\cdot\mathbf{d}_1)\mathbf{d}_1\\
&\quad= (\mathbf{c}_2\cdot(\mathbf{p}_2-\mathbf{p}_1))\mathbf{c}_1 + (\mathbf{c}_2\cdot((\mathbf{p}_1\cdot\mathbf{d}_1)\mathbf{d}_1-(\mathbf{p}_2\cdot\mathbf{d}_2)\mathbf{d}_2))\mathbf{c}_1\\
&\qquad+ (1-|\mathbf{c}_1|^2)(\mathbf{c}_2\cdot\mathbf{d}_1)\mathbf{d}_1\\
&\quad= (\mathbf{c}_2\cdot(\mathbf{p}_2-\mathbf{p}_1))\mathbf{c}_1 + (\mathbf{c}_2\cdot((\mathbf{p}_1\cdot\mathbf{d}_1)\mathbf{d}_1))\mathbf{c}_1 + (1-|\mathbf{c}_1|^2)(\mathbf{c}_2\cdot\mathbf{d}_1)\mathbf{d}_1\\
&\quad= (\mathbf{c}_2\cdot(\mathbf{p}_2-\mathbf{p}_1))\mathbf{c}_1 + (\mathbf{c}_2\cdot\mathbf{d}_1)\big((\mathbf{p}_1\cdot\mathbf{d}_1)\mathbf{c}_1 + (1-|\mathbf{c}_1|^2)\mathbf{d}_1\big)\\
&\quad= (\mathbf{c}_2\cdot(\mathbf{p}_2-\mathbf{p}_1))\mathbf{c}_1\\
&\qquad+ (\mathbf{c}_2\cdot\mathbf{d}_1)\big((\mathbf{p}_1\cdot\mathbf{d}_1)(\mathbf{p}_1-(\mathbf{p}_1\cdot\mathbf{d}_1)\mathbf{d}_1) + (1-|\mathbf{p}_1|^2+(\mathbf{p}_1\cdot\mathbf{d}_1)^2)\mathbf{d}_1\big)\\
&\quad= (\mathbf{c}_2\cdot(\mathbf{p}_2-\mathbf{p}_1))\mathbf{c}_1 + (\mathbf{c}_2\cdot\mathbf{d}_1)\big((\mathbf{p}_1\cdot\mathbf{d}_1)\mathbf{p}_1 + (1-|\mathbf{p}_1|^2)\mathbf{d}_1\big)
\end{aligned}$$

By Lemma 2.3, the y-coordinate of $(\mathbf{p}_1 \cdot \mathbf{d}_1)\mathbf{p}_1 + (1 - |\mathbf{p}_1|^2)\mathbf{d}_1$ is zero, so the y-coordinate of $(|\mathbf{c}_2|^2 - \mathbf{c}_1 \cdot \mathbf{c}_2)\mathbf{c}_1 + (1 - |\mathbf{c}_1|^2)(\mathbf{c}_2 \cdot \mathbf{d}_1)\mathbf{d}_1$ is the same as the y-coordinate of $(\mathbf{c}_2 \cdot (\mathbf{p}_2 - \mathbf{p}_1))\mathbf{c}_1$. Similarly, the y-coordinate of $(|\mathbf{c}_1|^2 - \mathbf{c}_1 \cdot \mathbf{c}_2)\mathbf{c}_2 + (1 - |\mathbf{c}_2|^2)(\mathbf{c}_1 \cdot \mathbf{d}_2)\mathbf{d}_2$ is the same as the y-coordinate of $(\mathbf{c}_1 \cdot (\mathbf{p}_1 - \mathbf{p}_2))\mathbf{c}_2$. Thus, the y-coordinate of $\mathbf{a}$ is the same as the y-coordinate of $(\mathbf{c}_2 \cdot (\mathbf{p}_2 - \mathbf{p}_1))\mathbf{c}_1 + (\mathbf{c}_1 \cdot (\mathbf{p}_1 - \mathbf{p}_2))\mathbf{c}_2$.

When nonzero, $\mathbf{p}_2 - \mathbf{p}_1$ points in the y-direction, so we can verify that the y-coordinate of $\mathbf{a}$ is 0 by computing $\mathbf{a} \cdot (\mathbf{p}_2 - \mathbf{p}_1) = (\mathbf{c}_2 \cdot (\mathbf{p}_2 - \mathbf{p}_1))(\mathbf{c}_1 \cdot (\mathbf{p}_2 - \mathbf{p}_1)) + (\mathbf{c}_1 \cdot (\mathbf{p}_1 - \mathbf{p}_2))(\mathbf{c}_2 \cdot (\mathbf{p}_2 - \mathbf{p}_1)) = 0$. If $\mathbf{p}_2 - \mathbf{p}_1 = \mathbf{0}$, then $t_1 = t_2$. By a continuity argument, since the y-coordinate of $\mathbf{a}$ is 0 when $t_1 \neq t_2$, it's also 0 when $t_1 = t_2$. □

Proposition 2.6. *With* $\mathbf{c}_i$ *and* $\mathbf{d}_i$ *as in Proposition 2.2, the* z*-coordinate of* $\mathbf{a}$ *is* 0.

Proof. First, we compute the z-coordinate of $(|\mathbf{c}_2|^2 - \mathbf{c}_1 \cdot \mathbf{c}_2)\mathbf{c}_1 + (1 - |\mathbf{c}_1|^2)(\mathbf{c}_2 \cdot \mathbf{d}_1)\mathbf{d}_1$. Since $\mathbf{c}_1 = \mathbf{p}_1 - (\mathbf{p}_1 \cdot \mathbf{d}_1)\mathbf{d}_1$, and $\mathbf{p}_1$ lies in the x-y plane, the z-coordinate of $(|\mathbf{c}_2|^2 - \mathbf{c}_1 \cdot \mathbf{c}_2)\mathbf{c}_1 + (1 - |\mathbf{c}_1|^2)(\mathbf{c}_2 \cdot \mathbf{d}_1)\mathbf{d}_1$ is the same as the z-coordinate of

$$\begin{aligned}
&\mathbf{d}_1 \big(-(|\mathbf{c}_2|^2 - \mathbf{c}_1 \cdot \mathbf{c}_2)(\mathbf{p}_1 \cdot \mathbf{d}_1) + (1 - |\mathbf{c}_1|^2)(\mathbf{c}_2 \cdot \mathbf{d}_1)\big)\\
&\quad = \mathbf{d}_1 \big(-(|\mathbf{c}_2|^2 - (\mathbf{p}_1 \cdot \mathbf{c}_2) + (\mathbf{p}_1 \cdot \mathbf{d}_1)(\mathbf{d}_1 \cdot \mathbf{c}_2))(\mathbf{p}_1 \cdot \mathbf{d}_1)\\
&\qquad +(1 - |\mathbf{p}_1|^2 + (\mathbf{p}_1 \cdot \mathbf{d}_1)^2)(\mathbf{c}_2 \cdot \mathbf{d}_1)\big)\\
&\quad = \mathbf{d}_1 \big((-|\mathbf{c}_2|^2 + (\mathbf{p}_1 \cdot \mathbf{c}_2))(\mathbf{p}_1 \cdot \mathbf{d}_1) + (1 - |\mathbf{p}_1|^2)(\mathbf{c}_2 \cdot \mathbf{d}_1)\big)\\
&\quad = \mathbf{d}_1 \big((-|\mathbf{p}_2|^2 + (\mathbf{p}_2 \cdot \mathbf{d}_2)^2 + (\mathbf{p}_1 \cdot \mathbf{p}_2) - (\mathbf{p}_2 \cdot \mathbf{d}_2)(\mathbf{p}_1 \cdot \mathbf{d}_2))(\mathbf{p}_1 \cdot \mathbf{d}_1)\\
&\qquad +(1 - |\mathbf{p}_1|^2)((\mathbf{p}_2 \cdot \mathbf{d}_1) - (\mathbf{p}_2 \cdot \mathbf{d}_2)(\mathbf{d}_1 \cdot \mathbf{d}_2))\big)\\
&\quad = \mathbf{d}_1 \big((\mathbf{p}_2 \cdot (\mathbf{p}_1 - \mathbf{p}_2))(\mathbf{p}_1 \cdot \mathbf{d}_1) + (1 - |\mathbf{p}_1|^2)(\mathbf{p}_2 \cdot \mathbf{d}_1)\\
&\qquad +(\mathbf{p}_2 \cdot \mathbf{d}_2)(\mathbf{p}_1 \cdot \mathbf{d}_1)((\mathbf{p}_2 - \mathbf{p}_1) \cdot \mathbf{d}_2) - (1 - |\mathbf{p}_1|^2)(\mathbf{p}_2 \cdot \mathbf{d}_2)(\mathbf{d}_1 \cdot \mathbf{d}_2)\big)
\end{aligned}$$

Now, we compute the z-coordinate of this, using Lemma 2.3. The z-coordinate is

$$\begin{aligned}
&\frac{\sin\beta_1}{\sqrt{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}}\Bigg(\frac{(t_1t_2 - t_2^2)\sin^2\alpha(1-t_1^2)\cos\alpha\sin\alpha\cos\beta_1}{\sqrt{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}}\\
&\quad + \frac{(1-t_1^2)\sin^2\alpha(1-t_1t_2)\cos\alpha\sin\alpha\cos\beta_1}{\sqrt{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}}\\
&\quad + \frac{(1-t_2^2)\cos\alpha\sin\alpha\cos\beta_2(\mathbf{p}_1 \cdot \mathbf{d}_1)(t_1t_2 - t_2^2)\cos\alpha\sin\alpha\cos\beta_2}{(1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2)}\\
&\quad - \frac{(1-t_1^2)\sin^2\alpha(1-t_2^2)\cos\alpha\sin\alpha\cos\beta_2(\mathbf{d}_1 \cdot \mathbf{d}_2)}{\sqrt{1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2}}\Bigg)
\end{aligned}$$

Factoring out common terms, this becomes

$$\begin{aligned}
&= \frac{\sin\beta_1\cos\alpha\sin^2\alpha}{\sqrt{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}}\Bigg(\frac{(1-t_1^2)\sin\alpha\cos\beta_1((t_1t_2-t_2^2)+(1-t_1t_2))}{\sqrt{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}}\\
&\quad + \frac{(1-t_2^2)\cos\beta_2(\mathbf{p}_1\cdot\mathbf{d}_1)(t_1t_2-t_2^2)\cos\alpha\cos\beta_2}{(1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2)}\\
&\quad - \frac{(1-t_1^2)(1-t_2^2)\sin\alpha\cos\beta_2(\mathbf{d}_1\cdot\mathbf{d}_2)}{\sqrt{1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2}}\Bigg)\\
&= \frac{(1-t_2^2)\sin\beta_1\cos\alpha\sin^2\alpha}{\sqrt{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}}\Bigg(\frac{(1-t_1^2)\sin\alpha\cos\beta_1}{\sqrt{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}}\\
&\quad + \frac{\cos\beta_2(\mathbf{p}_1\cdot\mathbf{d}_1)(t_1t_2-t_2^2)\cos\alpha\cos\beta_2}{(1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2)} - \frac{(1-t_1^2)\sin\alpha\cos\beta_2(\mathbf{d}_1\cdot\mathbf{d}_2)}{\sqrt{1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2}}\Bigg)
\end{aligned}$$

Continuing to substitute expressions from Lemma 2.3 and simplify gives

$$\begin{aligned}
&= \frac{(1-t_2^2)\sin\beta_1\cos\alpha\sin^2\alpha}{\sqrt{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}}\Bigg(\frac{(1-t_1^2)\sin\alpha\cos\beta_1}{\sqrt{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}}\\
&+ \frac{\cos\beta_2(1-t_1^2)\cos\alpha\sin\alpha\cos\beta_1(t_1t_2-t_2^2)\cos\alpha\cos\beta_2}{(1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2)\sqrt{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}}\\
&- \frac{(1-t_1^2)\sin\alpha\cos\beta_2((\sin^2\alpha+t_1t_2\cos^2\alpha)\cos\beta_1\cos\beta_2+\sin\beta_1\sin\beta_2)}{(1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2)\sqrt{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}}\Bigg)\\
&= \frac{(1-t_1^2)(1-t_2^2)\sin\beta_1\cos\alpha\sin^3\alpha}{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}\Bigg(\cos\beta_1 + \frac{\cos^2\beta_2\cos^2\alpha\cos\beta_1(t_1t_2-t_2^2)}{(1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2)}\\
&- \frac{\cos\beta_2((\sin^2\alpha+t_1t_2\cos^2\alpha)\cos\beta_1\cos\beta_2+\sin\beta_1\sin\beta_2)}{1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2}\Bigg)\\
&= \frac{(1-t_1^2)(1-t_2^2)\sin\beta_1\cos\alpha\sin^3\alpha}{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}\Bigg(\cos\beta_1 - \frac{\cos^2\beta_2\cos^2\alpha\cos\beta_1t_2^2}{1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2}\\
&- \frac{\cos\beta_2(\sin^2\alpha\cos\beta_1\cos\beta_2+\sin\beta_1\sin\beta_2)}{1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2}\Bigg)\\
&= \frac{(1-t_1^2)(1-t_2^2)\sin\beta_1\cos\alpha\sin^3\alpha}{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}\Bigg(\frac{\cos\beta_1-\cos\beta_1\cos^2\alpha\cos^2\beta_2}{1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2}\\
&- \frac{\sin^2\alpha\cos\beta_1\cos^2\beta_2+\cos\beta_2\sin\beta_1\sin\beta_2}{1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2}\Bigg)\\
&= \frac{(1-t_1^2)(1-t_2^2)\sin\beta_1\cos\alpha\sin^3\alpha}{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}\Bigg(\frac{\cos\beta_1-\cos\beta_1\cos^2\beta_2-\cos\beta_2\sin\beta_1\sin\beta_2}{1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2}\Bigg)\\
&= \frac{(1-t_1^2)(1-t_2^2)\sin\beta_1\cos\alpha\sin^3\alpha}{1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1}\Bigg(\frac{\cos\beta_1\sin^2\beta_2-\cos\beta_2\sin\beta_1\sin\beta_2}{1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2}\Bigg)
\end{aligned}$$

$$= \frac{(1-t_1^2)(1-t_2^2)\sin\beta_1\sin\beta_2\cos\alpha\sin^3\alpha\sin(\beta_2-\beta_1)}{(1-(1-t_1^2)\cos^2\alpha\cos^2\beta_1)(1-(1-t_2^2)\cos^2\alpha\cos^2\beta_2)}$$

That was the z-coordinate of $(|\mathbf{c}_2|^2-\mathbf{c}_1\cdot\mathbf{c}_2)\mathbf{c}_1+(1-|\mathbf{c}_1|^2)(\mathbf{c}_2\cdot\mathbf{d}_1)\mathbf{d}_1$. Similarly, we can compute the z-coordinate of $(|\mathbf{c}_1|^2-\mathbf{c}_1\cdot\mathbf{c}_2)\mathbf{c}_2+(1-|\mathbf{c}_2|^2)(\mathbf{c}_1\cdot\mathbf{d}_2)\mathbf{d}_2$, which will be the same, except the $\sin(\beta_2-\beta_1)$ will become $\sin(\beta_1-\beta_2)$. Since sin is an odd function, the total z-coordinate of $\mathbf{a}$ is 0. □

Proposition 2.7. *If each of ℓ_1 and ℓ_2 is a line in D^3 or a point on ∂D^3 and neither ℓ_1 nor ℓ_2 contains the other, then their common perpendicular passes through the origin if and only if $\mathbf{a}=\mathbf{0}$.*

Proof. If $\mathbf{a}=\mathbf{0}$, then $\mathbf{a}\cdot\mathbf{c}_1=\mathbf{a}\cdot\mathbf{c}_2=0$.

$$\begin{aligned}0=\mathbf{a}\cdot\mathbf{c}_1&=(|\mathbf{c}_2|^2-\mathbf{c}_1\cdot\mathbf{c}_2)|\mathbf{c}_1|^2+(|\mathbf{c}_1|^2-\mathbf{c}_1\cdot\mathbf{c}_2)(\mathbf{c}_1\cdot\mathbf{c}_2)+(1-|\mathbf{c}_2|^2)(\mathbf{c}_1\cdot\mathbf{d}_2)^2\\&=(|\mathbf{c}_1|^2|\mathbf{c}_2|^2-(\mathbf{c}_1\cdot\mathbf{c}_2)^2)+(1-|\mathbf{c}_2|^2)(\mathbf{c}_1\cdot\mathbf{d}_2)^2\end{aligned}$$

Since $(|\mathbf{c}_1|^2|\mathbf{c}_2|^2-(\mathbf{c}_1\cdot\mathbf{c}_2)^2)$ and $(1-|\mathbf{c}_2|^2)(\mathbf{c}_1\cdot\mathbf{d}_2)^2$ are both nonnegative, we have that $|\mathbf{c}_1|^2|\mathbf{c}_2|^2-(\mathbf{c}_1\cdot\mathbf{c}_2)^2=0$ and $(1-|\mathbf{c}_2|^2)(\mathbf{c}_1\cdot\mathbf{d}_2)=0$. Similarly, from $\mathbf{a}\cdot\mathbf{c}_2=0$, we see that $(1-|\mathbf{c}_1|^2)(\mathbf{c}_2\cdot\mathbf{d}_1)=0$.

If $\mathbf{c}_1=\mathbf{c}_2=\mathbf{0}$, then lines ℓ_1 and ℓ_2 intersect at the origin, so their common perpendicular passes through the origin. Thus, assume without loss of generality that $\mathbf{c}_1\neq\mathbf{0}$. From $|\mathbf{c}_1|^2|\mathbf{c}_2|^2-(\mathbf{c}_1\cdot\mathbf{c}_2)^2=0$, we see that $\mathrm{Span}(\mathbf{c}_1,\mathbf{c}_2)$ is one dimensional (i.e. a line passing through $\mathbf{0}$). This line is perpendicular to ℓ_1. If ℓ_2 is a point on ∂D^3, then ℓ_2 is an endpoint of $\mathrm{Span}(\mathbf{c}_1,\mathbf{c}_2)\cap D^3$ so is perpendicular to $\mathrm{Span}(\mathbf{c}_1,\mathbf{c}_2)$. If ℓ_2 is a line in D^3, then $1-|\mathbf{c}_2|^2\neq 0$, so $\mathbf{c}_1\cdot\mathbf{d}_2=0$. Then $\mathrm{Span}(\mathbf{c}_1,\mathbf{c}_2)$ is perpendicular to ℓ_2. Thus $\mathrm{Span}(\mathbf{c}_1,\mathbf{c}_2)$ is the common perpendicular to ℓ_1 and ℓ_2. This completes one direction of the proof.

Now assume that the common perpendicular to ℓ_1 and ℓ_2 passes through the origin. Then $\mathbf{c}_1$ and $\mathbf{c}_2$ are linearly dependent, so $\mathbf{c}_2\cdot\mathbf{d}_1=\mathbf{c}_1\cdot\mathbf{d}_2=0$. There is some vector $\mathbf{v}$ and scalars c_1 and c_2 such that $\mathbf{c}_i=c_i\mathbf{v}$.

$$\begin{aligned}\mathbf{a}&=(|\mathbf{c}_2|^2-\mathbf{c}_1\cdot\mathbf{c}_2)\mathbf{c}_1+(|\mathbf{c}_1|^2-\mathbf{c}_1\cdot\mathbf{c}_2)\mathbf{c}_2+\mathbf{0}+\mathbf{0}\\&=(c_2^2-c_1c_2)|\mathbf{v}|^2c_1\mathbf{v}+(c_1^2-c_1c_2)|\mathbf{v}|^2c_2\mathbf{v}=\mathbf{0}\end{aligned}$$ □

Proposition 2.8. *With $\mathbf{c}_i$ and $\mathbf{d}_i$ as in Proposition 2.2, the x-coordinate of $\mathbf{a}$ has the same sign as $\cos\alpha$.*

Proof. Proposition 2.7 proves the result in the case that $\alpha=\frac{\pi}{2}$, so assume $\alpha\neq\frac{\pi}{2}$.

The x-coordinate of $\mathbf{a}$ is a continuous function of the β_i and t_i. The common perpendicular to ℓ_1 and ℓ_2 doesn't pass through the origin. Then Proposition 2.7 verifies that the x-coordinate of $\mathbf{a}$ is not zero, unless ℓ_1 and ℓ_2 are identical, in which case either $(t_1, \beta_1) = (t_2, \beta_2)$ or $t_1 = t_2 = \pm 1$. Fixing α, we may continuously vary the t_i and β_i without changing the sign of the x-coordinate of $\mathbf{a}$, as long as we avoid the conditions $(t_1, \beta_1) = (t_2, \beta_2)$ or $t_1 = t_2 = \pm 1$.

Without loss of generality, we may then assume that $t_1 = 1$ and $t_2 = -1$. Then $\mathbf{c}_1 = (\cos\alpha, \sin\alpha, 0)$ and $\mathbf{c}_2 = (\cos\alpha, -\sin\alpha, 0)$. It is then easy to compute that $\mathbf{a} = 2(1 - \cos 2\alpha)(\cos\alpha, 0, 0)$. Since $0 < \alpha < \pi$, we have completed the proof. □

Proposition 2.9. *Given two distinct lines ℓ_1 and ℓ_2 in D^3 (represented as vectors $\mathbf{c}_i$ and $\mathbf{d}_i$), if they have a common endpoint, then it is a positive scalar multiple of* $\mathbf{a}$.

Proof. Let $\mathbf{p} \in \partial D^3$ be the common endpoint and let the direction vector of line ℓ_i be $\mathbf{d}_i$. The sign of $\mathbf{d}_i$ is irrelevant, so we are free to choose $\mathbf{d}_i$ so $\mathbf{p} \cdot \mathbf{d}_i > 0$. The closest point on ℓ_i to $\mathbf{0}$ is $\mathbf{c}_i = \mathbf{p} - (\mathbf{p} \cdot \mathbf{d}_i)\mathbf{d}_i$. With this, we compute that

$$|\mathbf{c}_2|^2 - (\mathbf{c}_1 \cdot \mathbf{c}_2) = \mathbf{c}_2 \cdot (\mathbf{c}_2 - \mathbf{c}_1) = \mathbf{c}_2 \cdot ((\mathbf{p} \cdot \mathbf{d}_1)\mathbf{d}_1 - (\mathbf{p} \cdot \mathbf{d}_2)\mathbf{d}_2) = (\mathbf{p} \cdot \mathbf{d}_1)(\mathbf{c}_2 \cdot \mathbf{d}_1)$$

Also, $1 - |\mathbf{c}_1|^2 = 1 - (|\mathbf{p}|^2 - (\mathbf{p} \cdot \mathbf{d}_1)^2) = (\mathbf{p} \cdot \mathbf{d}_1)^2$. Similarly, we can compute that $|\mathbf{c}_1|^2 - (\mathbf{c}_1 \cdot \mathbf{c}_2) = (\mathbf{p} \cdot \mathbf{d}_2)(\mathbf{c}_1 \cdot \mathbf{d}_2)$ and $1 - |\mathbf{c}_2|^2 = (\mathbf{p} \cdot \mathbf{d}_2)^2$. Then $\mathbf{a}$ is

$$\begin{aligned}
&(|\mathbf{c}_2|^2 - \mathbf{c}_1 \cdot \mathbf{c}_2)\mathbf{c}_1 + (|\mathbf{c}_1|^2 - \mathbf{c}_1 \cdot \mathbf{c}_2)\mathbf{c}_2 + (1 - |\mathbf{c}_1|^2)(\mathbf{c}_2 \cdot \mathbf{d}_1)\mathbf{d}_1 + (1 - |\mathbf{c}_2|^2)(\mathbf{c}_1 \cdot \mathbf{d}_2)\mathbf{d}_2 \\
&= (\mathbf{p} \cdot \mathbf{d}_1)(\mathbf{c}_2 \cdot \mathbf{d}_1)\mathbf{c}_1 + (\mathbf{p} \cdot \mathbf{d}_2)(\mathbf{c}_1 \cdot \mathbf{d}_2)\mathbf{c}_2 + (\mathbf{p} \cdot \mathbf{d}_1)^2(\mathbf{c}_2 \cdot \mathbf{d}_1)\mathbf{d}_1 + (\mathbf{p} \cdot \mathbf{d}_2)^2(\mathbf{c}_1 \cdot \mathbf{d}_2)\mathbf{d}_2 \\
&= (\mathbf{p} \cdot \mathbf{d}_1)(\mathbf{c}_2 \cdot \mathbf{d}_1)(\mathbf{c}_1 + (\mathbf{p} \cdot \mathbf{d}_1)\mathbf{d}_1) + (\mathbf{p} \cdot \mathbf{d}_2)(\mathbf{c}_1 \cdot \mathbf{d}_2)(\mathbf{c}_2 + (\mathbf{p} \cdot \mathbf{d}_2)\mathbf{d}_2) \\
&= (\mathbf{p} \cdot \mathbf{d}_1)(\mathbf{c}_2 \cdot \mathbf{d}_1)\mathbf{p} + (\mathbf{p} \cdot \mathbf{d}_2)(\mathbf{c}_1 \cdot \mathbf{d}_2)\mathbf{p}
\end{aligned}$$

We can compute that the scalar coefficient of $\mathbf{p}$ in the above expression is

$$\begin{aligned}
&(\mathbf{p} \cdot \mathbf{d}_1)(\mathbf{c}_2 \cdot \mathbf{d}_1) + (\mathbf{p} \cdot \mathbf{d}_2)(\mathbf{c}_1 \cdot \mathbf{d}_2) \\
&= (\mathbf{p} \cdot \mathbf{d}_1)((\mathbf{p} \cdot \mathbf{d}_1) - (\mathbf{p} \cdot \mathbf{d}_2)(\mathbf{d}_1 \cdot \mathbf{d}_2)) + (\mathbf{p} \cdot \mathbf{d}_2)((\mathbf{p} \cdot \mathbf{d}_2) - (\mathbf{p} \cdot \mathbf{d}_1)(\mathbf{d}_1 \cdot \mathbf{d}_2)) \\
&= (\mathbf{p} \cdot \mathbf{d}_1)^2 + (\mathbf{p} \cdot \mathbf{d}_2)^2 - 2(\mathbf{p} \cdot \mathbf{d}_1)(\mathbf{p} \cdot \mathbf{d}_2)(\mathbf{d}_1 \cdot \mathbf{d}_2) \\
&\geq (\mathbf{p} \cdot \mathbf{d}_1)^2 + (\mathbf{p} \cdot \mathbf{d}_2)^2 - 2(\mathbf{p} \cdot \mathbf{d}_1)(\mathbf{p} \cdot \mathbf{d}_2) = ((\mathbf{p} \cdot \mathbf{d}_1) - (\mathbf{p} \cdot \mathbf{d}_2))^2 \geq 0
\end{aligned}$$

By Proposition 2.7, $\mathbf{a} \neq \mathbf{0}$. Thus, $\mathbf{a}$ is a positive scalar multiple of $\mathbf{p}$, so $\mathbf{p}$ is also a positive scalar multiple of $\mathbf{a}$. □

Theorem 2.10. *Let each of ℓ_1 and ℓ_2 be a line in D^3 or a point on ∂D^3 (represented as vectors $\mathbf{c}_i$ and $\mathbf{d}_i$), neither of which contains the other. Let $\ell_{12}^{\perp}$ be their common perpendicular. The projection of $\mathbf{0}$ onto $\ell_{12}^{\perp}$ is a positive scalar multiple of the vector*

$$\mathbf{a} = (|\mathbf{c}_2|^2 - \mathbf{c}_1 \cdot \mathbf{c}_2)\mathbf{c}_1 + (|\mathbf{c}_1|^2 - \mathbf{c}_1 \cdot \mathbf{c}_2)\mathbf{c}_2 + (1 - |\mathbf{c}_1|^2)(\mathbf{c}_2 \cdot \mathbf{d}_1)\mathbf{d}_1 + (1 - |\mathbf{c}_2|^2)(\mathbf{c}_1 \cdot \mathbf{d}_2)\mathbf{d}_2$$

Proof. Proposition 2.9 proves the theorem in the case that $\ell_{12}^{\perp}$ is a point on ∂D^3. Thus, we may assume that $\ell_{12}^{\perp}$ is a line. Without loss of generality, we may rotate D^3 so $\ell_{12}^{\perp}$ lies in the x-y plane and is parallel to the y-axis. Propositions 2.5 and 2.6 prove that $\mathbf{a}$ lies along the x-axis. The projection of $\mathbf{0}$ onto $\ell_{12}^{\perp}$ is $(\cos\alpha, 0, 0)$. Proposition 2.8 then verifies that $\mathbf{a}$ and $(\cos\alpha, 0, 0)$ point in the same direction. □

Remark 2. It is worth noting that the expression in Theorem 2.10 is a polynomial expression in the entries of the vectors $\mathbf{c}_i$ and $\mathbf{d}_i$. In higher dimensions, it doesn't seem to be the case that the projection of $\mathbf{0}$ onto $\ell_{12}^{\perp}$ is a scalar multiple of such a simple expression. We speculate that is the reason why the main results of this paper can't be extended to higher dimensions.

3. Proof of duality

In this section, we prove the main results of the paper, that the coplanar surface of three distinct lines is (usually) the same as the coplanar surface of their pairwise common perpendiculars. A similar result applies to faces.

Proposition 3.1. *Let each of ℓ_1, ℓ_2, and ℓ_3 be a line in D^3 or a point on ∂D^3 (represented as vectors $\mathbf{c}_i$ and $\mathbf{d}_i$ as in Definition 2.1), none of which contains any of the others. If $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ are linearly dependent, then $\mathbf{a}_{12}$, $\mathbf{a}_{23}$, and $\mathbf{a}_{31}$ are also linearly dependent, where $\mathbf{a}_{ij}$ is computed using Definition 2.4 with lines ℓ_i and ℓ_j.*

In particular, if the three vectors $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ are linearly dependent, but no two of them are linearly dependent, then we can find nonzero numbers λ_i and μ_i such that $\sum_{i=1}^{3} \mu_i \mathbf{c}_i = \lambda_3 \mathbf{a}_{12} + \lambda_1 \mathbf{a}_{23} + \lambda_2 \mathbf{a}_{31} = \mathbf{0}$ and $\lambda_i \mu_i$ doesn't depend on i.

Proof. Without loss of generality, rotate D^3 about the origin so the vectors $\mathbf{c}_i$ all lie in the x-y plane. Then we may write the vectors $\mathbf{c}_i$ and $\mathbf{d}_i$ as $\mathbf{c}_i = (r_i \cos\theta_i, r_i \sin\theta_i, 0)$ and $\mathbf{d}_i = (-\sin\theta_i \sin\phi_i, \cos\theta_i \sin\phi_i, \cos\phi_i)$, where

$r_i \in [0,1]$, $\theta_i \in [0,2\pi)$, and $\phi_i \in [0,\pi)$. In what follows, we use cyclic indices mod 3.

The vector $(|\mathbf{c}_{i+1}|^2 - (\mathbf{c}_i \cdot \mathbf{c}_{i+1}))\mathbf{c}_i + (|\mathbf{c}_i|^2 - (\mathbf{c}_i \cdot \mathbf{c}_{i+1}))\mathbf{c}_{i+1}$ simplifies as

$$\begin{aligned}
(r_{i+1}^2 &- r_i r_{i+1}\cos(\theta_{i+1}-\theta_i))\mathbf{c}_i + (r_i^2 - r_i r_{i+1}\cos(\theta_{i+1}-\theta_i))\mathbf{c}_{i+1} \\
&= r_i r_{i+1}((r_{i+1} - r_i\cos(\theta_{i+1}-\theta_i))(\cos\theta_i, \sin\theta_i, 0) \\
&\quad + (r_i - r_{i+1}\cos(\theta_{i+1}-\theta_i))(\cos\theta_{i+1}, \sin\theta_{i+1}, 0)) \\
&= r_i r_{i+1}(r_{i+1}\sin(\theta_{i+1}-\theta_i)(\sin\theta_{i+1}, -\cos\theta_{i+1}, 0) \\
&\quad + r_i\sin(\theta_i - \theta_{i+1})(\sin\theta_i, -\cos\theta_i, 0)) \\
&= r_i r_{i+1}\sin(\theta_{i+1}-\theta_i)((\mathbf{c}_{i+1}-\mathbf{c}_i)\times\mathbf{k})
\end{aligned}$$

where $\mathbf{k} = (0,0,1)$. If we let $\lambda_i = r_i \sin(\theta_i - \theta_{i+2})\sin(\theta_{i+1}-\theta_i)$, then

$$\begin{aligned}
\sum_{i=1}^{3} &\lambda_{i+2}\left((|\mathbf{c}_{i+1}|^2 - (\mathbf{c}_i \cdot \mathbf{c}_{i+1}))\mathbf{c}_i + (|\mathbf{c}_i|^2 - (\mathbf{c}_i \cdot \mathbf{c}_{i+1}))\mathbf{c}_{i+1}\right) \\
&= \left(\prod_{j=1}^{3} r_j \sin(\theta_{j+1}-\theta_j)\right)\sum_{i=1}^{3}((\mathbf{c}_{i+1}-\mathbf{c}_i)\times\mathbf{k}) \\
&= \left(\prod_{j=1}^{3} r_j \sin(\theta_{j+1}-\theta_j)\right)\left(\left(\sum_{i=1}^{3}(\mathbf{c}_{i+1}-\mathbf{c}_i)\right)\times\mathbf{k}\right) = \mathbf{0}
\end{aligned}$$

Now we compute $\sum_{i=1}^{3}\lambda_{i+2}((1-|\mathbf{c}_i|^2)(\mathbf{c}_{i+1}\cdot\mathbf{d}_i)\mathbf{d}_i + (1-|\mathbf{c}_{i+1}|^2)(\mathbf{c}_i\cdot\mathbf{d}_{i+1})\mathbf{d}_{i+1})$

$$\begin{aligned}
\sum_{i=1}^{3} &\lambda_{i+2}\left((1-|\mathbf{c}_i|^2)(\mathbf{c}_{i+1}\cdot\mathbf{d}_i)\mathbf{d}_i + (1-|\mathbf{c}_{i+1}|^2)(\mathbf{c}_i\cdot\mathbf{d}_{i+1})\mathbf{d}_{i+1}\right) \\
&= \sum_{i=1}^{3}(1-|\mathbf{c}_i|^2)\left(\lambda_{i+2}(\mathbf{c}_{i+1}\cdot\mathbf{d}_i) + \lambda_{i+1}(\mathbf{c}_{i+2}\cdot\mathbf{d}_i)\right)\mathbf{d}_i \\
&= \sum_{i=1}^{3}(1-|\mathbf{c}_i|^2)\left(\lambda_{i+2}r_{i+1}\sin\phi_i\sin(\theta_{i+1}-\theta_i) + \lambda_{i+1}r_{i+2}\sin\phi_i\sin(\theta_{i+2}-\theta_i)\right)\mathbf{d}_i \\
&= \sum_{i=1}^{3}(1-|\mathbf{c}_i|^2)r_{i+1}r_{i+2}\sin\phi_i(\sin(\theta_{i+2}-\theta_{i+1})\sin(\theta_{i+3}-\theta_{i+2})\sin(\theta_{i+1}-\theta_i) \\
&\quad + \sin(\theta_{i+1}-\theta_{i+3})\sin(\theta_{i+2}-\theta_{i+1})\sin(\theta_{i+2}-\theta_i))\mathbf{d}_i \\
&= \sum_{i=1}^{3}\left((1-|\mathbf{c}_i|^2)r_{i+1}r_{i+2}\sin\phi_i\sin(\theta_{i+2}-\theta_{i+1})\sin(\theta_{i+1}-\theta_i)\right. \\
&\quad \left.\cdot(\sin(\theta_i-\theta_{i+2}) + \sin(\theta_{i+2}-\theta_i))\mathbf{d}_i\right) = \mathbf{0}
\end{aligned}$$

Then $\sum_{i=1}^{3} \lambda_{i+2}\mathbf{a}_{i,i+1}$ is

$$\sum_{i=1}^{3} \lambda_{i+2}\big(|\mathbf{c}_{i+1}|^2 - (\mathbf{c}_i \cdot \mathbf{c}_{i+1}))\mathbf{c}_i + (|\mathbf{c}_i|^2 - (\mathbf{c}_i \cdot \mathbf{c}_{i+1}))\mathbf{c}_{i+1} \\ + (1 - |\mathbf{c}_i|^2)(\mathbf{c}_{i+1} \cdot \mathbf{d}_i)\mathbf{d}_i + (1 - |\mathbf{c}_{i+1}|^2)(\mathbf{c}_i \cdot \mathbf{d}_{i+1})\mathbf{d}_{i+1}\big) = \mathbf{0}$$

As long as at least one of the λ_i is nonzero, we've proved that $\mathbf{a}_{12}$, $\mathbf{a}_{23}$, and $\mathbf{a}_{31}$ are linearly dependent. We still need to check the degenerate cases in which $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

If two or more of the r_i are 0, then without loss of generality we may assume that $r_1 = r_2 = 0$, so $\mathbf{c}_1 = \mathbf{c}_2 = \mathbf{0}$. Then $\mathbf{a}_{12} = \mathbf{0}$ so $\mathbf{a}_{12}$, $\mathbf{a}_{23}$, and $\mathbf{a}_{31}$ are linearly dependent.

If two or more of the r_i are nonzero, then without loss of generality we may assume that $r_1 \neq 0$ and $r_2 \neq 0$. Then from $\lambda_1 = \lambda_2 = 0$, we have that $\sin(\theta_1 - \theta_3)\sin(\theta_2 - \theta_1) = \sin(\theta_2 - \theta_1)\sin(\theta_3 - \theta_2) = 0$. This produces two possibilities: either $\sin(\theta_2 - \theta_1) = 0$ or $\sin(\theta_1 - \theta_3) = \sin(\theta_3 - \theta_2) = 0$. In the second case, we can still conclude that $\sin(\theta_2 - \theta_1) = 0$, so in either case, we have that $\sin(\theta_2 - \theta_1) = 0$. Then $\mathbf{c}_1$ and $\mathbf{c}_2$ are nonzero scalar multiples of each other. Then $(|\mathbf{c}_2|^2 - (\mathbf{c}_1 \cdot \mathbf{c}_2))\mathbf{c}_1 + (|\mathbf{c}_1|^2 - (\mathbf{c}_1 \cdot \mathbf{c}_2))\mathbf{c}_2 = \mathbf{0}$ and $\mathbf{c}_2 \cdot \mathbf{d}_1 = \mathbf{c}_1 \cdot \mathbf{d}_2 = 0$, so $\mathbf{a}_{12} = \mathbf{0}$. Then $\mathbf{a}_{12}$, $\mathbf{a}_{23}$, and $\mathbf{a}_{31}$ are linearly dependent.

Now we prove the remaining claim in the theorem. If no two of the $\mathbf{c}_i$ are linearly dependent, then let $\mu_i = \mathbf{c}_{i+1} \times \mathbf{c}_{i+2} \cdot \mathbf{k}$ where $\mathbf{k} = (0, 0, 1)$. Then $\sum_{i=1}^{3} \mu_i \mathbf{c}_i = \mathbf{0}$. Further $\mu_i = r_{i+1} r_{i+2} \sin(\theta_{i+2} - \theta_{i+1}) \neq 0$.

Note that

$$\begin{aligned} \lambda_i \mu_i &= r_i \sin(\theta_i - \theta_{i+2}) \sin(\theta_{i+1} - \theta_i) r_{i+1} r_{i+2} \sin(\theta_{i+2} - \theta_{i+1}) \\ &= \prod_{j=1}^{3} r_j \sin(\theta_{j+1} - \theta_j) \end{aligned}$$

so doesn't depend on i. If no two of the $\mathbf{c}_i$ are linearly dependent, then none of the $\mathbf{c}_i$ is $\mathbf{0}$ and no two of the θ_i differ by an integer multiple of π. Then $\lambda_i \mu_i \neq 0$, so the λ_i and μ_i are all nonzero. □

Theorem 3.2. *Let each of ℓ_1, ℓ_2, and ℓ_3 be a line in $\mathbb{H}^3$ or a point on $\partial\mathbb{H}^3$, none of which contains any of the others. Then their coplanar surface is contained in the coplanar surface of their pairwise common perpendiculars. If none of the pairwise common perpendiculars contain any other of the pairwise common perpendiculars, then the two coplanar surfaces are identical.*

Proof. We represent $\mathbb{H}^3$ as D^3 in the Klein model. Represent ℓ_i by vectors $\mathbf{c}_i$ and $\mathbf{d}_i$ as in Definition 2.1. Let p be any point on the coplanar surface of ℓ_1, ℓ_2, and ℓ_3. By performing a hyperbolic isometry, we may assume that p is the origin. Then $\pi_i(p) = \mathbf{c}_i$. Since p is on the coplanar surface of ℓ_1, ℓ_2, and ℓ_3, we have that $\mathbf{0}$, $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ are coplanar, so $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ are linearly dependent. By Proposition 3.1, $\mathbf{a}_{12}$, $\mathbf{a}_{23}$, and $\mathbf{a}_{31}$ are linearly dependent. By Theorem 2.10, the projections of p onto the three pairwise common perpendiculars are scalar multiples of $\mathbf{a}_{12}$, $\mathbf{a}_{23}$, and $\mathbf{a}_{31}$ respectively. Thus, p is on the coplanar surface of the pairwise common perpendiculars. This proves that the coplanar surface of ℓ_1, ℓ_2, and ℓ_3 is contained in the coplanar surface of their pairwise common perpendiculars.

To prove the second part of the theorem, note that if none of the pairwise common perpendiculars contain any other of the pairwise common perpendiculars, then the pairwise common perpendiculars also satisfy the hypotheses of the theorem.

The pairwise common perpendiculars to the pairwise common perpendiculars are ℓ_1, ℓ_2, and ℓ_3. Applying the portion of the theorem that we've already proved, the coplanar surface to the pairwise common perpendiculars is contained in the coplanar surface to the pairwise common perpendiculars to the pairwise common perpendiculars. □

Remark 3. Although a point which is coplanar with its projections onto the ℓ_i is also coplanar with its projections onto the pairwise common perpendiculars, the two planes aren't usually the same. One can readily compute that even if $\mathbf{c}_1$ and $\mathbf{c}_2$ lie in the x-y plane, $\mathbf{a}_{12}$ generally won't.

Proposition 3.3. *Let each of ℓ_1, ℓ_2, and ℓ_3 be a line in D^3 or a point on ∂D^3, none of which contains any of the others. Then the associated face does not include any points on any of the ℓ_i or $\ell_{ij}^{\perp}$.*

Proof. Given any point in the face, we may perform a hyperbolic isometry to move the point to the origin. Thus, without loss of generality, we prove that if the origin is on one of the ℓ_i or the $\ell_{ij}^{\perp}$, then it isn't in the face.

Suppose that $\mathbf{0}$ is on one of the ℓ_i, without loss of generality ℓ_1. Then $\pi_1(\mathbf{0}) = \mathbf{0}$. The only way $\pi_1(\mathbf{0})$ can lie in the relative interior of the convex hull of $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, and $\pi_3(\mathbf{0})$ is if that convex hull is one-dimensional. However, in that case, $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, and $\pi_3(\mathbf{0})$ aren't in general position. Thus, $\mathbf{0}$ isn't in the face.

Suppose that $\mathbf{0}$ is on one of the $\ell_{ij}^{\perp}$, without loss of generality $\ell_{12}^{\perp}$. Then $\mathbf{0}$, $\pi_1(\mathbf{0})$, and $\pi_2(\mathbf{0})$ are collinear. Again, the only way that $\pi_1(\mathbf{0})$ can lie in the relative interior of the convex of hull of $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, and $\pi_3(\mathbf{0})$ is if that convex hull is one-dimensional. Again, that would require that $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, and $\pi_3(\mathbf{0})$ aren't in general position. Thus, $\mathbf{0}$ isn't in the face. □

Proposition 3.4. *Let each of ℓ_1, ℓ_2, and ℓ_3 be a line in D^3 or a point on ∂D^3 (represented as vectors $\mathbf{c}_i$ and $\mathbf{d}_i$ as in Definition 2.1), none of which contains any of the others. If $\mathbf{0}$ lies on their coplanar surface, but not on any of the ℓ_i or $\ell_{ij}^{\perp}$, then the affine hull of $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, and $\pi_3(\mathbf{0})$ is two-dimensional, so there are unique (up to scaling) numbers μ_1, μ_2, μ_3 such that $\sum_{i=1}^{3} \mu_i \mathbf{c}_i = \mathbf{0}$*

Proof. Since $\mathbf{0}$ lies on the coplanar surface, $\pi_1(\mathbf{0}) = \mathbf{c}_1$, $\pi_2(\mathbf{0}) = \mathbf{c}_2$, and $\pi_3(\mathbf{0}) = \mathbf{c}_3$ are linearly dependent.

If the dimension of $\mathrm{Span}(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ is zero, then $\mathbf{c}_1 = \mathbf{c}_2 = \mathbf{c}_3 = \mathbf{0}$, so all three of the ℓ_i pass through the origin, violating the hypotheses.

Suppose that the dimension of $\mathrm{Span}(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ is one. If any of the $\mathbf{c}_i$ were $\mathbf{0}$, that would mean that one of the ℓ_i passed through the origin. Thus, the $\mathbf{c}_i$ are all nonzero. Then they are scalar multiples of each other, so each of the $\mathbf{c}_i$ is perpendicular to all of the $\mathbf{d}_j$. Thus, $\mathrm{Span}(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ is the common perpendicular to all three of the ℓ_i, violating the hypotheses.

Thus, $\mathrm{Span}(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ is two-dimensional. The μ_i exist and are unique up to scaling. □

Proposition 3.5. *Let each of ℓ_1, ℓ_2, and ℓ_3 be a line in D^3 or a point on ∂D^3, none of which contains any of the others. Then the point $\mathbf{0}$ is in the associated face if and only if there are positive constants μ_i (unique up to scaling) such that $\sum_{i=1}^{3} \mu_i \pi_i(\mathbf{0}) = \mathbf{0}$ and $\mathbf{0}$ is not on any of the lines ℓ_i or $\ell_{ij}^{\perp}$.*

Proof. For three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^3$ whose affine hull is two-dimensional and passes through $\mathbf{0}$, $\mathbf{0}$ is in the relative interior of their convex hull if and only if there are positive numbers μ_1, μ_2, μ_3 such that $\mathbf{0} = \sum_{i=1}^{3} \mu_i \mathbf{v}_i$.

The point $\mathbf{0}$ is in the face if and only if $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, and $\pi_3(\mathbf{0})$ are in general position and $\mathbf{0}$ lies in the relative interior of their convex hull. Then $\mathbf{0}$ is on the face if and only if the affine hull of $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, $\pi_3(\mathbf{0})$ is two-dimensional, and $\mathbf{0}$ lies in this affine hull, and there are positive numbers μ_1, μ_2, μ_3 such that $\mathbf{0} = \sum_{i=1}^{3} \mu_i \pi_i(\mathbf{0})$.

Suppose that $\mathbf{0}$ is on the face. Then by Proposition 3.3, $\mathbf{0}$ doesn't lie on any of the ℓ_i or the $\ell_{ij}^{\perp}$. By Proposition 3.4, the affine hull of $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, $\pi_3(\mathbf{0})$ is two-dimensional. Since $\mathbf{0}$ lies in the relative interior of the convex hull of $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, and $\pi_3(\mathbf{0})$, there are positive constants μ_i (unique up to scaling) such that $\sum_{i=1}^{3} \mu_i \pi_i(\mathbf{0}) = \mathbf{0}$. This completes one direction of the proof.

Suppose now that there are positive constants μ_i such that $\sum_{i=1}^{3} \mu_i \pi_i(\mathbf{0}) = \mathbf{0}$ and $\mathbf{0}$ is not on any of the lines ℓ_i or $\ell_{ij}^{\perp}$. Since $\sum_{i=1}^{3} \mu_i \pi_i(\mathbf{0}) = \mathbf{0}$, the vectors $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, $\pi_3(\mathbf{0})$ are linearly dependent, and thus $\mathbf{0}$ lies on the coplanar surface. By Proposition 3.4, the affine hull of $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, and $\pi_3(\mathbf{0})$ is two-dimensional. Since the μ_i are all positive, $\mathbf{0}$ must lie in the interior of the convex hull of $\pi_1(\mathbf{0})$, $\pi_2(\mathbf{0})$, and $\pi_3(\mathbf{0})$. □

Theorem 3.6. *Let each of ℓ_1, ℓ_2, and ℓ_3 be a line in $\mathbb{H}^3$ or a point on $\partial\mathbb{H}^3$, none of which contains any of the others. Let $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$ be their pairwise common perpendiculars. If none of the $\ell_{ij}^{\perp}$ contain any other of the $\ell_{ij}^{\perp}$, then the face associated with ℓ_1, ℓ_2, and ℓ_3 is the same as the face associated with $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$.*

Proof. Let p be a point on the face associated with ℓ_1, ℓ_2, and ℓ_3. Without loss of generality, we may assume that p is at the origin in the Klein model. Since p is a point on the face, it's also a point on the coplanar surface to ℓ_1, ℓ_2, and ℓ_3. Then Theorem 3.2 verifies that p is a point on the coplanar surface to $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$. From Propositions 3.3 and 3.5, we have that p is not on any of the lines ℓ_i or $\ell_{ij}^{\perp}$ and that there are positive constants μ_i such that $\sum_{i=1}^{3} \mu_i \pi_i(\mathbf{0}) = \mathbf{0}$. The μ_i are unique up to scaling, so may be assumed to be the same as the μ_i produced by Proposition 3.1. Since p is not on any of the ℓ_i or $\ell_{ij}^{\perp}$, Proposition 3.1 provides nonzero constants λ_i such that $\lambda_3 \mathbf{a}_{12} + \lambda_1 \mathbf{a}_{23} + \lambda_2 \mathbf{a}_{31} = \mathbf{0}$ and $\lambda_i \mu_i$ doesn't depend on i. Then the λ_i are all of the same sign. Without loss of generality, we may assume that the λ_i are all positive. Since the projection of $\mathbf{0}$ onto $\ell_{ij}^{\perp}$ is a positive scalar multiple of $\mathbf{a}_{ij}$, by Proposition 3.5, we have that p lies on the face associated with $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$.

Repeat the argument starting with the face associated to $\ell_{12}^{\perp}$, $\ell_{23}^{\perp}$, and $\ell_{31}^{\perp}$. □

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ANDREW PRZEWORSKI
DEPARTMENT OF MATHEMATICS
COLLEGE OF CHARLESTON
66 GEORGE STREET
CHARLESTON, SC 29424
UNITED STATES OF AMERICA

E-mail: przeworskia@cofc.edu
URL: http://przeworskia.people.cofc.edu

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