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Generalized near-derivations and their applications in Lie algebras

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Abstract. Let L be a Lie algebra. The aim of this paper is to investigate generalized near-derivations of L, which is a generalization of near-derivation initiated by Brešar in 2008. As an application we determine all linear maps $f : L \to L$ with the property that $[\dots[[f, \delta_1], \delta_2], \dots, \delta_n]$ is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of L, where n is a fixed positive integer.

1. Introduction

Let n be a fixed positive integer. Let R be a not necessarily associative algebra with multiplication $\cdot$. Recall that a linear map $\delta : R \to R$ is said to be a derivation if $\delta(x \cdot y) = \delta(x) \cdot y + x \cdot \delta(y)$ for all $x, y \in R$. A linear map $f : R \to R$ is said to be a *generalized derivation* if there exist linear maps $g, h : R \to R$ such that

$$f(x) \cdot y = g(x \cdot y) + x \cdot h(y) \quad \text{for all } x, y \in R.$$

Leger and Luks [4] investigated generalized derivations of Lie algebras. As it is well known, $[\delta', \delta] = \delta'\delta - \delta\delta'$ is a derivation whenever δ and δ' are derivations. Is it possible to determine all linear maps $f : R \to R$ with the property that $[f, \delta]$ is a derivation whenever δ is a derivation? Brešar [2] discussed this question in Lie algebras. He answered this question by introducing an interesting concept of near-derivations in Lie algebras.

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Let L be a Lie algebra over a field $\mathbb{F}$. By ad$x$ we denote the inner derivation induced by $x \in L$, i.e. $(\operatorname{ad} x)(y) = [x, y]$ for all $y \in L$. A linear map $f : L \to L$ is said to be a *near-derivation* of L if there exists a linear map $g : L \to L$ such that

$$(\operatorname{ad} x)f - g(\operatorname{ad} x)$$

is a derivation for every $x \in L$ (see [2, Section 1]). Note that every generalized derivation is a near-derivation. Brešar gave a description of f in certain Lie algebras that arise from associative ones. The typical result in [2] states that a near-derivation f of L is of the form $f = \delta + \gamma I + \tau$, where δ is a derivation, γ is an element in the center C of a certain associative algebra containing L, and τ is a linear map from L into C.

It is natural for us to consider the following general question: How to determine all linear maps $f : L \to L$ with the property that $[\dots[[f, \delta_1], \delta_2], \dots, \delta_n]$ is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of L?

We shall solve this problem on Lie algebras. For this purpose we shall extend the definition of near-derivations. Let Q be a unital associative algebra, containing L as its Lie subalgebra (the Q always exists). First, we give a slight generalization of near-derivations as follows: A linear map $f : L \to L$ is said to be a *near-derivation* if there exists a linear map $g : L \to Q$ such that

$$(\operatorname{ad} x)f - g(\operatorname{ad} x)$$

is a derivation for every $x \in L$. It is easy to see that the typical result $\delta + \gamma I + \tau$ in [2] is a near derivation in the above sense. Now, we say that a linear map $f : L \to L$ is a *generalized near-derivation* if there exists a linear map $g : L \to Q$ such that

$$(\operatorname{ad} x)f - g(\operatorname{ad} x)$$

is a near-derivation for every $x \in L$. It is easy to see that every near-derivation is a generalized near-derivation.

We shall apply the powerful theory of functional identities [3] to the descriptions of generalized near-derivations in certain Lie algebras that arise from associative ones.

2. Functional identities preliminaries

Let Q be a unital algebra and let S be a d-free subset of Q for some positive integer d. Denote by P as the set of all quasi-polynomials. We refer the reader to the recent book [3] for the basic properties of d-free subsets and quasi-polynomials, upon which the present paper heavily rests.

The following result is a slight generalization of [2, Lemma 2].

Lemma 2.1. *Let Q be a unital algebra and let S be a subset of Q. Let $B : S \times S \to Q$ be a skew-symmetric map. Suppose that*

$$[B(x, y), z] + [B(z, x), y] + [B(y, z), x] \in P$$

for all $x, y, z \in S$. If S is a 4-free subset of Q, then there exist $\lambda \in C$, the center of Q, a linear map $\mu : S \to C$, and a skew-symmetric map $\nu : S^2 \to C$ such that

$$B(x, y) = \lambda[x, y] + \mu(x)y - \mu(y)x + \nu(x, y) \quad \text{for all } x, y \in S.$$

Proof. Using [3, Theorem 4.13] it follows that B is a quasi-polynomial. This means that there exist $\lambda_1, \lambda_2 \in C$ and maps $\mu_1, \mu_2 : S \to C$, $\nu : S^2 \to C$ such that

$$B(x, y) = \lambda_1 xy + \lambda_2 yx + \mu_1(x)y + \mu_2(y)x + \nu(x, y).$$

Since $B(x, y) = -B(y, x)$ it follows that

$$(\lambda_1 + \lambda_2)(xy + yx) + (\mu_1 + \mu_2)(x)y + (\mu_1 + \mu_2)(y)x + \nu(x, y) + \mu(y, x) = 0.$$

But then $\lambda_1 = -\lambda_2$, $\mu_1 = -\mu_2$ and ν is skew-symmetric [3, Lemma 4.4]. Setting $\lambda = \lambda_1$ and $\mu = \mu_1$ we thus have

$$B(x, y) = \lambda[x, y] + \mu(x)y - \mu(y)x + \nu(x, y) \quad \text{for all } x, y \in S. \qquad \square$$

Let A be a prime associative algebra. By $Q_{ml}(A)$ we denote the maximal left ring of quotients of A (see [1, Chapter 2] or [3, Appendix A]). The center C of $Q_{ml}(A)$ is a field called the extended centroid of A. By $\deg(x)$ we denote the degree of the algebraicity of $x \in A$ over C. If x is not algebraic, then we write $\deg(x) = \infty$. Further, we set

$$\deg(A) = \sup\{\deg(x) \mid x \in A\}.$$

The condition that $\deg(A) = \infty$ is equivalent to the condition that A is not a PI-algebra, while the condition that $\deg(A) = n < \infty$ is equivalent to the condition that A is a PI-algebra satisfying the standard polynomial identity of degree $2n$, but not satisfying a polynomial identity of degree $< 2n$. If A is a central simple algebra, then $\deg(A) = \infty$ is the same as saying that A is infinite-dimensional over $\mathbb{F}$, while $\deg(A) = n < \infty$ is equivalent to $\dim_{\mathbb{F}} A = n^2$ [3, Appendix C]. If $\deg(A) \geq d + 1$, then every noncommutative Lie ideal L of A is a d-free subset of $Q_{ml}(A)$ [3, Corollary 5.16]. If $\deg(A) \geq 2d + 3$, A has an involution and K is the set of skew elements in A, then every noncentral Lie ideal L of K is a d-free subset of $Q_{ml}(A)$ [3, Corollary 5.19].

3. Generalized near-derivations on Lie algebras

We begin with the following crucial result.

Lemma 3.1. *Let L be a Lie algebra with $\mathrm{char}(\mathbb{F}) \neq 2, 3$ and let Q be a unital associative algebra, containing L as its Lie subalgebra. Let f be a generalized near-derivation of L. Suppose L is a 4-free subset of Q. Then there exist $\gamma \in C$, the center of Q, and a skew-symmetric bilinear map $\beta : L^2 \to C$ such that*
$$(f + \gamma I)([x, y]) = [f(x), y] + [x, f(y)] + \beta(x, y) \quad \textit{for all } x, y \in L.$$

Proof. Our assumption is that there exists $g : L \to Q$ such that $(\mathrm{ad}\, x)f - g(\mathrm{ad}\, x)$ is a near-derivation for every $x \in L$. In view of [2, Lemma 3.1] we have that there exists $\lambda_x \in C$ (depending on x) such that

$$\begin{aligned}((\mathrm{ad}\, x)f - g(\mathrm{ad}\, x) + \lambda_x I)([y, z]) - [((\mathrm{ad}\, x)f - g(\mathrm{ad}\, x))(y), z]& \\ - [y, ((\mathrm{ad}\, x)f - g(\mathrm{ad}\, x))(z)] \in C.&\end{aligned}$$

That is,

$$\begin{aligned}[x, f([y, z])] - g([x, [y, z]]) + \lambda_x[y, z] - [[x, f(y)] - g([x, y]), z]& \\ - [y, [x, f(z)] - g([x, z])] \in C \quad (1)&\end{aligned}$$

for all $x, y, z \in L$. Now define $\theta : L \to C$ by the rule

$$\theta(x) = \lambda_x \quad \text{for all } x \in L.$$

We claim that θ is well-defined. It is enough to show that $x = 0$ implies $\lambda_x = 0$. Picking $x = 0$ in (1) we obtain $\lambda_0[y, z] \in C$ for all $y, z \in L$. Applying [3, Lemma 4.4] we get $\lambda_0 = 0$ as desired. Hence, the identity (1) can be rewritten as

$$\begin{aligned}[x, f([y, z])] - g([x, [y, z]]) + \theta(x)[y, z] - [[x, f(y)] - g([x, y]), z]& \\ - [y, [x, f(z)] - g([x, z])] \in C \quad (2)&\end{aligned}$$

for all $x, y, z \in L$. In view of the Jacobi identity we have

$$g([x, [y, z]]) + g([z, [x, y]]) + g([y, [z, x]]) = 0;$$

according to (2) this can be rewritten as

$$\begin{aligned}[x, f([y, z])] &- [[x, f(y)], z] + [g([x, y]), z] - [y, [x, f(z)]] + [y, g([x, z])] \\ &+ [z, f([x, y])] - [[z, f(x)], y] + [g([z, x]), y] - [x, [z, f(y)]] \\ &+ [x, g([z, y])] + [y, f([z, x])] - [[y, f(z)], x] + [g([y, z]), x] \\ &- [z, [y, f(x)]] + [z, g([y, x])] + \theta(x)[y, z] \\ &+ \theta(z)[x, y] + \theta(y)[z, x] \in C.\end{aligned}$$

Rearranging the terms we get

$$\begin{aligned}[(2g-f)([y,z]) &- [f(y),z] - [y,f(z)],x] \\ &+ [(2g-f)([x,y]) - [f(x),y] - [x,f(y)],z] \\ &+ [(2g-f)([z,x]) - [f(z),x] - [z,f(x)],y] \\ &+ \theta(x)[y,z] + \theta(z)[x,y] + \theta(y)[z,x] \in C \end{aligned} \tag{3}$$

for all $x, y, z \in L$. That is,

$$\begin{aligned}[(2g-f)([y,z]) &- [f(y),z] - [y,f(z)],x] \\ &+ [(2g-f)([x,y]) - [f(x),y] - [x,f(y)],z] \\ &+ [(2g-f)([z,x]) - [f(z),x] - [z,f(x)],y] \in P \end{aligned}$$

for all $x, y, z \in L$. We are now in a position to apply Lemma 2.1. Thus there exist $\lambda \in C$, $\mu_1 : L \to C$, and a skew-symmetric map $\nu : L^2 \to C$ such that

$$(2g-f)([x,y]) - [f(x),y] - [x,f(y)] = \lambda[x,y] + \mu_1(x)y - \mu_1(y)x + \nu(x,y). \tag{4}$$

Substituting (4) into (3) we obtain

$$(2\mu_1(x) + \theta(x))[y,z] + (2\mu_1(z) + \theta(z))[x,y] + (2\mu_1(y) + \theta(y))[z,x] \in C.$$

Applying [3, Lemma 4.4] it follows that $2\mu_1(x) + \theta(x) = 0$ for all $x \in L$ and hence $\mu_1 = -\frac{1}{2}\theta$. Thus, the expression (4) becomes

$$(2g-f)([x,y]) - [f(x),y] - [x,f(y)] = \lambda[x,y] - \frac{1}{2}\theta(x)y + \frac{1}{2}\theta(y)x + \nu(x,y). \tag{5}$$

Set $h = 2g - f - \lambda I$. Then

$$h([x,y]) - [f(x),y] - [x,f(y)] = -\frac{1}{2}\theta(x)y + \frac{1}{2}\theta(y)x + \nu(x,y). \tag{6}$$

Since

$$h([x,[y,z]]) + h([z,[x,y]]) + h([y,[z,x]]) = 0$$

for all $x, y, z \in L$, we get from (6) that

$$\begin{aligned}[f(x),[y,z]] &+ [x,f([y,z])] + [f(z),[x,y]] + [z,f([x,y])] + [f(y),[z,x]] \\ &+ [y,f([z,x])] - \frac{1}{2}\theta(x)[y,z] + \frac{1}{2}\theta([y,z])x - \frac{1}{2}\theta(z)[x,y] \\ &+ \frac{1}{2}\theta([x,y])z - \frac{1}{2}\theta(y)[z,x] + \frac{1}{2}\theta([z,x])y \\ &+ \nu(x,[y,z]) + \nu(z,[x,y]) + \nu(y,[z,x]) = 0. \end{aligned}$$

We rewrite this as

$$\begin{aligned}
&[f([y,z]) - [f(y),z] - [y,f(z)],x] + [f([x,y]) - [f(x),y] - [x,f(y)],z] \\
&\quad + [f([z,x]) - [f(z),x] - [z,f(x)],y] - \frac{1}{2}\theta(x)[y,z] + \frac{1}{2}\theta([y,z])x \\
&\quad - \frac{1}{2}\theta(z)[x,y] + \frac{1}{2}\theta([x,y])z - \frac{1}{2}\theta(y)[z,x] + \frac{1}{2}\theta([z,x])y \\
&\quad + \nu(x,[y,z]) + \nu(z,[x,y]) + \nu(y,[z,x]) = 0. \qquad (7)
\end{aligned}$$

That is,

$$\begin{aligned}
&[f([y,z]) - [f(y),z] - [y,f(z)],x] + [f([x,y]) - [f(x),y] - [x,f(y)],z] \\
&\qquad + [f([z,x]) - [f(z),x] - [z,f(x)],y] \in P.
\end{aligned}$$

Applying Lemma 2.1 again we get

$$f([x,y]) - [f(x),y] - [x,f(y)] = \alpha[x,y] + \mu_2(x)y - \mu_2(y)x + \beta(x,y) \qquad (8)$$

for some $\alpha \in C$, $\mu_2 : L \to C$, and skew-symmetric map $\beta : L^2 \to C$. It is clear that the linearity of f implies the bilinearity of β.

Substituting (8) into (7) we obtain

$$\begin{aligned}
&(2\mu_2(x) - \frac{1}{2}\theta(x))[y,z] + (2\mu_2(z) - \frac{1}{2}\theta(z))[x,y] + (2\mu_2(y) - \frac{1}{2}\theta(y))[z,x] \\
&\qquad + \frac{1}{2}\theta([y,z])x + \frac{1}{2}\theta([x,y])z + \frac{1}{2}\theta([z,x])y \in C.
\end{aligned}$$

Applying [3, Lemma 4.4] it follows that $2\mu_2(x) - \frac{1}{2}\theta(x) = 0$ for all $x \in L$ and hence $\mu_2 = \frac{1}{4}\theta$. Thus, the expression (8) becomes

$$f([x,y]) - [f(x),y] - [x,f(y)] = \alpha[x,y] + \frac{1}{4}\theta(x)y - \frac{1}{4}\theta(y)x + \beta(x,y) \qquad (9)$$

We claim that $\theta = 0$. Indeed, subtracting (9) from (5) we get

$$2(g-f)([x,y]) = (\lambda - \alpha)[x,y] - \frac{3}{4}\theta(x)y + \frac{3}{4}\theta(y)x + \nu(x,y) - \beta(x,y).$$

Set $k = 2g - 2f - (\lambda - \alpha)I$. Then

$$k([x,y]) = -\frac{3}{4}\theta(x)y + \frac{3}{4}\theta(y)x + \nu(x,y) - \beta(x,y). \qquad (10)$$

Since

$$k([x,[y,z]])+k([z,[x,y]])+k([y,[z,x]])=0$$

by the Jacobi identity, we get from (10) that

$$-\theta(x)[y,z]+\theta([y,z])x-\theta(z)[x,y]+\theta([x,y])z-\theta(y)[z,x]+\theta([z,x])y\in C.$$

Applying [3, Lemma 4.4] it follows that $\theta(x)=0$ for all $x\in L$.

Setting $\gamma=-\alpha$ we can get from (9) that

$$(f+\gamma I)([x,y])=[f(x),y]+[x,f(y)]+\beta(x,y)\quad\text{for all } x,y\in L.$$

This proves the lemma. □

We denote by $\mathrm{H}_2(L,\mathbb{F})$ the second cohomology group of L. Applying Lemma 3.1 we have the following:

Theorem 3.1. *Let L be a Lie algebra with $\mathrm{char}(\mathbb{F})\neq 2,3$ and $H_2(L,\mathbb{F})=0$. Let Q be a unital associative algebra, containing L as its Lie subalgebra. Let f be a generalized near-derivation of L. Suppose L is a 4-free subset of Q. Then there exist $\gamma\in C$, the center of Q, a derivation $\delta: L\to Q$ and a linear map $\tau: L\to C$ such that $f=\delta+\gamma I+\tau$.*

Proof. By Lemma 3.1 the map $d=f-\gamma I: L\to CL\subseteq Q$ satisfies

$$d([x,y])-[d(x),y]-[x,d(y)]=\beta(x,y)\in C$$

for all $x,y\in L$. Consequently,

$$\beta(x,[y,z])=d([x,[y,z]])-[d(x),[y,z]]-[x,[d(y),z]]-[x,[y,d(z)]],$$

since $[x,\beta(y,z)]=0$. Using the Jacobi identity it readily follows that

$$\beta(x,[y,z])+\beta(z,[x,y])+\beta(y,[z,x])=0.$$

Since $\mathrm{H}^2(L,\mathbb{F})=0$ then exists a linear map $\tau: L\to C$ such that $\beta(x,y)=\tau([x,y])$ for all $x,y\in L$ (see [2, P. 3769]). That is,

$$d([x,y])-[d(x),y]-[x,d(y)]=\tau([x,y]).$$

It follows that $\delta=d-\tau$ is a derivation from L into Q. □

Let L be a Lie algebra over $\mathbb{F}$. We denote $\mathrm{End}(L)$ by the $\mathbb{F}$-linear space of all $\mathbb{F}$-linear transformations on L. Set

$$\mathrm{Cent}(L) = \{\chi \in \mathrm{End}(L) \mid \chi([x,y]) = [\chi(x), y] = [x, \chi(y)] \quad \text{for all } x, y \in L\}.$$

We call $\mathrm{Cent}(L)$ the *centroid* of L. Applying Lemma 3.1 we also have the following:

Theorem 3.2. *Let L be a Lie algebra with* $\mathrm{char}(\mathbb{F}) \neq 2, 3$. *Let Q be a unital associative algebra, containing L as its Lie subalgebra. Assume further that L has trivial center and $[L, L] = L$. Let f be a generalized near-derivation of L. Suppose that L is a 4-free subset of Q. Then there exist a derivation $\delta : L \to L$ and $\zeta \in \mathrm{Cent}(L)$, the centroid of L, such that $f = \delta + \zeta$.*

Proof. Lemma 3.1 implies that

$$\gamma[[x,y],z] = [[f(x),y],z] + [[x,f(y)],z] - [f([x,y]),z] \in L$$

for all $x, y, z \in L$. Since $[L, L] = L$, and hence also $[[L, L], L] = L$, it follows that $\gamma L \subseteq L$. That is, the map $\zeta : x \mapsto \gamma x$ maps L into L and so $\zeta \in \mathrm{Cent}(L)$. Further,

$$\beta(x,y) = f([x,y]) + \gamma[x,y] - [f(x),y] - [x,f(y)]$$

then lies in $L \cap C$ which is zero since L has trivial center. But then $\delta = f - \zeta$ is a derivation of L. $\square$

Applying Lemma 3.1 and using the same arguments as in the corresponding results in [2, Section 3], we can obtain the following results. We omit their proofs for brevity.

Corollary 3.1. *Let A be a central simple algebra with* $\mathrm{char}(\mathbb{F}) \neq 2, 3$. *Suppose that one of the following conditions is satisfied:*

(i) $\dim_{\mathbb{F}} A \geq 25$ *and* $1 \in L = [A, A]$;

(ii) *Suppose that A has an involution of the first kind and* $\dim_{\mathbb{F}} A \geq 121$. *Let K be the set of skew elements in A, and set $L = [K, K]$.*

Then every generalized near-derivation f of L is of the form $f = \delta + \gamma I$ where δ is a derivation of L and $\gamma \in \mathbb{F}$;

Corollary 3.2. *Let A be a prime algebra with* $\mathrm{char}(\mathbb{F}) \neq 2, 3$, *let C be the extended centroid of A. Suppose that one of the following conditions is satisfied:*

(i) *Let L be a noncommutative Lie ideal of A. Suppose that* $\deg(A) \geq 5$;

(ii) *Suppose that* A *has an involution and* $\deg(A) \geq 11$. *Let* K *be the skew elements of* A, *and let* L *be a noncentral Lie ideal of* K.

If f *is a generalized near-derivation of* L, *then there exist an (associative) derivation* $\delta : \langle L\rangle \to \langle L\rangle C + C$, $\gamma \in C$, *and a linear map* $\tau : L \to C$ *such that* $f = \delta + \gamma I + \tau$.

4. An application of generalized near-derivations

As an application of generalized near-derivations we shall answer the question posed in Section 1. More precisely, we have the following result.

Theorem 4.1. *Let* L *be a Lie algebra with* $\mathrm{char}(\mathbb{F}) \neq 2,3$ *and* $H_2(L,\mathbb{F}) = 0$. *Let* f *be a linear maps of* L *with the property that*

$$[\ldots[[f,\delta_1],\delta_2],\ldots,\delta_n]$$

is a derivation whenever $\delta_1,\delta_2,\ldots,\delta_n$ *are derivations of* L. *Suppose there exists a unital associative algebra* Q, *containing* L *as its Lie subalgebra, such that* L *is a* 4*-free subset of* Q. *Then there exist* $\gamma \in C$, *the center of* Q, *a derivation* $\delta : L \to Q$ *and a linear map* $\tau : L \to C$ *such that* $f = \delta + \gamma I + \tau$.

Proof. For every $x_1, x_2, \ldots, x_n \in L$ we get from our assumption that

$$[\ldots[[f,\operatorname{ad} x_1],\operatorname{ad} x_2],\ldots,\operatorname{ad} x_n]$$

is a derivation and so a near derivation. That is

$$[\ldots[[f,\operatorname{ad} x_1],\operatorname{ad} x_2],\ldots,\operatorname{ad} x_{n-1}]$$

is a generalized near-derivation. By Theorem 3.1 we get that

$$[\ldots[[f,\operatorname{ad} x_1],\operatorname{ad} x_2],\ldots,\operatorname{ad} x_{n-1}]$$

is also a near derivation. Following the same process we finally obtain that f is a near-derivation. Then the result follows from Theorem 3.1. □

Similarly, applying the corresponding results in the above section we can obtain the following results. We omit their proofs for brevity.

Theorem 4.2. *Let* L *be a Lie algebra with* $\mathrm{char}(\mathbb{F}) \neq 2,3$ *such that* L *has trivial center and* $[L,L] = L$. *Let* f *be a linear maps of* L *with the property that*

$$[\ldots[[f,\delta_1],\delta_2],\ldots,\delta_n]$$

is a derivation whenever $\delta_1,\delta_2,\ldots,\delta_n$ *are derivations of* L. *Suppose that there* *exists a unital associative algebra Q , containing L as its Lie subalgebra, such that L is a 4-free subset of Q. Then there exist a derivation $\delta : L \to L$ and $\zeta \in \mathrm{Cent}(L)$, the centroid of L, such that $f = \delta + \zeta$.*

Corollary 4.1. *Let A be a central simple algebra with $\mathrm{char}(\mathbb{F}) \neq 2, 3$. Suppose that one of the following conditions is satisfied:*

(i) $\dim_{\mathbb{F}} A \geq 25$ *and* $1 \in L = [A, A]$*;*

(ii) *Suppose that A has an involution of the first kind and $\dim_{\mathbb{F}} A \geq 121$. Let K be the set of skew elements in A, and $L = [K, K]$.*

If f is a linear map of L with the property that

$$[\dots[[f, \delta_1], \delta_2], \dots, \delta_n]$$

is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of L, then $f = \delta + \gamma I$ where δ is a derivation of L and $\gamma \in \mathbb{F}$.

Corollary 4.2. *Let A be a prime algebra with $\mathrm{char}(\mathbb{F}) \neq 2, 3$, let C be the extended centroid of A. Suppose that one of the following conditions is satisfied:*

(i) *Suppose that L is a noncommutative Lie ideal of A and $\deg(A) \geq 5$;*

(ii) *Suppose that A has an involution with $\deg(A) \geq 11$, K is the skew elements of A and let L be a noncentral Lie ideal of K.*

If f is a linear map of L with the property that

$$[\dots[[f, \delta_1], \delta_2], \dots, \delta_n]$$

is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of L, then there exist an (associative) derivation $\delta : \langle L\rangle \to \langle L\rangle C + C$, $\gamma \in C$, and a linear map $\tau : L \to C$ such that $f = \delta + \gamma I + \tau$.

Let us show that there exists a generalized near-derivation that is not a near-derivation.

Let L be the usual Lie algebra of all 4×4 strict upper triangular matrices over $\mathbb{F}$, i.e.,

$$L = \Big\{ \sum_{1 \leq i < j \leq 4} a_{ij} e_{ij} \mid a_{ij} \in \mathbb{F} \Big\}.$$

Let φ be a linear functional on L such that $\varphi(e_{14}) = 1$. Now define $f : L \to L$ by $f(y) = \varphi(y) e_{24}$. We claim that f is not a near-derivation of L. Indeed, for every linear map $g : L \to L$ we have

$$((\mathrm{ad}\, e_{12}) f - g(\mathrm{ad}\, e_{12}))([e_{13}, e_{34}]) = e_{14}.$$

On the other hand, we have

$$[((\operatorname{ad} e_{12})f - g(\operatorname{ad} e_{12}))(e_{13}), e_{34}] + [e_{13}, ((\operatorname{ad} e_{12}) - g(\operatorname{ad} e_{12}))(e_{34})] = 0.$$

This implies that $(\operatorname{ad} e_{12})f - g(\operatorname{ad} e_{12})$ is not a derivation of L. However, since $(\operatorname{ad} x)(\operatorname{ad} y)f = 0$ for all $x, y \in L$, we get that $(\operatorname{ad} y)f$ is a near-derivation for every $y \in L$ and so f is a generalized near-derivation.

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