Publ. Math. Debrecen
83/3 (2013), 275–289
DOI: 10.5486/PMD.2013.5442

The Gysin sequence for $\mathbb{S}^3$-actions on manifolds

By JOSÉ IGNACIO ROYO PRIETO (Bilbao)
and MARTINTXO SARALEGI ARANGUREN (Lens)

Abstract. Given a smooth action of $\mathbb{S}^3$ on a manifold M, we are interested in the relationship between the cohomologies of M and $M/\mathbb{S}^3$. If the action is free, we have indeed a principal $\mathbb{S}^3$-bundle, and this relationship is described by the classical Gysin sequence, which also exists when the action is semi-free (i.e., fixed points are allowed) [2]. In this work, we obtain a Gysin sequence for the case of a general smooth action. An exotic term appears, and we show that it is an obstruction for the duality of the second term of the de Rham spectral sequence associated to the action.

Let us consider a smooth action $\Phi : G \times M \to M$ of a compact Lie group on a manifold M. The action Φ induces naturally a filtration $\{F^i\Omega^*(M) \mid i \in \mathbb{N}\}$ of the complex of de Rham differential forms $\Omega^*(M)$, defined by:

$$F^i\Omega^{i+j}(M) = \{\omega \in \Omega^j(M) \mid i_{X_0} \cdots i_{X_j}\omega = 0 \\ \text{for each family } \{X_0, \dots, X_j\} \subset \mathfrak{X}_\Phi(M)\}.$$

Here, we have denoted by $\mathfrak{X}_\Phi(M)$ the orbit distribution of TM formed by the vector fields of M tangent to the orbits of Φ. This filtration defines the first quadrant *de Rham spectral sequence*, which converges to $H^*(M)$. The underlying motivation of this paper is the study of the Poincaré duality of the second term $E_2^{s,t}$ of this spectral sequence.

When Φ is free, we have the duality $E_2^{s,t} \cong E_2^{n-s,\ell-t}$, where $n = \dim M/G$ and $\ell = \dim G$. This property is lost when the action is no longer free.

Mathematics Subject Classification: 57R19, 57R30, 57S15.
Key words and phrases: Gysin sequence, basic cohomology, $\mathbb{S}^3$-actions, controlled forms.
This work has been partially should be substituted by "supported by the UPV/EHU grants EHU09/04 and EHU12/05, and by the Spanish MICINN grant MTM2010-15471.

Inspired by the work of Goresky and MacPherson, one expects to recover the Poincaré duality by using intersection cohomology. This is the case when the group G is the circle $\mathbb{S}^1$ (see [6]). The next natural group G to study is $\mathbb{S}^3$ (of rank 1 and not abelian). It has been proved in [7] that Poincaré duality still holds when the action Φ is semi-free.

What about the other $\mathbb{S}^3$-actions? Surprisingly, the second term of the above spectral sequence has not been computed yet in this context. The main result of this paper is in the following Gysin sequence, which computes this second term

$$\cdots \longrightarrow H^i\big(M\big) \xrightarrow{②} \underbrace{H^{i-3}\big(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\big)}_{E_2^{i-3,3}} \oplus \underbrace{\Big(H^{i-2}\big(M^{\mathbb{S}^1}\big)\Big)^{-\mathbb{Z}_2}}_{E_2^{i-2,2}} \xrightarrow{③}$$

$$\xrightarrow{③} \underbrace{H^{i+1}\big(M/\mathbb{S}^3\big)}_{E_2^{i+1,0}} \xrightarrow{①} H^{i+1}\big(M\big) \longrightarrow \cdots$$

with $E_2^{i,1} = 0$, where

- Σ is the subset of points of M whose isotropy group is infinite;
- our choice of maximal torus, $\mathbb{S}^1$, is $\big\{a + bi \mid a^2 + b^2 = 1\big\} \leq \mathbb{S}^3$;
- the $\mathbb{Z}_2$-action is induced by $j \in \mathbb{S}^3$,
- $(-)^{-\mathbb{Z}_2}$ denotes the subspace of antisymmetric elements (cf. (6)),
- ① is induced by the natural projection $\pi : M \to M/\mathbb{S}^3$,
- ② is induced by the integration along the fibers of π, and
- ③ involves the multiplication by the *Euler class* $[e] \in \mathbb{H}^4_{\bar{4}}(M/\mathbb{S}^3)$ (cf. [7])

(cf. Theorem 2.2 and paragraph 2.3).

Notice that the first floor $E_2^{i,1}$ always vanishes (even for any perversity!) whereas the second floor $E_2^{i,2}$ may not, as we show in Example 2.4. So, it follows that Poincaré duality does not work in the generic case, even considering intersection cohomology.

As a consequence, a new approach is needed in order to extend the Poincaré duality for general actions, and, thus, for Singular Riemannian Foliations, as is the partition induced by the orbits of a general $\mathbb{S}^3$-action. Results in this direction are being explored.

In fact, in [5] the duality of the spectral sequence associated to a Singular Riemannian Foliation is claimed for the special case of extreme perversities (which is tantamount to working only in the regular stratum or relatively to the singular strata). As the previous counterexample shows, a new approach is needed in order to extend the duality result for general perversities if one wants to consider the usual cohomology ($\overline{p} = \overline{0}$) case.

A different Gysin sequence relating the cohomology of M and the $\mathbb{S}^3$-equivariant cohomology of M was constructed in [3], also in the case of a general smooth $\mathbb{S}^3$-action.

In the sequel M is a connected, second countable, Hausdorff, without boundary and smooth (of class C^∞) manifold. We fix a smooth action $\Phi : \mathbb{S}^3 \times M \to M$.

We wish to thank the referees for the indications given in order to improve this paper.

1. Stratifications and differential forms

We describe the stratification arising from the action. We also introduce the controlled differential forms, defined by Verona, in order to compute the singular cohomology in this context.

1.1. Thom–Mather structure. There are three possibilities for the dimension of the isotropy subgroup[1] $\mathbb{S}^3_x$ of a point $x \in M$, namely: 0,1 and 3. So, we have the dimension-type filtration

$$F = \big\{x \in M \mid \dim \mathbb{S}^3_x = 3\big\} \subset \Sigma = \big\{x \in M \mid \dim \mathbb{S}^3_x \geq 1\big\} \subset M$$
$$= \big\{x \in M \mid \dim \mathbb{S}^3_x \geq 0\big\}.$$

In this section, we describe the geometry of the triple (M, Σ, F). The subset Σ is not necessarily a manifold, but the subsets $F = M^{\mathbb{S}^3}$, $\Sigma \backslash F = \{x \in M \mid \dim \mathbb{S}^3_x = 1\}$ and $M \backslash \Sigma = \{x \in M \mid \dim \mathbb{S}^3_x = 0\}$ are proper invariant submanifolds[2] of M. So, we can consider $\tau_0 : T_0 \to F$ and $\tau_1 : T_1 \to \Sigma \backslash F$ two invariant tubular neighborhoods in M. Over each connected component, the structure group is the orthogonal group. Associated to these tubular neighborhoods we have the following maps ($k = 0, 1$):

$\rightsquigarrow$ The *radius map* $\nu_k : T_k \to [0, \infty[$, defined fiberwise by $u \mapsto \|u\|$. It is an invariant smooth map.

$\rightsquigarrow$ The *dilatation map* $\partial_k : [0, \infty[\times T_k \to T_k$, defined fiberwise by $(t, u) \mapsto t \cdot u$. It is a smooth equivariant map.

[1] We refer the reader to [2] for the notions related with compact Lie group actions, such as isotropy, invariant tubular neighborhoods, . . .

[2] In fact, these manifolds may have connected components with different dimensions.

The family of tubular neighborhoods $\mathfrak{T}_M = \{T_0, T_1\}$ is a *Thom–Mather system* when:

$$(TM) \left\{ \begin{array}{c} \tau_0 = \tau_0 \circ \tau_1 \\ \nu_0 = \nu_0 \circ \tau_1 \end{array} \right\} \quad \text{on} \quad T_0 \cap T_1 = \tau_1^{-1}\big(T_0 \cap (\Sigma \backslash F)\big).$$

Lemma 1.1. *Thom–Mather systems exist.*

Proof. We fix an invariant tubular neighborhood $\tau_0 : T_0 \to F$. It exists since F is an invariant closed submanifold of M. Since the isotropy subgroup of any point of F is the whole $\mathbb{S}^3$, we can find[3] an atlas $\mathcal{A} = \{\varphi : U \times \mathbb{R}^n \to \tau_0^{-1}(U)\}$ of τ_0, having $O(n)$ as structure group, and an orthogonal action $\Psi : \mathbb{S}^3 \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi(x, \Psi(g, v)) = \Phi(g, \varphi(x, v)) \quad \forall x \in U, \ \forall v \in \mathbb{R}^n \text{ and } \forall g \in \mathbb{S}^3. \tag{1}$$

We write $\tau_0' : S_0 \to F$ the restriction of τ_0, where S_0 is the submanifold $\nu_0^{-1}(1)$. It is a fiber bundle. The restriction $\tau_0'' : (S_0 \cap (\Sigma \backslash F)) \to F$ is also a fiber bundle whose induced atlas is $\mathcal{A}'' = \{\varphi : U \times \mathbb{S}^{n-1}_\Sigma \to \tau_0''^{-1}(U)\}$, where $\mathbb{S}^{n-1}_\Sigma = \{w \in \mathbb{S}^{n-1} \mid \dim \mathbb{S}^3_w = 1\}$.

The map $\mathfrak{L}_0 : T_0 \backslash F \to S_0 \times]0, \infty[$, defined by $\mathfrak{L}_0(x) = (\partial_0(\nu_0(x)^{-1}, x), \nu_0(x))$, is an equivariant diffeomorphism. Under $\mathfrak{L}_0$:

$\rightsquigarrow$ the map τ_0 becomes $(y, t) \mapsto \tau_0'(y)$,

$\rightsquigarrow$ the map ν_0 becomes $(y, t) \mapsto t$, and

$\rightsquigarrow$ the manifold $T_0 \cap (\Sigma \backslash F)$ becomes $(S_0 \cap (\Sigma \backslash F)) \times]0, \infty[$.

Since the structure group of τ_0' is a compact Lie group, condition (1) allows us to construct an invariant Riemannian metric μ_0 on S_0 such that the fibers of τ_0' are totally geodesic submanifolds and $(T(S_0 \cap (\Sigma \backslash F)))^\perp \subset \ker(\tau_0')_*$. Then, if we consider the associated tubular neighborhood $\tau_1' : T_1' \to S_0 \cap (\Sigma \backslash F)$ we have $\tau_0' \circ \tau_1' = \tau_0'$.

We can construct now an invariant Riemannian metric μ on $M \backslash F$ such that under $\mathfrak{L}_0$:

$\rightsquigarrow$ the metric μ becomes $\mu_0 + dr^2$ on $S_0 \times]0, \infty[$.

We consider the associated tubular neighborhood $\tau_1 : T_1 \to \Sigma \backslash F$. Verification of the property (TM) must be done on $T_0 \cap T_1$, where using $\mathfrak{L}_0$, we get:

$\rightsquigarrow$ $T_0 \cap T_1$ becomes $T_1' \times]0, \infty[$.

$\rightsquigarrow$ τ_1 becomes $(y, t) \mapsto (\tau_1'(y), t)$.

A straightforward calculation gives (TM) and ends the proof. □

[3] For each connected component of F.

We fix a such system $\mathfrak{T}_M$. For each $k \in \{0,1\}$, we shall write $D_k \subset M$ the open subset $\nu_k^{-1}([0,1[)$ and call it the *soul* of the tubular neighborhood τ_k. We shall write $\Delta_0 = D_0 \cap \Sigma$.

1.2. Verona's differential forms. As it is shown in [8], the singular cohomology of M can be computed by using differential forms on $M \backslash \Sigma$. This is the tool we use in this work. The complex of *controlled forms* (or *Verona's forms*) of M is defined by

$$\Omega_V^*(M) = \left\{ \omega \in \Omega^*\big(M\backslash\Sigma\big) \mid \exists \omega_1 \in \Omega^*(\Sigma\backslash F) \text{ and } \omega_0 \in \Omega^*\big(F\big) \right.$$
$$\left. \text{with} \left\{ \begin{array}{l} (a)\ \ \tau_1^*\omega_1 = \omega \ \text{ on } \ D_1\backslash\Sigma \\ (b)\ \ \tau_0^*\omega_0 = \omega \ \text{ on } \ D_0\backslash\Sigma \\ (c)\ \ \tau_0^*\omega_0 = \omega_1 \ \text{ on } \ \Delta_0\backslash F \end{array} \right\} \right\}.$$

Following [8] we know that the cohomology of the complex $\Omega_V^*(M)$ is the singular cohomology $H^*\big(M\big)$.

We also use in this work the complex $\Omega_V^*(\Sigma) = \{\gamma \in \Omega^*\big(\Sigma\backslash F\big) \mid \exists \gamma_0 \in \Omega^*(F)$ with $\tau_0^*\gamma_0 = \gamma$ on $\Delta_0\backslash F\}$ and the *relative complexes* $\Omega_V^*(M,\Sigma) = \{\omega \in \Omega_V^*(M) \mid \omega_1 \equiv 0\}$ and $\Omega_V^*(\Sigma, F) = \{\gamma \in \Omega_V^*(\Sigma) \mid \gamma_0 \equiv 0\}$.

Since M is a manifold, controlled forms are in fact differential forms on M.

Lemma 1.2. *Any controlled form of M is the restriction of a differential form of M.*

Proof. First, we construct a section σ of the restriction $\rho : \Omega_V^*(M) \to \Omega_V^*(\Sigma)$ defined by $\rho(\omega) = \omega_1$. Let us consider a smooth function $f :]\,0,\infty[\to [0,1]$ verifying $f \equiv 0$ on $[3,\infty[$ and $f \equiv 1$ on $]0,2]$. Notice that the compositions $f\circ\nu_0 : M \to [0,1]$ and $f\circ\nu_1 : M\backslash F \to [0,1]$ are smooth invariant maps. So, for each $\gamma \in \Omega_V^*(\Sigma)$ we have

$$\sigma(\gamma) = (f\circ\nu_0)\cdot\tau_0^*\gamma_0 + (1-(f\circ\nu_0))\cdot(f\circ\nu_1)\tau_1^*\gamma \in \Omega^*(M). \tag{2}$$

This differential form is a controlled form since

(a) Since $(f\circ\nu_1) \equiv 1$ on D_1, $(f\circ\nu_0) \equiv 0$ on $M\backslash T_0$ and (TM) then we have

$$\sigma(\gamma) = (f\circ\nu_0)\cdot\tau_1^*\tau_0^*\gamma_0 + (1-(f\circ\nu_0))\cdot\tau_1^*\gamma = \tau_1^*\left((f\circ\nu_0)\cdot\tau_0^*\gamma_0 + (1-(f\circ\nu_0))\cdot\gamma\right)$$

on $D_1\backslash\Sigma$. This gives $(\sigma(\gamma))_1 = (f\circ\nu_0)\cdot\tau_0^*\gamma_0 + (1-(f\circ\nu_0))\cdot\gamma$. Since $\tau_0^*\gamma_0 = \gamma$ on $\Delta_0\backslash F$ then $(\sigma(\gamma))_1 = (f\circ\nu_0)\cdot\gamma + (1-(f\circ\nu_0))\cdot\gamma = \gamma$.

(b) Since $(f \circ \nu_0) \equiv 1$ on D_0 then we have $\sigma(\gamma) = \tau_0^* \gamma_0$ on $D_0 \backslash \Sigma$. This gives $(\sigma(\gamma))_0 = \gamma_0$.

(c) We have $(\sigma(\gamma))_1 = \gamma = \tau_0^* \gamma_0 = \tau_0^* (\sigma(\gamma))_0$ on $\Delta_0 \backslash F$.

This map σ is a section of ρ since $\rho(\sigma(\gamma)) = (\sigma(\gamma))_1 = \gamma$.

In particular, $\rho(\omega - \sigma(\rho(\omega))) = 0$ for each $\omega \in \Omega^*_V(M)$. As $\sigma(\rho(\omega)) \in \Omega^*(M)$ (cf. (2)) and coincides with ω in the open set $(D_0 \cup D_1) \backslash \Sigma$ we conclude that ω can be extended to M. $\square$

1.3. Invariant forms. We fix $\{u_1, u_2, u_3\}$ a basis of the Lie algebra of $\mathbb{S}^3$ with $[u_1, u_2] = u_3$, $[u_2, u_3] = u_1$ and $[u_3, u_1] = u_2$. We denote by $X_i \in \mathfrak{X}_\Phi\big(M\big)$ the *fundamental vector field* associated to u_i, $i = 1, 2, 3$.

A controlled form ω of M is an *invariant form* when $L_{X_i}\omega = 0$ for each $i = 1, 2, 3$. The complex of invariant forms is denoted by $\underline{\Omega}^*_V(M)$. The inclusion $\underline{\Omega}^*_V(M) \hookrightarrow \Omega^*_V(M)$ induces an isomorphism in cohomology. This is a standard argument based on the fact that $\mathbb{S}^3$ is a connected compact Lie group (cf. [4, Theorem I, Ch. IV, vol. II]). So,

$$H^*\big(\underline{\Omega}^{\cdot}_V(M)\big) = H^*\big(\Omega^{\cdot}_V(M)\big) = H^*\big(M\big). \tag{3}$$

1.4. Basic forms. A controlled form ω of M is a *basic form* when $i_X\omega = i_X d\omega = 0$ for each $X \in \mathfrak{X}_\Phi\big(M\big)$. The complex of the basic forms is denoted by $\Omega^*_V(M/\mathbb{S}^3)$. In a similar fashion we define $\Omega^*_V(\Sigma/\mathbb{S}^3)$. In this work, we shall use the following relative versions of these complexes: $\Omega^*_V(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3) = \Omega^*_V(M/\mathbb{S}^3) \cap \Omega^*_V(M, \Sigma)$, as well as $\Omega^*_V(\Sigma/\mathbb{S}^3, F) = \Omega^*_V(\Sigma/\mathbb{S}^3) \cap \Omega^*_V(\Sigma, F)$.

Lemma 1.3.

$$H^*(\Omega^{\cdot}_V(M/\mathbb{S}^3)) = H^*\big(M/\mathbb{S}^3\big) \quad \textit{and} \quad H^*(\Omega^{\cdot}_V(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3)) = H^*\big(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\big).$$

Proof. The orbit space $M/\mathbb{S}^3$ is a stratified pseudomanifold. The family of tubular neighborhoods $\mathfrak{T}_{M/\mathbb{S}^3} = \{\pi(T_0), \pi(T_1)\}$ is a *Thom–Mather system.* Here, $\pi : M \to M/\mathbb{S}^3$ denotes the canonical projection. Using this projection, we identify the complex of controlled forms of $M/\mathbb{S}^3$ with $\Omega^{\cdot}_V(M/\mathbb{S}^3)$, and the same holds for Σ.

Since $H^*(\Omega^{\cdot}_V(X)) = H^*\big(X\big)$ for any stratified pseudomanifold X, then $H^*(\Omega^{\cdot}_V(M/\mathbb{S}^3)) = H^*\big(M/\mathbb{S}^3\big)$ and $H^*(\Omega^{\cdot}_V(\Sigma/\mathbb{S}^3)) = H^*\big(\Sigma/\mathbb{S}^3\big)$ (cf. [8]). In fact, the orbit spaces $M/\mathbb{S}^3$ and $\Sigma/\mathbb{S}^3$ are triangulable [9], and by [10], both of them possess good coverings. Moreover, any open covering of $M/\mathbb{S}^3$ (resp. $\Sigma/\mathbb{S}^3$) possesses a subordinated partition of unity made up of controlled functions. So, we can proceed as in [1] and construct a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^p(\Omega^{\cdot}_{V}(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3)) \arrow[r] & H^p(\Omega^{\cdot}_{V}(M/\mathbb{S}^3)) \arrow[r] \arrow[d, "f_M"] & H^p(\Omega^{\cdot}_{V}(\Sigma/\mathbb{S}^3)) \arrow[r] \arrow[d, "f_\Sigma"] & H^{p+1}(\Omega^{\cdot}_{V}(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3)) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^p(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3) \arrow[r] & H^p(M/\mathbb{S}^3) \arrow[r] & H^p(\Sigma/\mathbb{S}^3) \arrow[r] & H^{p+1}(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3) \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical arrows are isomorphisms and the horizontal rows are the long exact sequences associated to the pair $(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3)$. This gives
$H^*(\Omega^{\cdot}_{V}(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3)) = H^*\big(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\big)$[4]. □

2. Gysin sequence

We construct the long exact sequence associated to the action $\Phi : \mathbb{S}^3 \times M \to M$ relating the cohomology of M and $M/\mathbb{S}^3$. First of all, we shall use strongly that Φ is *almost free*[5] in $M\backslash\Sigma$ to get a better description of the controlled forms of M.

2.1. Decomposition of a differential form. We endow $M\backslash\Sigma$ with an $\mathbb{S}^3$-invariant Riemannian metric μ_0, which exists because $\mathbb{S}^3$ is compact. We also fix a bi-invariant Riemannian metric ν on the Lie group $\mathbb{S}^3$. Consider now the μ_0-orthogonal $\mathbb{S}^3$-invariant decomposition $T(M\backslash\Sigma) = \mathcal{D} \oplus \xi$, where $\mathcal{D}$ is the distribution generated by Φ. Since the action Φ is almost free on $M\backslash\Sigma$, for each point $x \in M\backslash\Sigma$, the family $\{X_1(x), X_2(x), X_3(x)\}$ is a basis of $\mathcal{D}_x$. We define the $\mathbb{S}^3$-Riemannian metric μ on $M\backslash\Sigma$ by putting

$$\mu(w_1, w_2) = \begin{cases} \mu_0(w_1, w_2) & \text{if } \ w_1, w_2 \in \xi_x \\ 0 & \text{if } \ w_1 \in \xi_x, \ w_2 \in \mathcal{D}_x \\ \delta_{i,j} & \text{if } \ w_1 = X_i(x), \ w_2 = X_j(x) \end{cases}$$

We denote by $\chi_i = i_{X_i}\mu \in \Omega^1(M\backslash\Sigma)$ the *characteristic form* associated to u_i, $i = 1, 2, 3$. Since $\chi_j(X_i) = \mu(X_i, X_j) = \delta_{ij}$, each differential form $\omega \in \Omega^*(M\backslash\Sigma)$ possesses a unique writing,

$$\omega = {}_0\omega + \sum_{p=1}^{3} \chi_p \wedge {}_p\omega + \sum_{1 \le p < q \le 3} \chi_p \wedge \chi_q \wedge {}_{pq}\omega + \chi_1 \wedge \chi_2 \wedge \chi_3 \wedge {}_{123}\omega,$$

where the coefficients ${}_\bullet\omega$ are *horizontal forms*, that is, they verify $i_X({}_\bullet\omega) = 0$ for each $X \in \mathfrak{X}_\Phi(M)$. This is the *canonical decomposition* of ω. For example,

[4] Notice that this is not the five lemma.

[5] All the isotropy subgroups are finite groups.

$d\beta = {}_0(d\beta) + \chi_1 \wedge L_{X_1}\beta + \chi_2 \wedge L_{X_2}\beta + \chi_3 \wedge L_{X_3}\beta$, for any horizontal form β (notice that this formula is no longer true if β is not horizontal). Since $L_{X_i}\chi_j = \chi_{[u_i,u_j]}$, with $1 \leq i, j \leq 3$, then

$$\begin{aligned} &L_{X_1}\chi_1 = L_{X_2}\chi_2 = L_{X_3}\chi_3 = 0, && L_{X_1}\chi_2 = -L_{X_2}\chi_1 = \chi_3 \\ &L_{X_1}\chi_3 = -L_{X_3}\chi_1 = -\chi_2 && L_{X_2}\chi_3 = -L_{X_3}\chi_2 = \chi_1, \end{aligned} \tag{4}$$

and we have the canonical decompositions

$$\begin{cases} d\chi_1 = e_1 - \chi_2 \wedge \chi_3 \\ d\chi_2 = e_2 + \chi_1 \wedge \chi_3 \\ d\chi_3 = e_3 - \chi_1 \wedge \chi_2. \end{cases} \tag{5}$$

Here, the forms e_i are basic for $i = 1, 2, 3$. Notice that $e_1 - \chi_2 \wedge \chi_3$ is the *Euler form* of the action of the maximal torus with fundamental vector X_1, and that $e_1^2 + e_2^2 + e_3^2$ is the Euler form of the action Φ (see section (2.3)).

Consider $U \subset M \backslash \Sigma$ an equivariant open subset. If $\omega \in \Omega^*\big(M\backslash\Sigma, U\big)$ then the coefficients of its canonical decomposition are horizontal forms of $\Omega^*\big(M\backslash\Sigma, U\big)$. The following Lemma is the key for the construction of the Gysin sequence. Given an action of $\mathbb{Z}_2$ on a vector space E generated by the morphism $h : E \to E$, we shall write

$$E^{-\mathbb{Z}_2} = \{e \in E \mid h(e) = -e\}, \tag{6}$$

the subspace of *antisymmetric elements*. Notice that $j \in \mathbb{S}^3$ acts naturally on $M^{\mathbb{S}^1}$.

Lemma 2.1.

$$H^*\left(\frac{\Omega^{\cdot}_V(M)}{\Omega^{\cdot}_V(M/\mathbb{S}^3)}\right) = H^{*-3}\big(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\big) \oplus \left(H^{*-2}\big(M^{\mathbb{S}^1}\big)\right)^{-\mathbb{Z}_2}$$

Proof. We consider the integration operator:

$$\fint : \frac{\Omega^*_V(M)}{\Omega^*_V(M/\mathbb{S}^3)} \longrightarrow \Omega^{*-3}_V(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3),$$

given by:

$$\fint (\langle\omega\rangle) = (-1)^{\deg \omega}\ i_{X_3} i_{X_2} i_{X_1} \omega.$$

It is a well defined differential operator since

- the tubular neighborhoods of the Thom–Mather's structure $\mathfrak{T}$ are invariant,
- the operator $i_{X_3} i_{X_2} i_{X_1}$ vanishes on Σ, and

– $i_X i_{X_3} i_{X_2} i_{X_1}\omega = i_X d i_{X_3} i_{X_2} i_{X_1}\omega = 0$ for each $X \in \mathfrak{X}_\Phi\big(M\big)$[6].

Every form $\gamma \in \Omega_V^{*-3}(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3)$ vanishes in a neighborhood of Σ. So, the product $\chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \gamma$ belongs to $\underline{\Omega}_V^*(M)$ (cf. (4)). Since $i_{X_3} i_{X_2} i_{X_1}(\chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \gamma) = \gamma$ then we have the short exact sequence

$$0 \longrightarrow \mathrm{Ker}^* \, f \longrightarrow \frac{\underline{\Omega}_V^*(M)}{\Omega_V^*(M/\mathbb{S}^3)} \xrightarrow{\ f\ } \Omega_V^{*-3}(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3) \longrightarrow 0 \qquad (7)$$

By Lemma 1.3, it suffices to prove the following:

(a) $H^*\big(\mathrm{Ker}^* \, f\big) = \Big(H^{*-2}\big(M^{\mathbb{S}^1}\big)\Big)^{-\mathbb{Z}_2}$.

(b) The associated connecting homomorphism δ vanishes.

(a)

For the sake of simplicity we put $\mathcal{A}^*\big(M\big) = \mathrm{Ker}^* \, f$. In fact we have $\mathcal{A}^*\big(M\big) = \frac{\{\omega \in \underline{\Omega}_V^*(M) \mid i_{X_3} i_{X_2} i_{X_1}\omega = 0\}}{\Omega_V^*(M/\mathbb{S}^3)}$. Analogously, we define $\mathcal{A}^*\big(M, \Sigma\big)$, $\mathcal{A}^*\big(\Sigma\big)$ and $\mathcal{A}^*\big(\Sigma, F\big)$. To get (a), it suffices to prove the following facts:

(a1) $H^*\big(\mathcal{A}^*\big(M\big)\big) = H^*\big(\mathcal{A}^*\big(\Sigma\big)\big)$.

(a2) $H^*\big(\mathcal{A}^{\cdot}\big(\Sigma\big)\big) = H^*\big(\mathcal{A}^{\cdot}\big(\Sigma, F\big)\big)$.

(a3) $H^*\big(\mathcal{A}^{\cdot}\big(\Sigma, F\big)\big) = \Big(H^{*-2}\big(M^{\mathbb{S}^1}\big)\Big)^{-\mathbb{Z}_2}$.

(a1)

Consider the inclusion $L : \mathcal{A}^*\big(M, \Sigma\big) \longrightarrow \mathcal{A}^*\big(M\big)$ and the restriction $R : \mathcal{A}^*\big(M\big) \to \mathcal{A}^*\big(\Sigma\big)$, which are differential morphisms. This gives the short sequence

$$0 \longrightarrow \mathcal{A}^*\big(M, \Sigma\big) \xrightarrow{\ L\ } \mathcal{A}^*\big(M\big) \xrightarrow{\ R\ } \mathcal{A}^*\big(\Sigma\big) \longrightarrow 0.$$

Notice that $R \circ L = 0$. This short sequence is exact since:

• *The operator R is an onto map.* Consider $\gamma \in \underline{\Omega}_V^*(\Sigma)$. We know that $\sigma(\gamma) \in \Omega_V^*(M)$ (cf. Lemma 1.2). The result comes from:

$\rightsquigarrow$ $\sigma(\gamma) \in \underline{\Omega}_V^*(M)$. Since τ_0, τ_1 are equivariant and $f \circ \nu_0$, $f \circ \nu_1$ are invariant.

$\rightsquigarrow$ $i_{X_3} i_{X_2} i_{X_1}\sigma(\gamma) = 0$. Since τ_0, τ_1 are equivariant and rank$\{X_1(x), X_2(x), X_3(X)\} \leq 2$ for any $x \in \Sigma$.

$\rightsquigarrow$ $R\left(\langle\sigma(\gamma)\rangle\right) = \langle(\sigma(\gamma))_1\rangle = \langle\gamma\rangle$.

[6] $L_A i_B = i_B L_A + i_{[A,B]}$, $\forall A, B \in \mathfrak{X}(M)$.

• $\operatorname{Ker} R \subset \operatorname{Im} L$. Consider $\omega \in \underline{\Omega}^*_V(M)$ with $i_{X_3} i_{X_2} i_{X_1}\omega = 0$ and $i_{X_j}\omega_1 = 0$ for $j \in \{1,2,3\}$. Since τ_0 and τ_1 are equivariant and $X_j = 0$ on F then $i_{X_j}\sigma(\omega_1) = 0$ for $j \in \{1,2,3\}$. This gives $\langle \sigma(\omega_1)\rangle = 0$. Finally, we have $\langle\omega\rangle = \langle \omega - \sigma(\omega_1)\rangle = L(\langle \omega - \sigma(\omega_1)\rangle)$ since $(\omega - \sigma(\omega_1))_1 = \omega_1 - (\sigma(\omega_1))_1 = \omega_1 - \omega_1 = 0$.

Now, we will get (a1) by proving that $H^*\big(\mathcal{A}^{\cdot}\big(M,\Sigma\big)\big) = 0$. By definition of Verona's forms we have $\mathcal{A}^*\big(M,\Sigma\big) = \mathcal{A}^*\big(M,D\big) \overset{\text{excision}}{=\!=\!=} \mathcal{A}^*\big(M\backslash\Sigma, D\backslash\Sigma\big)$, where $D = D_0 \cup D_1$. A straightforward calculation gives:

$$H^*\big(\mathcal{A}^{\cdot}\big(M\backslash\Sigma, D\backslash\Sigma\big)\big)$$
$$= \frac{\{\omega \in \underline{\Omega}^*(M\backslash\Sigma, D\backslash\Sigma) \mid i_{X_3} i_{X_2} i_{X_1}\omega = 0 \text{ and } i_{X_j} d\omega = 0 \text{ for } j \in \{1,2,3\}\}}{\Omega^*((M\backslash\Sigma)/\mathbb{S}^3, (D\backslash\Sigma)/\mathbb{S}^3) + \{d\beta \mid \beta \in \underline{\Omega}^{*-1}(M\backslash\Sigma, D\backslash\Sigma) \text{ and } i_{X_3} i_{X_2} i_{X_1}\beta = 0\}}$$

Let ω be a differential form of $\underline{\Omega}^*(M\backslash\Sigma, D\backslash\Sigma)$ verifying $i_{X_3} i_{X_2} i_{X_1}\omega = 0$ and $i_{X_j} d\omega = 0$ for $j \in \{1,2,3\}$. Then

$$\omega = d\underbrace{(\chi_1 \wedge i_{X_3} i_{X_2}\omega - \chi_2 \wedge i_{X_3} i_{X_1}\omega + \chi_3 \wedge i_{X_2} i_{X_1}\omega)}_{\beta}$$
$$+ \underbrace{-e_1 \wedge i_{X_3} i_{X_2}\omega + e_2 \wedge i_{X_3} i_{X_1}\omega - e_3 \wedge i_{X_2} i_{X_1}\omega + {}_0\omega}_{\alpha}$$

(cf. (5)) with $\beta \in \underline{\Omega}^{*-1}(M\backslash\Sigma, D)$, verifying $i_{X_3} i_{X_2} i_{X_1}\beta = 0$, and $\alpha \in \Omega^*((M\backslash\Sigma)/\mathbb{S}^3, D/\mathbb{S}^3)$. This implies $H^*\big(\mathcal{A}^{\cdot}\big(M\backslash\Sigma, D\backslash\Sigma\big)\big) = 0$ and then $H^*\big(\mathcal{A}^{\cdot}\big(M,\Sigma\big)\big) = 0$.

(a2)

Consider the inclusion $L : \mathcal{A}^*\big(\Sigma, F\big) \hookrightarrow \mathcal{A}^*\big(\Sigma\big)$ which is a differential morphism. It suffices to prove that L is an onto map.

Let us consider a smooth function $f :]0,\infty[\to [0,1]$ verifying $f \equiv 0$ on $[3,\infty[$ and $f \equiv 1$ on $]0,2]$. Notice that the composition $f \circ \nu_0 : M \to [0,1]$ is a smooth invariant map. So, for each $\gamma \in \Omega^*\big(F\big)$ we have $\sigma(\gamma) = (f\circ\nu_0)\tau_0^*\gamma \in \Omega^*(M)$. This differential form verifies

$\rightsquigarrow$ $\sigma(\gamma) \in \Omega^*_V(\Sigma)$. Since $(f\circ\nu_0) \equiv 1$ on Δ_0 then $\sigma(\gamma) = \tau_0^*\gamma$ on $\Delta_0\backslash F$. This gives $(\sigma_0(\gamma))_0 = \gamma$.

$\rightsquigarrow$ $\sigma(\gamma) \in \underline{\Omega}^*_V(\Sigma)$. Since τ_0 is an equivariant map and $f\circ\nu_0$ is an invariant map.

$\rightsquigarrow$ $i_{X_j}\sigma(\gamma) = 0$ for $j \in \{1,2,3\}$ since τ_0 is an equivariant map and $X_j = 0$ on F.

Then $\langle\sigma(\gamma)\rangle = 0$ on $\mathcal{A}^*\big(\Sigma\big)$.

Let $\langle\omega\rangle$ be a class of $\mathcal{A}^*\big(\Sigma\big)$. We can write: $\langle\omega\rangle = \langle\omega - \sigma((\omega)_0)\rangle = L(\langle\omega - \sigma((\omega)_0)\rangle)$ since $(\omega - \sigma((\omega)_0))_0 = \omega_0 - (\sigma(\omega_0))_0 = \omega_0 - \omega_0 = 0$. This proves that L is an onto map.

(a3)

By definition of Verona's differential forms we have

$$\mathcal{A}^*\big(\Sigma, F\big) = \mathcal{A}^*\big(\Sigma, \Delta_0\big) \overset{\text{excision}}{=\!=\!=} \mathcal{A}^*\big(\Sigma\backslash F, \Delta_0\backslash F\big) = \frac{\underline{\Omega}^*(\Sigma\backslash F, \Delta_0\backslash F)}{\Omega^*((\Sigma\backslash F)/\mathbb{S}^3, (\Delta_0\backslash F)/\mathbb{S}^3)}.$$

The isotropy subgroup of a point of $\Sigma\backslash F$ is conjugated to $\mathbb{S}^1$ or $N(\mathbb{S}^1)$ (cf. [2, Theorem 8.5, pag. 153]). We consider the manifold $\Gamma = (\Sigma\backslash F)^{\mathbb{S}^1}$. A straightforward calculation gives that $\Sigma\backslash F$ is G-equivariant diffeomorphic to

$$\mathbb{S}^3 \times_{N(\mathbb{S}^1)} \Gamma = \big(\mathbb{S}^3/\mathbb{S}^1\big) \times_{N(\mathbb{S}^1)/\mathbb{S}^1} \Gamma = \mathbb{S}^2 \times_{\mathbb{Z}_2} \Gamma.$$

Notice that $\Gamma/\mathbb{Z}_2 = (\Sigma\backslash F)/\mathbb{S}^3$. Let Γ_0 be the open subset $\Gamma\cap\Delta_0$ of Γ. Analogously we have $\Delta_0\backslash F = \mathbb{S}^2 \times_{\mathbb{Z}_2} \Gamma_0$ and $\Gamma_0/\mathbb{Z}_2 = (\Delta_0\backslash F)/\mathbb{S}^3$.

The $\mathbb{Z}_2$-action on $\mathbb{S}^2$ is generated by $(x_0, x_1, x_2) \mapsto (-x_0, -x_1, -x_2)$[7]. Then, the $\mathbb{Z}_2$-action on $H^0\big(\mathbb{S}^2\big)$ (resp. $H^2\big(\mathbb{S}^2\big)$) is the identity Id (resp. $-\,\mathrm{Id}$). The $\mathbb{Z}_2$-action on Γ is induced by $\Phi(j, -)$. The Künneth formula gives

$$\begin{aligned}
H^*\big(\underline{\Omega}^*(\Sigma\backslash F, \Delta_0\backslash F)\big) &= H^*\big(\underline{\Omega}^{\cdot}(\mathbb{S}^2 \times_{\mathbb{Z}_2} \Gamma, \mathbb{S}^2 \times_{\mathbb{Z}_2} \Gamma_0)\big) = H^*\Big(\underline{\Omega}^{\cdot}\left(\mathbb{S}^2 \times \Gamma, \mathbb{S}^2 \times \Gamma_0\right)^{\mathbb{Z}_2}\Big) \\
&= H^*\big(\underline{\Omega}^{\cdot}(\mathbb{S}^2 \times \Gamma, \mathbb{S}^2 \times \Gamma_0)\big)^{\mathbb{Z}_2} = \big(H^*(\mathbb{S}^2) \otimes H^*(\Gamma, \Gamma_0)\big)^{\mathbb{Z}_2} \\
&= \big(H^0(\mathbb{S}^2) \otimes H^*(\Gamma, \Gamma_0)\big)^{\mathbb{Z}_2} \oplus \big(H^2(\mathbb{S}^2) \otimes H^{*-2}(\Gamma, \Gamma_0)\big)^{\mathbb{Z}_2} \\
&= \left(H^*(\Gamma, \Gamma_0)\right)^{\mathbb{Z}_2} \oplus \big(H^{*-2}(\Gamma, \Gamma_0)\big)^{-\mathbb{Z}_2} \\
&= H^*(\Gamma/\mathbb{Z}_2, \Gamma_0/\mathbb{Z}_2) \oplus \big(H^{*-2}(\Gamma, \Gamma_0)\big)^{-\mathbb{Z}_2} \\
&= H^*((\Sigma\backslash F)/\mathbb{S}^3, (\Delta_0\backslash F)/\mathbb{S}^3) \oplus \big(H^{*-2}(\Gamma, \Gamma_0)\big)^{-\mathbb{Z}_2},
\end{aligned}$$

and then

$$\begin{aligned}
H^*\big(\mathcal{A}^{\cdot}\left(\Sigma\backslash F, \Delta_0\backslash F\right)\big) &= H^*\Big(\frac{\underline{\Omega}^{\cdot}(\Sigma\backslash F, \Delta_0\backslash F)}{\Omega^{\cdot}((\Sigma\backslash F)/\mathbb{S}^3, (\Delta_0\backslash F)/\mathbb{S}^3)}\Big) = \big(H^{*-2}(\Gamma, \Gamma_0)\big)^{-\mathbb{Z}_2} \\
&= \Big(H^{*-2}((\Sigma\backslash F)^{\mathbb{S}^1}, (\Delta_0\backslash F)^{\mathbb{S}^1})\Big)^{-\mathbb{Z}_2} \\
&\overset{\text{excision}}{=\!=\!=} \Big(H^{*-2}(\Sigma^{\mathbb{S}^1}, \Delta_0^{\mathbb{S}^1})\Big)^{-\mathbb{Z}_2} \overset{\text{retraction}}{=\!=\!=} \Big(H^{*-2}(\Sigma^{\mathbb{S}^1}, F^{\mathbb{S}^1})\Big)^{-\mathbb{Z}_2}.
\end{aligned}$$

[7]This map is induced by $j : \mathbb{S}^3 \to \mathbb{S}^3$ defined by $j(u) = u \cdot j$ (see [1, Example 17.23]).

Consider the long exact sequence associated to the $\mathbb{Z}_2$-invariant pair $\big(\Sigma^{\mathbb{S}^1}, F^{\mathbb{S}^1}\big)$:

$$\cdots \to \big(H^{i-1}(F^{\mathbb{S}^1})\big)^{-\mathbb{Z}_2} \to \big(H^i(\Sigma^{\mathbb{S}^1}, F^{\mathbb{S}^1})\big)^{-\mathbb{Z}_2} \to \big(H^i(\Sigma^{\mathbb{S}^1})\big)^{-\mathbb{Z}_2} \to \big(H^i(F^{\mathbb{S}^1})\big)^{-\mathbb{Z}_2} \to \cdots .$$

Since the action of $\mathbb{Z}_2$ on $F^{\mathbb{S}^1} = F$ is trivial, then $\big(H^i(F^{\mathbb{S}^1})\big)^{-\mathbb{Z}_2} = 0$. On the other hand, we have $\Sigma^{\mathbb{S}^1} = M^{\mathbb{S}^1}$. This gives $\big(H^{*-2}(\Sigma^{\mathbb{S}^1}, F^{\mathbb{S}^1})\big)^{-\mathbb{Z}_2} = \big(H^{*-2}(M^{\mathbb{S}^1})\big)^{-\mathbb{Z}_2}$.

(b)

Notice that the connecting morphism δ is defined by
$\delta([\zeta]) = \pm\big[\langle d(\chi_1 \wedge \chi_2 \wedge \chi_3) \wedge \zeta\rangle\big]$. We have $\delta \equiv 0$ since $\zeta_1 = 0$ (cf. (a1)). □

Theorem 2.2. *Given any smooth action $\Phi : \mathbb{S}^3 \times M \longrightarrow M$ we have the Gysin sequence*

$$\cdots \longrightarrow H^i\big(M\big) \longrightarrow H^{i-3}\big(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\big) \oplus \big(H^{i-2}\big(M^{\mathbb{S}^1}\big)\big)^{-\mathbb{Z}_2} \longrightarrow$$
$$\longrightarrow H^{i+1}\big(M/\mathbb{S}^3\big) \longrightarrow H^{i+1}\big(M\big) \longrightarrow \cdots$$

where Σ is the subset of points of M whose isotropy group is infinite, the $\mathbb{Z}_2$-action is induced by $j \in \mathbb{S}^3$ and $(-)^{-\mathbb{Z}_2}$ denotes the subspace of antisymmetric elements.

Proof. Consider the short exact sequence

$$0 \longrightarrow \Omega^*_V(M/\mathbb{S}^3) \longrightarrow \underline{\Omega}^*_V(M) \longrightarrow \frac{\underline{\Omega}^*_V(M)}{\Omega^*_V(M/\mathbb{S}^3)} \longrightarrow 0, \tag{8}$$

take its associated long exact sequence and then, apply Lemma 1.3, (3) and Lemma 2.1. □

2.2. Example. Consider the connected sum $M = \mathbb{CP}^2 \# \mathbb{CP}^2 \cong \big(\mathbb{S}^3 \times [0,1]\big)/\sim$, with

$$((z_1, z_2), i) \sim ((z \cdot z_1, z \cdot z_2), i), \quad i = 0, 1,$$

for all $z \in \mathbb{S}^1$ and $(z_1, z_2) \in \mathbb{S}^3$ in complex coordinates. The product of $\mathbb{S}^3$ induces on M the action:

$$g \cdot [h, t] = [g \cdot h, t], \quad \forall g, h \in \mathbb{S}^3, \forall t \in [0,1].$$

For this action, we have:

$$\Sigma = \big(\mathbb{S}^3 \times \{0,1\}\big)/\sim \;\cong \mathbb{S}^2 \times \{0,1\}, \quad F = \emptyset,$$

$$M/\mathbb{S}^3 \cong [0,1], \quad \Sigma/\mathbb{S}^3 \cong \{0,1\}, \quad M^{\mathbb{S}^1} \cong \{N, S\} \times \{0,1\},$$

where N and S stand for the North and South poles of $\mathbb{S}^2$. The $\mathbb{Z}_2$- action on $M^{\mathbb{S}^1}$ is determined by $j \in \mathbb{S}^3$, which induces the antipodal map on $\mathbb{S}^2$, and so, interchanges its poles. Thus, the exotic term that appears in the central part of the Gysin Sequence is not trivial:

$$H^2(M) \xrightarrow{\cong} \left(H^0\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = \left(H^0\left(\{N,S\} \times \{0,1\}\right)\right)^{-\mathbb{Z}_2} \cong \mathbb{R} \oplus \mathbb{R}.$$

2.3. Morphisms. We describe the morphisms of the Gysin sequence.

$$① : H^*\left(M/\mathbb{S}^3\right) \longrightarrow H^*\left(M\right)$$

It is the pull-back π^* of the canonical projection $\pi : M \to M/\mathbb{S}^3$ (cf. Lemma 1.3).

$$② : H^*\left(M\right) \longrightarrow H^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \oplus \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$$

We have already seen that the first component of this morphism is induced by $⨍_{\mathbb{S}^3}[\omega] = [i_{X_3} i_{X_2} i_{X_1} \omega]$. For the second component we keep track of the isomorphisms given by Lemma 2.1 and we get that it is defined by: $[\omega] \mapsto$ class $\left(⨍_{\mathbb{S}^2}\left(\omega_1 - \sigma(\iota^*\omega_1)\right)\right)$.

$$③ : H^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \oplus \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \longrightarrow H^{*+1}\left(M/\mathbb{S}^3\right)$$

A straightforward calculation using sequences (7) and (8) gives that the connecting morphism ③ of the Gysin sequence sends:

- $[\zeta] \in H^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right)$ to -$\left[(e_1^2 + e_2^2 + e_3^2) \wedge \zeta\right]$, and
- $[\xi] \in \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = \left(H^{*-2}(\Sigma^{\mathbb{S}^1}, F^{\mathbb{S}^1})\right)^{-\mathbb{Z}_2}$ to $[d\sigma \wedge \epsilon \wedge \tau_1^* \xi]$ where ϵ is an Euler form of the restriction $\Phi_1 : \mathbb{S}^1 \times \left(\tau_1^{-1}(\Sigma^{\mathbb{S}^1}) \backslash \Sigma^{\mathbb{S}^1}\right) \to \left(\tau_1^{-1}(\Sigma^{\mathbb{S}^1}) \backslash \Sigma^{\mathbb{S}^1}\right)$ of Φ.

Since $e_1^2 + e_2^2 + e_3^2$ is not a Verona's form, then it does not define a class of $H^4\left(M/\mathbb{S}^3\right)$. Nevertheless, it does generate a class in the intersection cohomology group $\mathbb{H}^4_{\bar{4}}(M/\mathbb{S}^3)$ (as in the semi-free case of [7]).

2.4. Remarks. (a) We have $\left(H^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = H^*\left(M^{\mathbb{S}^1}\right) \Big/ H^*\left(M^{\mathbb{S}^1}/\mathbb{Z}_2\right)$. Let us see that. The correspondence $\omega \mapsto \left(\frac{\omega + j^*\omega}{2}, \frac{\omega - j^*\omega}{2}\right)$ establishes the isomorphism $\Omega^*\left(M^{\mathbb{S}^1}\right) = \left(\Omega^*\left(M^{\mathbb{S}^1}\right)\right)^{\mathbb{Z}_2} \oplus \left(\Omega^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = \Omega^*\left(M^{\mathbb{S}^1}/\mathbb{Z}_2\right) \oplus \left(\Omega^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$ and hence, $H^*\left(M^{\mathbb{S}^1}\right) = H^*\left(M^{\mathbb{S}^1}/\mathbb{Z}_2\right) \oplus \left(H^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$. This gives the claim.

(b) Let us suppose that the action is semi-free, almost free or free. Then, j acts trivially on $M^{\mathbb{S}^1} = F$, and hence, we have a long exact sequence

$$\cdots \to H^i\left(M\right) \to H^{i-3}\left(M/\mathbb{S}^3, F\right) \to H^{i+1}\left(M/\mathbb{S}^3\right) \to H^{i+1}\left(M\right) \to \cdots.$$

(c) Let us suppose that there is not a point of M whose isotropy subgroup is conjugated to $\mathbb{S}^1$. Then, we have a long exact sequence

$$\cdots \to H^i\big(M\big) \to H^{i-3}\big(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\big) \to H^{i+1}\big(M/\mathbb{S}^3\big) \to H^{i+1}\big(M\big) \to \cdots .$$

since j acts trivially on $M^{\mathbb{S}^1} = \big\{x \in M \mid \mathbb{S}^3_x = \mathbb{S}^3 \text{ or } N(\mathbb{S}^1)\big\}$.

2.5. Actions over $\mathbb{S}^1$. Using the Gysin sequence we have constructed, we now give a list of all the different cohomologies of a $\mathbb{S}^3$-manifold M having the circle as orbit space[8]. By geometrical reasons, the orbit space is composed by just one stratum, the whole circle. Following the nature of the orbits, we distinguish four cases.

(a) All orbits are of dimension 3. We have $P_M = 1 + t + t^3 + t^4$. This is the case of the manifold $\mathbb{S}^3 \times \mathbb{S}^1$, where $\mathbb{S}^3$ acts by multiplication on the left factor.

(b) All orbits are isomorphic to $\mathbb{S}^2$. We distinguish two cases following wether the covering $M^{\mathbb{S}^1} \to M^{\mathbb{S}^1}/\mathbb{Z}_2 = M/M^{\mathbb{S}^1}$ is trivial or not. In the first case we have $P_M = 1 + t + t^2 + t^3$. This is the case of the manifold $\mathbb{S}^2 \times \mathbb{S}^1$, where $\mathbb{S}^3$ acts by multiplication on the left factor. In the second case we have $P_M = 1 + t$, as is the case of the manifold $\mathbb{S}^2 \times_{\mathbb{Z}^2} \mathbb{S}^1$ where $\mathbb{S}^3$ acts by multiplication on the left factor.

(c) All orbits are isomorphic to $\mathbb{RP}^2$. In this case, we have $P_M = 1 + t$. This is the case of the manifold $\mathbb{RP}^2 \times \mathbb{S}^1$ where $\mathbb{S}^3$ acts by multiplication on the left factor.

(d) All orbits are points. We have $P_M = 1 + t$. This corresponds to the manifold $\mathbb{S}^1$ where $\mathbb{S}^3$ acts ineffectively.

References

[1] R. Bott and L. W. Tu, Differential forms in Algebraic Topology, Vol. 82, Graduate Texts in Mathematics, *Springer–Verlag*, *New York – Berlin*, 1982.

[2] G. E. Bredon, Introduction to compact transformation groups, Vol. 46, Pure and Applied Mathematics, *Academic Press*, *New York – London*, 1972.

[3] Hua Chen, An analytic model for $\mathbb{S}^3$-equivariant cohomology, Topology Hawaii (Honolulu, HI, 1990), *World Sci. Publ.*, *River Edge, NJ*, 1992, 53–61.

[4] W. Greub, S. Halperin and R. Vanstone, Connections, curvature and cohomology, Vol. II: Lie groups, principal bundles, and characteristic classes, Vol. 47-II, Pure and Applied Mathematics, *Academic Press*, *New York and London*, 1972.

[8] In fact, we give the Poincaré polynomial P_M of M.

[5] X. Masa, Duality and minimality for Riemannian foliations on open manifolds, Differential Geometry – Proceedings of the VIII International Colloquium, (Álvarez López & E. García-Río), eds.), *World Scientific*, 2009, 102–103.

[6] J. I. Royo Prieto, Estudio Cohomológico de flujos riemannianos, *Ph.D. Thesis, University of the Basque Country UPV/EHU*, 2003.

[7] M. Saralegi, A Gysin sequence for semi-free actions of $\mathbb{S}^3$, *Proc. Amer. Math. Soc.* **118**, no. 4 (1993), 1335–1345.

[8] A. Verona, Le théorème de de Rham pour les préstratifications abstraites, *C. R. Acad. Sci. Paris Sér. A–B* **273** (1971), 886–889.

[9] A. Verona, Stratified mappings-structures and triangulability, Vol. 1102, Lecture Notes in Mathematics, *Springer–Verlag*, *Berlin*, 1984.

[10] C. T. Yang, The triangulability of the orbit space of a differentiable transformation group, *Bull. Amer. Math. Soc.* **69** (1963), 405–408.

JOSÉ IGNACIO ROYO PRIETO
MATEMATIKA APLIKATUA SAILA
UNIVERSITY OF THE BASQUE COUNTRY UPV/EHU
ALAMEDA DE URQUIJO S/N
48013 BILBAO
SPAIN

E-mail: joseignacio.royo@ehu.es
URL: http://www.ehu.es/joseroyo

MARTINTXO SARALEGI ARANGUREN
UNIVERSITÉ D'ARTOIS
LABORATOIRE DE
MATHÉMATIQUES DE LENS EA 2462
FÉDÉRATION CNRS NORD-PAS-DE-CALAIS FR 2956
FACULTÉ DES SCIENCES JEAN PERRIN
RUEJEAN SOUVRAZ, S.P. F-62 300 LENS
FRANCE

E-mail: saralegi@euler.univ-artois.fr
URL: http://www.univ-artois.fr/recherche/unites-de-recherche/lml/equipe/ms

(Received February 2, 2012; revised October 28, 2012)