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Homotopy finiteness theorems for Finsler manifolds

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Abstract. In this paper, we consider generalized LGC spaces whose metrics are nonreversible and show the compactness of such spaces in the generalized Gromov–Hausdorff topology. On the basis of these, we derive some homotopy finiteness theorems for Finsler manifolds, which are extensions of Yamaguchi's results.

1. Introduction

Finiteness theorems are theorems giving bounds on certain geometrical quantities such that the family of manifolds admitting metrics which satisfy the bounds is finite up to homotopy equivalence, or homeomorphism, of diffeomorphism. The finiteness problem in Riemannian geometry has been studied extensively (cf. [1], [4], [8], [9], [10], [11], [26]).

In [26], Yamaguch considered the contractibility radius c_M of a Riemannian manifold M (cf. [11], [15]) and found that two Riemannian manifolds have the same homotopy type if they are close enough in the Gromov–Hausdorff metric and have the same lower bound for the contractibility radii (see [26, Theorem 2]). Then he obtained the following homotopy finiteness results:

1. Given positive constants k, D and R, the class of closed Riemannian n-manifolds (M, g) with $\mathbf{Ric}_M \geq -(n-1)k^2$, $\mathrm{diam}(M) \leq D$ and $c_M \geq R$ contains at most finitely many homotopy types.

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2. Given positive constants i and V, the class of closed Riemannian n-manifolds with $\mathfrak{i}_M \geq i$ and $\mathrm{Vol}(M) \leq V$ contains at most finitely many homotopy types.

In [10], Grove, Petersen and Wu considered LGC spaces (cf. [8], [16]) and investigated the convergence of such spaces in the Gromov–Hausdorff topology. Then they gave different proofs of Yamaguch's results. See [8], [10], [15], [16] for more details.

Finsler geometry is Riemannian geometry without quadratic restriction. Instead of a Euclidean norm on each tangent space one endows Minkowski norms on every tangent space of a differentiable manifold. A reversible Finsler manifold is a metric space in a usual sense. Using the Gromov–Hausdorff distance and the theory of LGC spaces, Shen in [22], [24] considered the topological finiteness problem for a special class of reversible Finsler manifolds. The purpose of present paper is to continue the discussion on this topic in the general case.

There are infinitely many nonreversible (or non-Riemannian) Finsler metrics. For example, a Randers metric in the form $F = \alpha + \beta$ is non-reversible, where α is a Riemannian metric and β is a 1-form. Nonreversible Finsler manifolds are not metric spaces but general metric spaces (see Section 2), and therefore, one cannot investigate their homotopy types by the ordinary Gromov–Hausdorff distance.

The *reversibility* λ_F ([18]) and the *uniformity constant* Λ_F ([6]) of a Finsler metric F are defined by

$$\lambda_F := \sup_{y \in SM} F(-y), \quad \Lambda_F := \sup_{X,Y,Z \in SM} \frac{g_Z(X,X)}{g_Y(X,X)}.$$

It is easy to see that $1 \leq \lambda_F \leq \sqrt{\Lambda_F}$. $\lambda_F = 1$ if and only if F is reversible; $\Lambda_F = 1$ if and only if F is Riemannian.

We now extend the concept of contractibility radius to Finsler manifolds, which is a natural generalization of the original definition ([7]). Given a forward complete Finsler manifold (M, F), let d denote the general metric induced by F. A point $q \in M$ is *regular* for $r_p(\cdot) := d(p, \cdot)$ if and only if there exists a unit vector $v \in S_qM$ and a $\eta > 0$ such that

$$r_p(\gamma_v(t)) \geq r_p(\gamma_v(0)) + \eta \cdot t$$

for all sufficiently small $t > 0$, where $\gamma_v(t) := \exp_q(tv)$. If $p \in M$ is not a regular point for r_p, then it is called a *critical point*. The *contractibility radius* c_M is defined as the supremum of r such that every forward metric ball of radius r contains no critical points of the distance function from the center. Let $\mathbf{S}$ be the S-curvature of a volume form and let $\mu(M)$ denote either the Busemann–Hausdorff volume or the Holmes–Thompson volume of M. Then we have the following

Theorem 1.1. *Given n and positive numbers k, D, R, δ, the class of closed Finsler n-manifolds (M, F) with $\mathbf{Ric}_M \geq -(n-1)k^2$, $\operatorname{diam}(M) \leq D$, $c_M \geq R$ and $\Lambda_F \leq \delta$, contains at most finitely many homotopy types.*

Theorem 1.2. *Given n and positive numbers k, h, D, R, δ, the class of closed Finsler n-manifolds (M, F) with $\mathbf{Ric}_M \geq -(n-1)k^2$, $\mathbf{S}_M \geq (n-1)h$, $\operatorname{diam}(M) \leq D$, $c_M \geq R$ and $\lambda_F \leq \delta$, contains at most finitely many homotopy types.*

Theorem 1.3. *Given n and positive numbers i, V, δ, the class of closed reversible Finsler n-manifolds (M, F) with $\mathrm{i}_M \geq i$, $\mu(M) \leq V$ and $\Lambda_F \leq \delta$, contains at most finitely many homotopy types.*

In the Riemannian case, $\lambda_F = \Lambda_F = 1$ and $\mathbf{S}_M = 0$. Hence, Theorem 1.1, 1.2 and 1.3 imply Yamaguch's results.

The arrangement of contents of this paper is as follows. In Section 2, we study the properties of general metric spaces and the generalized Gromov–Hausdorff distance. In Section 3, we extend Hausdorff dimension to general metric spaces and discuss the relation between covering dimension and generalized Hausdorff dimension. In Section 4, we consider general metric spaces which are LGC and investigate the homotopy types and the convergence of such spaces in the generalized Gromov–Hausdorff topology. On the base of these, we prove Theorem 1.1–1.3 in Section 5.

2. General metric space

To understand how geometric constraints on Finsler manifolds give rise to topological constraints, we considered general metric spaces in [20]. In this section, we continue to study the properties of such spaces. For more details we refer to [3], [11], [20].

A *general metric space* is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}^+ \cup \{\infty\}$, called a metric, is a function, satisfying the following two conditions for all $x, y, z \in X$:

(i) $d(x, y) \geq 0$, with equality $\Leftrightarrow x = y$;

(ii) $d(x, y) + d(y, z) \geq d(x, z)$.

All the spaces under our consideration are general metric spaces, and they are simply called metric spaces. In a metric space X we define the *forward* (resp. *backward*) *ε-ball*, $\varepsilon > 0$, about $x \in X$ to be $B^+_x(\varepsilon) := \{y \in X \mid d(x,y) < \varepsilon\}$ (resp. $B^-_x(\varepsilon) := \{y \in X \mid d(y,x) < \varepsilon\}$). A subset $U \subset X$ is said to be *open* if, for each point $x \in U$, there is an forward ε-ball about x contained in U. Then we get the topology on X. We always assume that the metric d of a metric space (X,d) is continuous with respect to the product topology on $X \times X$. Thus, every backward ε-ball is open and the metric space is a Hausdorff (T_2-) space.

The *reversibility* λ_X of X is defined by

$$\lambda_X := \sup_{\substack{x,y\in X,\\ x\neq y}} \frac{d(x,y)}{d(y,x)}.$$

The metric spaces with reversibility 1 are called *reversible (metric) spaces.*

Let $\mathcal{M}^\delta$ be the collection of compact metric spaces with reversibility $\leq \delta$. Given a subset A in a metric space, the diameter of A is defined as $\operatorname{diam}(A) := \sup_{x,y\in A} d(x,y)$. Then we have the following

Lemma 2.1. *Given a metric space $X \in \mathcal{M}^\delta$, let $\{U_\alpha\}$ be a finite open covering of X. There exists $\eta > 0$ such that for each subset $A \subset X$ with $\operatorname{diam}(A) < \eta$, $A \subset U_\alpha$ for some α.*

Proof. If not, there would be a sequence of subsets $\{A_i\}$ with $\operatorname{diam}(A_i) < 1/i$ such that A_i is not contained in any U_α. Select a point $x_i \in A_i$. By [20, Theorem 2.11], we can suppose that $\{x_i\}$ converges to a point $x \in X$. Then there exists some U_β such that $x \in U_\beta$. Since U_β is open, $r = d(x, X - U_\beta) > 0$. By [20, Proposition 2.2], there is $N > 0$ such that for $i > N$, $d(x,x_i) + 1/i < r/2$. Hence, for $i > N$, $A_i \subset B^+_x(r/2) \subset U_\beta$, which is a contradiction. □

Proposition 2.2. *Let X, Y be two metric spaces with $X \in \mathcal{M}^\delta$ and $\lambda_Y < \infty$. If $f : X \to Y$ is a continuous map, then f is uniformly continuous.*

Proof. Given $\epsilon > 0$. For each $x \in X$, there exists $\eta_x > 0$ such that $f(B^+_x(\eta_x)) \subset B^+_{f(x)}\left(\frac{\epsilon}{2\lambda_Y}\right)$. Since X is compact, there exists finitely many points $x_1, \ldots, x_n$ such that $\{B^+_{x_i}(\eta_{x_i})\}_{i=1}^n$ is an open covering of X. The conclusion now follows from the lemma above. □

Given a metric space X and two subsets $A, B \subset X$, the *Hausdorff distance* between A and B is defined by

$$d_H(A,B) := \inf\{\epsilon : A \subset B^+(B,\epsilon),\ B \subset B^+(A,\epsilon)\}.$$

where $B^+(A,\epsilon) := \{x \in X : d(A,x) < \epsilon\}$ and $d(A,B) := \inf\{d(a,b) : a \in A, b \in B\}$.

Given two metric spaces X, Y in $\mathcal{M}^\delta$. An *admissible metric* on the disjoint union $X \sqcup Y$ is a metric that extends the given metrics on X and Y. A *δ-admissible metric* is an admissible metric with reversibility $\leq \delta$. The *δ-Gromov–Hausdorff distance* between X and Y is defined as

$$d^\delta_{GH}(X,Y) := \inf\{d_H(X,Y) : \delta\text{-admissible metrics on } X \sqcup Y\}.$$

By [20, Proposition 3.7], $(\mathcal{M}^\delta, d^\delta_{GH})$ is a pseudometric space. Moreover, if we consider equivalence classes of isometric spaces, then it becomes a reversible metric space.

A sequence $\{X_n\}_{n=1}^\infty \subset M^\delta$ converges to a compact metric space $X \in M^\delta$ if $d^\delta_{GH}(X_n, X) \to 0$ as $n \to \infty$. In this case, we will write $X_n \xrightarrow{\delta-GH} X$ and call X *the δ-Gromov–Hausdorff limit* of $\{X_n\}$. First, we have the following observation.

Proposition 2.3. *If a sequence $\{X_n\}_{n=1}^\infty \subset M^\delta$ converges to a compact metric space $X \in M^\delta$, then there exists a δ-admissible metric d on $Y = \sqcup_i X_i \sqcup X$ such that (Y,d) is forward complete.*

Proof. Since $X_i \xrightarrow{\delta-GH} X$, we can choose a sequence $\{\epsilon_i\}$ converging to 0 such that $d^\delta_{GH}(X_i, X) < \epsilon_i$. Thus, for each i, there exists a δ-admissible metric d^i on $X \sqcup X_i$ with $d^i_H(X, X_i) < \epsilon_i$. Define a δ-admissible metric d on Y by

$$d(x_i,x_j) := \min_{x\in X}\{d^i(x_i,x) + d^j(x,x_j)\},\ d(x_j,x_i) := \min_{x\in X}\{d^j(x_j,x) + d^i(x,x_i)\},$$

for all $x_i \in X_i$, $x_j \in X_j$.

Given a forward Cauchy sequence $\{x_\alpha\}$. If some X_i (or X) contains infinitely many x_α, then $\{x_\alpha\}$ must converge to some point $x \in X_i$ (or $x \in X$) (see [20, Theorem 2.11]). Hence, we assume that each X_i and X contain finitely many x_α. Then we can choose a subsequence $\{x_{\alpha_i}\}$ of $\{x_\alpha\}$ such that $x_{\alpha_i} \in X_{\alpha_i}$ and $\alpha_i \nearrow +\infty$. Since $d_H(X_i, X) < \epsilon_i$, there exists a sequence $\{z_{\alpha_i}\} \subset X$ with $d(z_{\alpha_i}, x_{\alpha_i}) < \epsilon_{\alpha_i}$. Without loss of generality, we can suppose that $\{z_{\alpha_i}\}$ converges to a point $z \in X$. The triangle inequality yields $d(z, x_{\alpha_i}) \to 0$ as $i \to +\infty$. Hence, $\{x_\alpha\}$ converges to z. □

Moreover, we have the following theorem.

Theorem 2.4. *The "metric space" $(\mathcal{M}^\delta, d^\delta_{GH})$ is complete.*

PROOF. It suffices to show there exists a convergent subsequence in an arbitrary Cauchy sequence $\{X_n\}$. Select a subsequence $\{X_i\}$ such that $d^{\delta}_{GH}(X_i, X_{i+1}) < 2^{-i}$ for all i. Then choose δ-admissible metrics $d_{i,i+1}$ on $X_i \sqcup X_{i+1}$ such that its Hausdorff distance between X_i and X_{i+1} is less than 2^{-i}. Now define a metric $d_{i,i+j}$ on $X_i \sqcup X_{i+j}$ by

$$d_{i,i+j}(x_i, x_{i+j}) := \min_{\{x_{i+k} \in X_{i+k}\}} \left(\sum_{k=0}^{j-1} d_{i+k,i+k+1}(x_{i+k}, x_{i+k+1}) \right),$$

$$d_{i,i+j}(x_{i+j}, x_i) := \min_{\{x_{i+k} \in X_{i+k}\}} \left(\sum_{k=0}^{j-1} d_{i+k,i+k+1}(x_{i+k+1}, x_{i+k}) \right),$$

for all $x_i \in X_i$, $x_{i+j} \in X_{i+j}$. Clearly, $d_{i,i+j}$ is a δ-admissible metric on $X_i \sqcup X_{i+j}$. In fact we have defined a δ-admissible metric, say d, on $Y := \sqcup_i X_i$. Let d_H denote the Hausdorff distance on Y induced by d. It is easy to check that $d_H(X_i, X_{i+j}) \leq 2^{-i+1}$ for all $X_i, X_{i+j} \subset Y$.

Let $\widehat{X} := \{\{x_i\} \mid x_i \in X_i \text{ and } d(x_i, x_j) \to 0 \text{ as } i, j \to \infty\}$. This space has a pseudometric defined by $d(\{x_i\}, \{y_i\}) := \lim_{i\to\infty} d(x_i, y_i)$. The definition is well-defined. One can verify that $d(\{x_i\}, \{y_i\})$ satisfies the triangle inequality and its reversibility is $\leq \delta$. Define an equivalence relation $\sim$ on $\widehat{X}$ by

$$\{x_i\} \sim \{y_i\} \Leftrightarrow d(\{x_i\}, \{y_i\}) = 0.$$

Then we have a δ-admissible metric on the quotient space $X := \widehat{X}/\sim$ and let $\bar{X}$ be the completion of X. Now we can extend the metric on Y to one on $\bar{X} \sqcup Y$ by declaring

$$d(y, \{x_k\}) := \lim_{k\to\infty} d(y, x_k), \ d(\{x_k\}, y) := \lim_{k\to\infty} d(x_k, y),$$

for all $y \in Y$ and $\{x_k\} \in X$. It is easy to check that this metric is a δ-admissible metric on $\bar{X} \sqcup Y$. A similar argument to the one given in [15, Proposition 43] yields $\lim_{i\to\infty} d^{\delta}_{GH}(X_i, \bar{X}) = 0$. By [20, Theorem 2.11], one can easily show that $\bar{X}$ is compact. □

The above result in the reversible case is due to PETERSEN (cf. [15], [16]).

By [20, Proposition 3.7], if $X, Y \in \mathcal{M}^{\delta}$ satisfy $d^{\delta}_{GH}(X, Y) = 0$, then X is isometric to Y. Furthermore, we have the following proposition.

Proposition 2.5. *Let $\{X_i\}$ and $\{A_i\}$ be two sequences of metric spaces in $\mathcal{M}^{\delta}$ with $A_i \subset X_i$ for all i. If $X_i \xrightarrow{\delta-GH} X$ and $A_i \xrightarrow{\delta-GH} A$, then A is (isometric to) a subset of X.*

Proof. We can select a sequence $\{\epsilon_i\}$ converging to 0 such that $d^\delta_{GH}(X_i, X) < \epsilon_i$ and $d^\delta_{GH}(A_i, A) < \epsilon_i$ for all i. Hence, for each i, there exist δ-admissible metrics $\hat{d}^i, \tilde{d}^i$ on $X_i \sqcup X$ and $A_i \sqcup A$, respectively, such that $\hat{d}^i_H(X_i, X) < \epsilon_i$ and $\tilde{d}^i_H(A_i, A) < \epsilon_i$. Let $\iota_i : A_i \hookrightarrow X_i$ be the injective map. Define a δ-admissible metric $\bar{d}^i$ on $A_i \sqcup X_i$ by

$$\bar{d}^i(x, y) = \epsilon_i + d(\iota_i(x), y),\ \bar{d}^i(y, x) = \epsilon_i + d(y, \iota_i(x)),\quad \forall (x, y) \in A_i \times X_i.$$

Then we obtain a sequence of δ-admissible metrics $\{d^i\}$ on $A \sqcup X$, where

$$d^i(x, y) := \inf_{z\in A_i,\, w\in X_i} (\tilde{d}^i(x, z) + \bar{d}^i(z, w) + \hat{d}^i(w, y)),$$
$$d^i(y, x) := \inf_{z\in A_i,\, w\in X_i} (\hat{d}^i(y, w) + \bar{d}^i(w, z) + \tilde{d}^i(z, x)),\quad \forall (x, y) \in A \times X.$$

Given $a \in A$. For each i, there are $a_i \in A_i$ and $x_i \in X$ such that $\tilde{d}^i(a_i, a) \le \epsilon_i$ and $\hat{d}^i(x_i, \iota_i(a_i)) \le \epsilon_i$. Thus, we obtain a sequence of points $\{x_i\}_{i=1}^\infty \subset X$. Since X is compact, there exists a subsequence of $\{x_i\}$ converging to a point $x(a) \in X$.

Let $\mathcal{A} := \overline{\{x(a) : a \in A\}} \subset X$. By the construction above, it is easy to check that $d^\delta_{GH}(\mathcal{A}, A) = 0$. Then the result follows from [20, Proposition 3.7]. □

Recall the *covering* $\mathrm{Cov}(X, \epsilon)$ of a compact metric space X is minimum number of forward ϵ-balls it takes to cover X. Given an decreasing (possibly discontinuous) function $N : (0, \beta) \to (0, \infty)$. Let $\mathscr{C}(N) := \{X \in \mathcal{M}^\delta : \mathrm{Cov}(X, \epsilon) \le N(\epsilon)$ for all $\epsilon \in (0, \beta)\}$. Then we have the following result (compared [20, Proposition 3.12]).

Theorem 2.6. *The class $\mathscr{C}(N)$ is compact in the δ-Gromov–Hausdorff topology.*

Proof. *Step 1: $\mathscr{C}(N)$ is precompact.*

Given a sequence $\{X_i\} \subset \mathscr{C}(N)$ and a positive number ϵ. Since $\mathrm{Cov}(X_i, \epsilon) \le N(\epsilon) < \infty$, we can select a positive number $M \le N(\epsilon)$ such that there is an infinite subsequence $\{X_\alpha\}$ of $\{X_i\}$ where each X_α is covered by exactly M forward ϵ-balls. Denote by x^s_α, $s = 1, \ldots, M$, the centers of these forward balls in X_α. Since $d(x^i_\alpha, x^j_\alpha) \le \mathrm{diam}\, X_\alpha \le (1 + \delta)\epsilon M \le (1 + \delta)\epsilon N(\epsilon)$, using the Cantor diagonal procedure, we can obtain a subsequence $\{X_{\alpha_\beta}\}$ of $\{X_\alpha\}$ such that $\{d(x^i_{\alpha_\beta}, x^j_{\alpha_\beta})\}_{\beta=1}^\infty$ converge for all i, j. By [20, Proposition 3.8], there exists $L > 0$ such that $\beta, \gamma \ge L$ implies $d^\delta_{GH}(S^\epsilon_{\alpha_\beta}, S^\epsilon_{\alpha_\gamma}) < \epsilon$, where $S^\epsilon_{\alpha_\beta} := \{x^i_{\alpha_\beta}\}_{i=1}^M$ is a forward ϵ-net of X_{α_β}. Thus, the triangle inequality yields $d^\delta_{GH}(X_{\alpha_\beta}, X_{\alpha_\gamma}) < 3\epsilon$. The conclusion then follows from Theorem 2.4.

Step 2: $\mathscr{C}(N)$ is closed.

Let $\{X_i\} \subset \mathscr{C}(N)$ be a sequence converging to a space $X \in \mathcal{M}^\delta$. Given $\epsilon > 0$, for each i, there exists a finite forward ϵ-net S_i^ϵ of X_i with $\sharp S_i^\epsilon \leq N(\epsilon)$. Since $N(\epsilon)$ is a decreasing function, $\{S_i^\epsilon\} \subset \mathscr{C}(N')$, where $N' := N|_{(0,\epsilon)}$. By Step 1, $\{S_i^\epsilon\}$ is precompact and therefore contain a subsequence $\{S_\alpha^\epsilon\}$ converging to a space S^ϵ.

Given $\eta > 0$, there exists S_α^ϵ with $d_{GH}^\delta(S_\alpha^\epsilon, S^\epsilon) < \eta$. Thus, one can select a δ-admissible metric d on $S_\alpha^\epsilon \sqcup S^\epsilon$ such that $d_H(S_\alpha^\epsilon, S^\epsilon) < \eta$. Hence, for every $x \in S^\epsilon$, there exists some $x_\alpha^s \in S_\alpha^\epsilon$ with $x \in B_{x_\alpha^s}^+(\eta)$. Now we have shown that for each $\eta > 0$, there exists a covering $\{U_s\}$ of S^ϵ such that $\mathrm{diam}(U_s) < (1+\delta)\eta$ and $\sharp\{U_s\} \leq N(\epsilon)$. This implies $\sharp S^\epsilon \leq N(\epsilon)$.

By using Proposition 2.5, one can check that S^ϵ is a forward ϵ-net of X. Hence, $X \in \mathscr{C}(N)$. □

The theorem above in reversible case is due to Petersen (see [15], [16]).

3. Generalized Hausdorff dimension

In this section, we recall some definitions and properties of covering dimension and extend Hausdorff dimension to general metric spaces. See [3], [13], [19] for details. The *covering dimension* of a space is said to be $\leq n$ if for each open covering, there is a refinement of order $\leq n+1$. The covering dimension of X, denote by $\dim(X)$, is the smallest integer n such that X have covering dimension $\leq n$.

Now, we generalize Hausdorff measure to general metric spaces.

Definition 3.1. Given a (general) metric space X and a non-negative real number r, define

$$\mu_\epsilon^r(X) := \inf \sum_{i=1}^{+\infty} [\mathrm{diam}(A_i)]^r ,$$

where the infimum is taken over all countable covering $\{A_i\}$ of X with $\mathrm{diam}(A_i) < \epsilon$ for all i, and

$$[\mathrm{diam}(A_i)]^0 := \begin{cases} 0 & A_i = \emptyset, \\ 1 & A_i \neq \emptyset. \end{cases}$$

Let

$$\mu^r(X) := \sup_{\epsilon > 0} \mu_\epsilon^r(X) = \lim_{\epsilon \to 0} \mu_\epsilon^r(X).$$

$\mu^r(X)$ is called r(-dimensional) generalized Hausdorff measure of X.

By [19], it is easy to check that μ^r is a (outer) measure. Moreover, we have the following proposition, whose proof is trivial.

Proposition 3.2. *Given a metric space X.*

(i) $\mu^r(X) = 0$ *for any* $r \geq 0$ *iff* $X = \emptyset$.

(ii) $\mu^0(X) = n$ *if* $X = \{p_1, \dots, p_n\}$.

(iii) $\mu^0(X) = \infty$ *if* X *is an infinite set.*

(iv) *If* $r_1 < r_2$, *then* $\mu^{r_1}(X) \geq \mu^{r_2}(X)$. *In fact,* $\mu^{r_1}(X) < \infty$ *implies* $\mu^{r_2}(X) = 0$, *and* $0 < \mu^{r_2}(X)$ *implies* $\mu^{r_1}(X) = +\infty$.

Definition 3.3. Given a (general) metric space X, the generalized Hausdorff dimension of X is defined by

$$\dim_{\mathrm{H}}(X) := \sup\{r : r > 0,\ \mu^r(X) > 0\}.$$

By Proposition 3.2, we have the following corollary.

Corollary 3.4. *For each metric space X, $\mu^r(X) = 0$ for all $r > \dim_{\mathrm{H}}(X)$ and $\mu^r(X) = +\infty$ for all $r < \dim_{\mathrm{H}}(X)$.*

Proposition 3.5. *For each compact metric space $(X, d) \in \mathcal{M}^\delta$, $\dim_{\mathrm{H}}(X) \geq \dim(X)$.*

Proof. Define a reversible metric $\tilde{d}$ on X by

$$\tilde{d}(x, y) := \frac{d(x, y) + d(y, x)}{2}, \tag{3.1}$$

for all $x, y \in X$. Thus, we have

$$\frac{2}{1+\delta}\tilde{d}(x, y) \leq d(x, y) \leq \frac{2}{1+\delta^{-1}}\tilde{d}(x, y), \quad \forall x, y \in X.$$

Hence, for each $r \geq 0$,

$$\left(\frac{2}{1+\delta}\right)^r \mu^r((X, \tilde{d})) \leq \mu^r((X, d)) \leq \left(\frac{2}{1+\delta^{-1}}\right)^r \mu^r((X, \tilde{d})).$$

Corollary 3.4 then yields $\dim_{\mathrm{H}}((X, d)) = \dim_{\mathrm{H}}((X, \tilde{d}))$.

Note that (X, d) is separable (see [20, Corollary 2.12]) and (X, d) is homeomorphic to $(X, \tilde{d})$. Hence, by [13], we have

$$\dim((X, d)) = \dim((X, \tilde{d})) \leq \dim_{\mathrm{H}}((X, \tilde{d})) = \dim_{\mathrm{H}}((X, d)). \qquad \square$$

The following result is an extension of a result due to PONTRJAGIN and SCHNIRELMANN (see [17]).

Theorem 3.6. *For each metric space $X \in \mathcal{M}^\delta$,*

$$\dim_{\mathrm{H}}(X) \leq \liminf_{\epsilon\to 0^+} \frac{\log \mathrm{Cov}(X,\epsilon)}{-\log \epsilon}. \tag{3.2}$$

PROOF. The result is trivial if $\dim_{\mathrm{H}}(X) = 0$. Hence, we suppose that $\dim_{\mathrm{H}}(X) > 0$. Given $\epsilon \in (0,1)$, there exists a covering $\{\overline{B^+_{x_i}(\epsilon)}\}_{i=1}^{\mathrm{Cov}(X,\epsilon)}$ of X. For each $r \geq 0$,

$$\sum_{i=1}^{\mathrm{Cov}(X,\epsilon)} \left[\mathrm{diam}\left(B^+_{x_i}(\epsilon)\right)\right]^r \leq \mathrm{Cov}(X,\epsilon)\cdot[(1+\delta)\epsilon]^r,$$

which implies that

$$r + \frac{r\log(1+\delta)}{\log\epsilon} + \frac{\log \mu^r_{(1+\delta)\epsilon}(X)}{-\log\epsilon} \leq \frac{\log \mathrm{Cov}(X,\epsilon)}{-\log\epsilon}.$$

It follows from Corollary 3.4 that $\mu^r(X) = +\infty$, for $r < \dim_{\mathrm{H}}(X)$. Hence, for all $r \in [0, \dim_{\mathrm{H}}(X))$,

$$\liminf_{\epsilon\to 0^+}\left[\frac{\log \mu^r_{(1+\lambda)\epsilon}(X)}{-\log\epsilon} + \frac{r\log(1+\delta)}{\log\epsilon}\right] \geq 0.$$

which implies (3.2). □

4. LGC spaces

In this section, we consider general metric spaces which are LGC and investigate the homotopy types and the compactness of such spaces in the δ-Gromov–Hausdorff topology. See [8], [10], [16] for more details on reversible metric spaces.

A *contractibility function* $\rho : [0,r) \to [0,+\infty)$ is a function satisfying: (a) $\rho(0) = 0$, (b) $\rho(\epsilon) \geq \epsilon$, (c) $\rho(\epsilon) \to 0$, as $\epsilon \to 0$, (d) ρ is nondecreasing.

A metric space X is LGC(ρ) for some contractibility function ρ, if for every $\epsilon \in [0,r)$ and $x \in X$, the forward ball $B^+_x(\epsilon)$ is contractible inside $B^+_x(\rho(\epsilon))$. And X is said to be LGC(n,ρ) for some integer $n \geq 0$, if for each integer $k \in \{0,\dots,n\}$, $\epsilon \in [0,r)$, $x \in X$ and a continuous map $f : \mathbb{S}^k \to B^+_x(\epsilon)$, there exists an continuous map $\bar{f} : \mathbb{D}^{k+1} \to B_x^+(\rho(\epsilon))$ which is an extension of f, where $\mathbb{S}^k = \partial\mathbb{D}^{k+1}$ is the k-sphere bounding the $k+1$-disk $\mathbb{D}^{k+1}$. Clearly, if X is LGC(ρ), then X is LGC(n, ρ) for all n.

A map $f : X \to Y$ between metric spaces is *ϵ-continuous* if there exists an $\eta > 0$ such that $d(x_1, x_2) < \eta$ implies $d(f(x_1), f(x_2)) < \epsilon$. A map $g : X \to Y$ between metric spaces is *continuous at a subset* $A \subset X$ if for each $x \in A$ and $\epsilon > 0$, there exists $\eta = \eta(x) > 0$ such that $g(B_x^+(\eta)) \subset B_{g(x)}^+(\epsilon)$, where $B_x^+(\eta) \subset X$.

Given a contractibility function ρ, define $\rho_0(\epsilon) := \rho(\epsilon)$ and $\rho_i(\epsilon) := \rho(\epsilon + \rho_{i-1}(\epsilon))$ for $i \geq 1$.

The following lemma is an extension of a result due to Petersen (cf. [16]).

Lemma 4.1. *Let X and Y be two metric spaces where $X \in \mathcal{M}^\delta$ and Y is* LGC$(n-1, \rho)$. *Suppose that $A \subset X$ is a closed subset with* $\dim(X - A) \leq n$ *and $f : X \to Y$ is a (possibly discontinuous) map satisfying*

(i) *f is continuous at A,*

(ii) *f is ϵ-continuous.*

If $\epsilon + \rho_{n-1}(\epsilon) < r$, then there exists a continuous map $g : X \to Y$ with $g|_A = f|_A$ and for all $x \in X$, $d(f(x), g(x)) \leq \epsilon + \rho_n(\epsilon)$.

Proof. Since f is ϵ-continuous, there exists $\eta > 0$ such that $d(x_1, x_2) < \eta$ implies $d(f(x_1), f(x_2)) < \epsilon$. For each $x \in X - A$, set

$$r(x) := \min\left(\frac{\eta}{10\delta}, \frac{d(x, A)}{10\delta}\right).$$

Note that X is separable. Hence, $X - A$ is a Lindelöf space. Since $\dim(X - A) \leq n - 1$, the covering $\{B_x^+(r(x))\}_{x \in X - A}$ has a countable refinement $\{U_\alpha\}$ of order $\leq n$. The triangle inequality yields

$$\operatorname{diam}(U_\alpha) \leq \min\left(\frac{1+\delta}{10\delta - 1} d(U_\alpha, A), \frac{1+\delta}{10\delta}\eta\right), \quad \forall \alpha.$$

Let $\mathscr{N}$ denote the nerve of $\{U_\alpha\}$ and $\Phi : X - A \to \mathscr{N}$ be a barycentric map. Define a map $\iota : X \to \mathscr{N} \cup A$ by

$$\iota(x) := \begin{cases} x & x \in A, \\ \Phi(x) & x \in X - A. \end{cases}$$

Denote by d_X the metric of X. Define a new metric $\tilde{d}_X$ on X by (3.1). Since the topology of (X, d_X) coincides with that of $(X, \tilde{d}_X)$, by [12], one can topologize $\mathscr{N} \cup A$ so that ι is continuous.

We now construct a continuous map $\bar{g} : \mathscr{N} \cup A \to Y$ by induction on the skeleton $\mathscr{N}^0 \cup A, \dots, \mathscr{N}^n \cup A = \mathscr{N} \cup A$.

For each vertex $U_\alpha \in N$, select a point $x_\alpha \in U_\alpha$ and define $\bar{g}_0(U_\alpha) := f(x_\alpha)$ and $\bar{g}_0|_A := f|_A$. By (i), $\bar{g}_0 : \mathscr{N}^0 \cup A \to Y$ is continuous. Moveover, if $U_\alpha \cap U_\beta \neq \emptyset$, then $d(x_\alpha, x_\beta) < \eta$ for all $x_\alpha \in U_\alpha$, $x_\beta \in U_\beta$, which implies $d(\bar{g}_0(U_\alpha), \bar{g}_0(U_\beta)) < \epsilon$.

Let $\triangle_{\alpha_0,\dots,\alpha_k}$ denote the k-simplex of $\mathscr{N}$ spanned by $U_{\alpha_0}, \dots, U_{\alpha_k}$ if $U_{\alpha_0} \cap \dots \cap U_{\alpha_k} \neq \emptyset$. Suppose that we have constructed a continuous map $\bar{g}_k : \mathscr{N}^k \cup A \to Y$ satisfying

(1) $\bar{g}_k|_{\mathscr{N}^i \cup A} = \bar{g}_i$ for all $0 \le i \le k$,

(2) for each $\triangle_{\alpha_0,\dots,\alpha_k}$, $\bar{g}_k(\triangle_{\alpha_0,\dots,\alpha_k}) \subset B^+_{\bar{g}_k(U_{\alpha_i})}(\rho_k(\epsilon))$, for all $0 \le i \le k$.

Thus, for each $(k+1)$-simplex $\triangle_{\alpha_0,\dots,\alpha_{k+1}}$ of $\mathscr{N}$, it is easy to check that $\bar{g}_k(\partial\triangle_{\alpha_0,\dots,\alpha_{k+1}}) \subset B^+_{\bar{g}_k(U_{\alpha_i})}(\epsilon + \rho_k(\epsilon))$. Since $\partial\triangle_{\alpha_0,\dots,\alpha_{k+1}} \approx \mathbb{S}^k$ and Y is LGC$(n-1,\rho)$, there exists an extension $\bar{g}_{k+1} : \mathscr{N}^{k+1} \cup A \to Y$ of $\bar{g}_k$ such that

$$\bar{g}_{k+1}(\triangle_{\alpha_0,\dots,\alpha_{k+1}}) \subset B^+_{\bar{g}_{k+1}(U_{\alpha_i})}(\rho_{k+1}(\epsilon)), \quad \text{for all } 0 \le i \le k+1.$$

Now we obtain a continuous map $\bar{g}(= \bar{g}_n) : \mathscr{N} \cup A \to Y$ such that for each $\triangle_{\alpha_0,\dots,\alpha_n} \in \mathscr{N}$, $\bar{g}(\triangle_{\alpha_0,\dots,\alpha_n}) \subset B^+_{\bar{g}(U_{\alpha_i})}(\rho_n(\epsilon))$ for all $0 \le i \le n$. Set $g := \bar{g} \circ \iota$. Given $x \in X - A$, there is U_α such that $\iota(x)$ is contained in some simplex with U_α as its vertex. Then $d(f(x), g(x)) \le d(f(x), \bar{g}(U_\alpha)) + d(\bar{g}(U_\alpha), \bar{g} \circ \iota(x)) \le \epsilon + \rho_n(\epsilon)$. □

Corollary 4.2. *Let $X \in \mathcal{M}^\delta$ have covering dimension $\le n$ and let $Y \in \mathcal{M}^\delta$ be* LGC(n,ρ). *Suppose $f_i : X \to Y$, $i = 1, 2$, are continuous maps with $d(f_1, f_2) < \epsilon/\delta$. If $\epsilon + \rho_n(\epsilon) < r$, then f_1 is $2\epsilon + \rho_{n+1}(\epsilon)$-controlled homotopy equivalent to f_2.*

Proof. Let $Z = [0,1] \times X$ and $A = \{0,1\} \times X$. The metric d_Z is defined as $d_Z := d_{[0,1]} + d_X$. Thus, $(Z, d_Z) \in \mathcal{M}^\delta$. Since $X \in \mathcal{M}^\delta$, X is a normal compact space. Hence, $\dim(Z - A) \le n + 1$ (cf. [14]). Define a map $h : Z \to Y$ by

$$h(t,x) := \begin{cases} f_1(x) & (t,x) \in \left[0, \dfrac{1}{2}\right] \times X, \\ f_2(x) & (t,x) \in \left(\dfrac{1}{2}, 1\right] \times X. \end{cases}$$

It follows from Proposition 2.2 that h is ϵ-continuous. Lemma 4.1 then yields a continuous map $H : [0,1] \times X \to Y$ such that $f_1 \overset{H}{\simeq} f_2$ and $d(h, H) \le \epsilon + \rho_{n+1}(\epsilon)$. It is easy to check that $d(f_2(x), H(t,x)) \le 2\epsilon + \rho_{n+1}(\epsilon)$ and $d(f_1(x), H(t,x)) \le \frac{(1+\delta)\epsilon}{\delta} + \rho_{n+1}(\epsilon)$ for all $(t,x) \in [0,1] \times X$. □

Proposition 4.3. *Suppose that $X, Y \in \mathcal{M}^\delta$ are* LGC(n,ρ) *and have covering dimension* $\leq n$. *If* $d^\delta_{GH}(X,Y) < \epsilon/\delta$ *and* $(2(1+2\delta)\epsilon + 2\delta\rho_n((1+\delta)\epsilon/\delta)) + \rho_n((2(1+2\delta)\epsilon + 2\delta\rho_n((1+\delta)\epsilon/\delta))) < r$, *then X and Y are homotopy equivalent.*

Proof. Since $d^\delta_{GH}(X,Y) < \epsilon/\delta$, we can select a δ-admissible metric d on $X \sqcup Y$ such that $d_H(X,Y) < \eta < \epsilon/\delta$.

We now construct a (possibly discontinuous) map $\bar{f} : X \to Y$. For each $x \in X$, there exists $y \in Y$ such that $d(y,x) < \eta$ and define $\bar{f}(x) := y$. Thus, $d(\bar{f}, \mathrm{id}_X) < \eta$, where $\mathrm{id}_X : X \to X$ is the identity map of X. For every two points $x_1, x_2 \in X$, if $d(x_1,x_2) < \epsilon/\delta - \eta$, then

$$d(\bar{f}(x_1), \bar{f}(x_2)) \leq d(\bar{f}(x_1), x_1) + d(x_1,x_2) + d(x_2, \bar{f}(x_2)) < \frac{(1+\delta)\epsilon}{\delta}.$$

Lemma 4.1 now yields a continuous map $f : X \to Y$ with $d(\bar{f}, f) \leq \frac{(1+\delta)\epsilon}{\delta} + \rho_n\big(\frac{(1+\delta)\epsilon}{\delta}\big)$. Hence, $d(\mathrm{id}_X, f) \leq \frac{(1+2\delta)\epsilon}{\delta} + \rho_n\big(\frac{(1+\delta)\epsilon}{\delta}\big)$. Similarly, one can obtain a continuous map $g : Y \to X$ with $d(\mathrm{id}_Y, g) \leq \frac{(1+2\delta)\epsilon}{\delta} + \rho_n\big(\frac{(1+\delta)\epsilon}{\delta}\big)$. Thus, for each $x \in X$,

$$d(x, g\circ f(x)) \leq d(x, f(x)) + d(f(x), g\circ f(x)) \leq 2\left[\frac{(1+2\delta)\epsilon}{\delta} + \rho_n\left(\frac{(1+\delta)\epsilon}{\delta}\right)\right].$$

It follows from Corollary 4.2 that $\mathrm{id}_X \simeq g\circ f$. Likewise, $\mathrm{id}_Y \simeq f \circ g$. □

From above, we have the following results for LGC general metric spaces. In the reversible case, these results are due to Grove, Petersen and Wu (see [10], [16]).

Theorem 4.4. *Let $\{X_i\}$ be a sequence in $(\mathcal{M}^\delta, d^\delta_{GH})$ converging to X in the δ-Gromov–Hausdorff topology. If X_i is* LGC(n,ρ) *for all i, then X is* LGC(n,ρ) *as well.*

Proof. Since $X_i \xrightarrow{\delta-GH} X$, we can choose a sequence $\{\epsilon_i\}$ with $\epsilon_i \searrow 0$ and $d^\delta_{GH}(X_i, X) < \epsilon_i/\delta$. By Proposition 2.3, there exists a δ-admissible metric d on $Y := \sqcup_i X_i \sqcup X$ such that (Y,d) is forward complete.

Given a point $p \in X$, $\epsilon > 0$ and a map $f : \mathbb{S}^k \to B^+_p(\epsilon) \subset X$ for $0 \leq k \leq n$. Since $d_H(X, X_i) < \epsilon_i/\delta$, there exists $p_i \in X_i$ with $d(p_i, p) < \epsilon_i/\delta$. Select $\eta < \epsilon$ such that $f(\mathbb{S}^k) \subset B^+_p(\eta)$. Using the argument in the proof of Proposition 4.3, one can find a continuous map $f_i : \mathbb{S}^k \to X_i$ such that $d(f, f_i) \leq \frac{(1+2\delta)\epsilon_i}{\delta} + \rho_n\big(\frac{(1+\delta)\epsilon_i}{\delta}\big) \to 0$, as $i \to +\infty$. For simplicity, set $\varepsilon_i := \frac{(1+2\delta)\epsilon_i}{\delta} + \rho_n\big(\frac{(1+\delta)\epsilon_i}{\delta}\big)$.

From above, we obtain a sequence of points $\{p_i\}$ and a sequence of continuous maps $\{f_i : \mathbb{S}^k \to X_i\}$ with $d(p_i, p) < \frac{\varepsilon_i}{1+2\delta}$ and $d(f, f_i) \leq \varepsilon_i$. Thus, there is $N > 0$

such that $(3+\delta)\varepsilon_N+\rho_k((3+\delta)\varepsilon_N)\le r$ and $i\ge N$ implies $f_i(\mathbb{S}^k)\subset B^+_{p_i}(\eta)$. Since X_N is LGC(n,ρ), there is an extension $\bar f_N:\mathbb{D}^{k+1}\to B^+_{p_N}(\rho(\eta))$ of f_N.

Now, we extend f over $\mathbb{D}^{k+1}$ inside $B^+_p(\rho(\eta))$.

Since $d_H(X_N,X_{N+1})<\frac{2\varepsilon_N}{1+2\delta}$, for each $t\in\mathrm{Int}(\mathbb{D}^{k+1})$, there exists $y_t\in X_{N+1}$ with $d(y_t,\bar f_N(t))<\frac{2\varepsilon_N}{1+2\delta}$. Define a map $g_{N+1}:\mathbb{D}^{k+1}\to X_{N+1}$ by

$$g_{N+1}(t):=\begin{cases} f_{N+1}(t) & t\in\mathbb{S}^k,\\ y_t & t\in\mathrm{Int}(\mathbb{D}^{k+1}).\end{cases}$$

Using Proposition 2.2 and the triangle inequality, one can verify that g_{N+1} is $(3+\delta)\varepsilon_N$-continuous map.

By Lemma 4.1 and $(3+\delta)\varepsilon_N+\rho_k((3+\delta)\varepsilon_N)\le r$, we obtain a continuous map $\bar f_{N+1}:\mathbb{D}^{k+1}\to X_{N+1}$ such that $d(g_{N+1},\bar f_{N+1})\le(3+\delta)\varepsilon_N+\rho_{k+1}((3+\delta)\varepsilon_N)$ and $\bar f_{N+1}|_{\mathbb{S}^k}=g_{N+1}|_{\mathbb{S}^k}=f_{N+1}$. Hence, $\bar f_{N+1}$ is an extension of f_{N+1}. And $d(\bar f_N,g_{N+1})\le(1+\delta)\varepsilon_N$ implies

$$d(\bar f_N,\bar f_{N+1})\le d(\bar f_N,g_{N+1})+d(g_{N+1},\bar f_{N+1})\le(4+2\delta)\varepsilon_N+\rho_{k+1}((3+\delta)\varepsilon_N).$$

Using $d(p_{N+1},p_N)\le(1+\delta)\varepsilon_N$ and $\bar f_N(\mathbb{D}^{k+1})\subset B^+_{p_N}(\rho(\eta))$, we obtain

$$\bar f_{N+1}(\mathbb{D}^{k+1})\subset B^+_{p_{N+1}}\left(\rho(\eta)+(5+3\delta)\varepsilon_N+\rho_{k+1}((3+\delta)\varepsilon_N)\right).$$

After possibly passing to a subsequence of $\{X_i\}$, we can suppose that

$$\sum_{s=N}^{+\infty}((5+2\delta)\varepsilon_s+\rho_{k+1}((3+\delta)\varepsilon_s))\le\min\{\epsilon-\eta,r,\rho(\epsilon)-\rho(\eta)\}.$$

Then we can construct inductively the extensions

$$\bar f_i:\mathbb{D}^{k+1}\to B^+_{p_i}\left(\rho(\eta)+\sum_{s=N}^{i}(5+3\delta)\varepsilon_s+\rho_{k+1}((3+\delta)\varepsilon_s)\right)$$

of $f_i:\mathbb{S}^k\to B^+_p(\eta)$ for all $i\ge N$. In particular, for $j>i\ge N$,

$$d(\bar f_i,\bar f_j)\le\sum_{s=i}^{j}((4+2\delta)\varepsilon_s+\rho_{k+1}((3+\delta)\varepsilon_s)).$$

Proposition 2.3 together with [20, Theorem 2.16, Theorem 2.15] implies that the sequence $\{f_s\}_{s\ge N}$ converges uniformly to a continuous map $\bar f:\mathbb{D}^{k+1}\to Y$. The proof of Proposition 2.3 yields $\bar f(\mathbb{D}^{k+1})\subset X$. Since $\bar f_i|_{\mathbb{S}^k}=f_i$ and $f_i\rightrightarrows f$, $\bar f$ is an extension of f. From above,

$$\bar f(\mathbb{D}^{k+1})\subset B^+_p\left(\rho(\eta)+\sum_{s=N}^{\infty}(5+2\delta)\varepsilon_s+\rho_{k+1}((3+\delta)\varepsilon_s)\right)\subset B^+_p(\rho(\epsilon)),$$

which implies that X is LGC(n,ρ). □

Theorem 4.5. *Given a covering function $N:(0,\alpha)\to(0,\infty)$ with $\limsup_{\epsilon\to 0^+}\epsilon^n N(\epsilon)<\infty$ and a contractibility function $\rho:[0,r)\to[0,\infty)$. Then the class*

$$\mathscr{C}(N,n,\rho):=\{X\in\mathcal{M}^\delta : X \text{ is } \mathrm{LGC}(n,\rho),\ \mathrm{Cov}(X,\epsilon)\le N(\epsilon) \text{ for all } \epsilon\in(0,\alpha)\}$$

is compact and contains only finitely many homotopy types.

Proof. Proposition 3.5 together with Theorem 3.6 furnishes

$$\dim X\le \dim_{\mathrm{H}}(X)\le \limsup_{\epsilon\to 0^+}\frac{\log\mathrm{Cov}(X,\epsilon)}{-\log\epsilon}\le \limsup_{\epsilon\to 0^+}\frac{\log N(\epsilon)}{-\log\epsilon}\le n,$$

for each $X\in\mathscr{C}(N,n,\rho)$. The conclusion now follows from Theorem 2.6, Theorem 4.4 and Proposition 4.3. □

Corollary 4.6. *Fix a covering function $N:(0,\alpha)\to(0,\infty)$ with $\limsup \epsilon\to 0^+\epsilon^n N(\epsilon)<\infty$ and a contractibility function $\rho:[0,r)\to[0,\infty)$. The class*

$$\mathscr{C}(N,\rho):=\{X\in\mathcal{M}^\delta : X \text{is } \mathrm{LGC}(\rho),\ \mathrm{Cov}(X,\epsilon)\le N(\epsilon) \text{ for all } \epsilon\in(0,\alpha)\}$$

contains only finitely many homotopy types.

5. Finsler manifold

In this section, we show Theorem 1.1-Theorem 1.3. We recall some definitions and properties of Finsler manifolds first. See [2], [22] for more details.

Let (M,F) be a (connected) Finsler manifold with Finsler metric $F:TM\to[0,\infty)$. Let $(x,y)=(x^i,y^i)$ be local coordinates on TM, and $\pi:TM\to M$ be the natural projection. Define

$$g_{ij}(x,y):=\frac{1}{2}\frac{\partial^2F^2(x,y)}{\partial y^i\partial y^j},\quad A_{ijk}(x,y):=\frac{F}{4}\frac{\partial^3F^2(x,y)}{\partial y^i\partial y^j\partial y^k},$$
$$\gamma^i_{jk}:=\frac{1}{2}g^{il}\left(\frac{\partial g_{jl}}{\partial x^k}+\frac{\partial g_{kl}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^l}\right),$$
$$N^i_j:=\left(\gamma^i_{jk}\ell^j-A^i_{jk}\gamma^k_{rs}\ell^r\ell^s\right)F,\quad \ell^i:=\frac{y^i}{F}.$$

The *Chern connection* is defined on the pulled-back bundle π^*TM and its forms are characterized by the following structure equations:

(1) Torsion freeness: $dx^j \wedge \omega_j^i = 0$;
(2) Almost g-compatibility: $dg_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_j^k = 2\frac{A_{ijk}}{F}(dy^k + N_l^k dx^l)$.
From above, it's easy to obtain $\omega_j^i = \Gamma^i_{jk}dx^k$, and $\Gamma^i_{jk} = \Gamma^i_{kj}$.
The curvature form of the Chern connection is defined as

$$\Omega_j^i := d\omega_j^i - \omega_j^k \wedge \omega_k^i =: \frac{1}{2}R^i_{j\,kl}dx^k \wedge dx^l + P^i_{j\,kl}dx^k \wedge \frac{dy^l + N_s^l dx^s}{F}.$$

Given a non-zero vector $V \in T_xM$, the *flag curvature* $\mathbf{K}(y, V)$ on $(x, y) \in TM\backslash 0$ is defined as

$$\mathbf{K}(y, V) := \frac{V^i y^j R_{jikl} y^l V^k}{g_y(y, y)g_y(V, V) - [g_y(y, V)]^2},$$

where $R_{jikl} := g_{is}R^s_{j\,kl}$. And the *Ricci curvature* of y is defined by

$$\mathbf{Ric}(y) := \sum_i K(y, e_i),$$

where $e_1, \ldots, e_n$ is a g_y-orthonormal base on $(x, y) \in TM\backslash 0$.

Let $d\mu$ be the volume form on M. In a local coordinate system (x^i), express $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$. For $y \in T_xM\backslash 0$, define the *distortion* of (M, F) as

$$\tau(y) := \log \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

And we define *S-curvature* $\mathbf{S}$ as

$$\mathbf{S}(y) := \frac{d}{dt}[\tau(\dot{\gamma}(t))]|_{t=0},$$

where $\gamma(t)$ is the geodesic with $\dot{\gamma}(0) = y$.

By [2], [24], the admissible paths of a Finsler manifold (M, F) are all Lipschitz continuous paths $\gamma : [a, b] \to M$, and the length structure

$$d(p, q) := \inf \int_a^b F(\dot{\gamma}(t)),$$

where the infimum is taken over all Lipschitz continuous paths $\gamma : [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$. Then d is an intrinsic metric and the manifold topology coincides with the metric topology.

The *reversibility* λ_F of (M, F) is defined by ([18])

$$\lambda_F := \sup_{(x,y)\in TM\backslash 0} \frac{F(x, -y)}{F(x, y)}.$$

Clearly $\lambda_F \geq 1$ and $\lambda_F = 1$ if and only if F is reversible. From above, it is easy to see that if $\lambda_F < \infty$, then $\lambda_d \leq \lambda_F$, where λ_d is the reversibility of (M, d).

According to [6], the *uniformity constant* of (M, F) is defined by

$$\Lambda_F := \sup_{X,Y,Z \in SM} \frac{g_X(Y,Y)}{g_Z(Y,Y)}.$$

Clearly, $\lambda_F \leq \sqrt{\Lambda_F}$ and $\Lambda_F = 1$ if and only if F is Riemannian.

Now we extend the concept of critical point of distance function to Finsler manifolds. We refer to [7], [15] for more details in the Riemannian case. Let (M, F) be a forward complete Finsler manifold. Given a point $p \in M$, we define $r_p(\cdot) := d(p, \cdot)$ and

$$\begin{aligned}\Gamma_p(x) := \{w \in S_xM :\ &\text{there is a unit speed minimal geodesic } \gamma : [0, r_p(x)] \to M,\\ &\text{such that } \gamma(0) = p,\ \gamma(r_p(x)) = x,\ \dot\gamma(r_p(x)) = w\}.\end{aligned}$$

In the case where r_p is smooth at x, we have $\Gamma_p(x) = \{\nabla r_p(x)\}$. At other points, $\Gamma_p(x)$ may contain more vectors (cf. [2], [22]).

Definition 5.1. r_p is *noncritical*, or *regular*, at x if there exists a vector $v \in S_xM$, such that

$$g_w(w, v) > 0, \quad \forall w \in \Gamma_p(x). \tag{5.1}$$

When F is Riemannian, (5.1) holds if and only if $\Gamma_p(x)$ is contained in an open hemisphere of S_xM, which is the definition of regular point in Riemann geometry (cf. [7], [15]).

Remark 1. A simple argument based on the first variation of arc length formula (see [2]) and Proposition 5.3 shows Definition 5.1 is equivalent to the following.

A point $q \in M$ is *regular* for r_p if and only if there exists a unit vector $v \in S_qM$ and a $\eta > 0$ such that

$$r_p(\gamma_v(t)) \geq r_p(\gamma_v(0)) + \eta \cdot t$$

for all sufficiently small $t > 0$, where $\gamma_v(t) := \exp_q(tv)$.

Firstly, we have the following Finsler version of Berger's lemma.

Proposition 5.2. *If r_p has a local maximum at q, then q is a critical point.*

PROOF. Select an arbitrary C^1-curve $\gamma(t) : [0,\epsilon] \to M$ such that $f(s) = r_p \circ \gamma(s)$ has maximum at $q = \gamma(0)$. For each unit speed minimal geodesic $c(s)$ from p to q, one can construct a variation $c : [0,\epsilon] \times [0, r_p(q)] \to M$ such that $c(s,0) = p$, $c(0,t) = c(t)$ and $c_s(t) = c(s,t)$ is a minimal geodesic from p to $\gamma(s)$. By the first variation formula, we obtain

$$0 \geq \left.\frac{d}{ds}\right|_{s=0} L(c_s) = g_{\dot{c}}(\dot{c}, \dot{\gamma}(0)),$$

which implies that q is a critical point. □

Define $\mathcal{G}r_p(x) := \{v \in T_xM : g_w(w,v) > 0, \forall w \in \Gamma_p(x)\}$. Given a subset $U \subset M$, a vector field X is call *gradient-like* for r_p on U, if $X(x) \in \mathcal{G}r_p(x)$ for all $x \in U$.

Proposition 5.3. *Let (M,F) and $r_p(\cdot) = d(p,\cdot)$ be as above. Then*

(1) $\cup_{x\in K}\Gamma_p(x)$ *is compact, for any compact set* $K \subset M$.

(2) *The set of regular points is open in* M.

(3) $\mathcal{G}r_p(x)$ *is convex.*

(4) *If U is an open set of regular points, then there exists a unit gradient-like vector field X on U. Furthermore, given a compact subset $K \subset U$, there exists a small positive constant ϵ such that for each integral curve $\gamma : [a,b] \to K$ of X,*

$$r_p(\gamma(t)) - r_p(\gamma(s)) > (t-s)\cos\left(\frac{\pi}{2} - \epsilon\right),\ \forall a \leq s < t \leq b.$$

PROOF. (1) Let $\{w_n\}$ be a sequence of unit vectors in $\cup_{q\in K}\Gamma_p(q)$ converging to some unit vector w. Set $x_n := \pi_1(w_n)$ and $x := \pi_1(w)$, where $\pi_1 : SM \to M$ is the natural projection. Thus, for each w_n, there exits a unit speed minimal geodesic $\sigma_n : [0, r_p(x_n)] \to M$ from p to x_n with $\dot{\sigma}_n(r_p(x_n)) = w_n$. Since $\{x_n\} \subset K$, there is a constant C such that $L_F(\sigma_n) = r_p(x_n) < C$ for all n. Set $v_n := \dot{\sigma}_n(0)$. Then $\sigma_n(t) = \exp_p(t \cdot v_n)$. Without loss of generality, we assume $\{v_n\}$ converges to some vector v. Thus, by the generalized Arzelá–Ascoli theorem ([21, Theorem 6.1]), $\sigma(t) = \exp_p(t\cdot v)$, $0 \leq t \leq r_p(x)$, is the limit of σ_n. Since $r_p(x_n) \leq i_{v_n}$, $r_p(x) \leq i_v$. Hence, $\sigma(t) = \exp_p(t\cdot v)$, $0 \leq t \leq r_p(x)$ is a unit speed minimal geodesic from p to x. Note that

$$w = \lim_{n\to\infty} w_n = \lim_{n\to\infty} \left(\exp_p\right)_{*r_p(x_n)\cdot v_n} v_n = \left(\exp_p\right)_{*r_p(x)\cdot v} v.$$

Thus, $w \in \Gamma_p(x)$, which implies $\cup_{q\in K}\Gamma_p(q)$ is closed and therefore compact.

(2) Let $\{x_n\}$ be a sequence of critical points with $x_n \to x$. We show x is a critical point as well. For each vector $v \in S_xM$, we can choose a sequence of unit vectors $\{v_n\}$ such that $v_n \in S_{x_n}M$ and $v_n \to v$. By Definition 5.1, for each n, there exists $w_n \in \Gamma_p(x_n)$ with $g_{w_n}(w_n, v_n) \leq 0$. Without loss of generality, we assume that w_n converges to some unit vector $w \in S_xM$. Thus, $g_w(w, v) \leq 0$. It follows from (1) that $w \in \Gamma_p(x)$. Therefore, x is a critical point.

(3) Given v_1, $v_2 \in \mathcal{G}r_p(x)$ and $t \in [0, 1]$. For each $w \in \Gamma_p(x)$, we have

$$g_w(w, (1-t)v_1 + tv_2) = (1-t)g_w(w, v_1) + tg_w(w, v_2) > 0,$$

which implies that $(1-t)v_1 + tv_2 \in \mathcal{G}r_p(x)$.

(4) For each point $x \in U$, there exists $v \in \mathcal{G}r_p(x)$. It follows from the proof of (2) that one can obtain a gradient-like vector field V_x on a small open neighborhood U_x of x by extending v. Now, let $\{U_i\}$ be a locally finite collection of U_x's and $\{\rho_i\}$ be a corresponding partition of unit. Define a vector field on U by $Y = \sum \rho_i V_i$. It follows from (3) that $X = Y/F(Y)$ is also gradient-like.

By (1), there exists a small constant $\epsilon > 0$ such that for any $x \in K$

$$g_w(w, X) > \cos\left(\frac{\pi}{2} - \epsilon\right), \quad \forall w \in \Gamma_p(x). \tag{5.2}$$

Set $\lambda(K) := \sup_{x \in K} \lambda(x) = \sup_{y \in SK} F(-y) < \infty$. Let γ be a integral curve for X. Clearly,

$$d(\gamma(t_1), \gamma(t_2)) \leq \lambda(K) \cdot |t_1 - t_2|, \quad \forall t_1, t_2 \in [a, b].$$

This implies that $r_p \circ \gamma$ is differentiable almost everywhere. By the proof of Proposition 5.2 and (5.2), we have

$$\frac{d}{dt} r_p \circ \gamma(t) > \cos\left(\frac{\pi}{2} - \epsilon\right),$$

for almost every $t \in [a, b]$. □

From above, we have the following important lemma.

Lemma 5.4. *Let (M, F) and $r_p(\cdot)$ be as above. Suppose that all the points in $r_p^{-1}([a, b])$ are regular. Then $r_p^{-1}((-\infty, b])$ deformation retracts onto $r_p^{-1}((-\infty, a])$.*

Proof. Note that $r_p^{-1}([a, b])$ is compact. Thus, by (Proposition 5.3, (2)), there exist two bounded open subsets Ω and U of M such that $\bar{U}$ is a set of regular points and

$$r_p^{-1}([a, b]) \subset \Omega \subset \bar{\Omega} \subset U.$$

(Proposition 5.3, (4)) furnishes a unit gradient-like X on U. Since $p \notin U$, we can assume that there is a small open subset O such that $r_p^{-1}((-\infty, a]) - \bar{U} \supset O$. Choose a cut-off function $\psi : M \to [0,1]$ such that $\psi|_{\bar{\Omega}} \equiv 1$ and $\psi|_{M-U} \equiv 0$. Now, we define a vector field V on M by $V(x) = \psi(x) \cdot X(x)$, for all $x \in M$. Let φ_t denote the flow for V. For each $x \in r_p^{-1}([a,b])$, there exists a maximum $\eta_x > 0$ such that $\varphi_t(x) \in \bar{\Omega}$ for $t \in [-\eta_x, 0]$. Note that $\varphi_t(x)$ is an integral curve for X for $t \in [-\eta_x, 0]$. By (Proposition 5.3, (4)), one can select a constant $\epsilon > 0$ such that

$$r_p(\varphi_t(x)) - r_p(\varphi_s(x)) > (t-s) \cdot \cos\left(\frac{\pi}{2} - \epsilon\right), \quad \forall -\eta_x \leq s < t \leq 0. \tag{5.3}$$

Denote by $t_x \in [0, \eta_x]$ the first time on $\varphi_{-t}(x)$ to hit $r_p^{-1}(a)$. Since ϵ is independent of x (see Proposition 5.3, (4)),

$$0 \leq t_x \leq \frac{b-a}{\cos\left(\frac{\pi}{2} - \epsilon\right)}, \quad \forall x \in r_p^{-1}([a,b]). \tag{5.4}$$

We now claim that the function

$$t : \ r_p^{-1}([a,b]) \to \mathbb{R}, \quad x \mapsto t_x$$

is continuous. If t_x is discontinuous at some point q, then there would be a constant $\delta > 0$ and a sequence of points $\{q_n\} \subset r_p^{-1}([a,b])$ such that $q_n \to q$ and $|t_{q_n} - t_q| \geq \delta$. (5.4) implies that there exists a convergent subsequence of $\{t_{q_n}\}$. Without loss of generality, we suppose that $\{t_{q_n}\}$ converges themselves to $\mathcal{T}$. Thus,

$$\varphi_{-\mathcal{T}}(q) = \lim_{n\to\infty} \varphi_{-t_{q_n}}(q_n) \in r_p^{-1}(a).$$

Since $\{\varphi_{s\cdot(-t_{q_n})}(q_n) : s \in [0,1]\} \subset r_p^{-1}([a,b])$,

$$\{\varphi_{-t}(q) : t \in [0, \mathcal{T}]\} = \{\varphi_{s\cdot(-\mathcal{T})}(q) : s \in [0,1]\} \subset r_p^{-1}([a,b]). \tag{5.5}$$

It follows from (5.3) and (5.5) that $\mathcal{T}$ is first time for which $\varphi_{-\mathcal{T}}(q) \in r_p^{-1}(a)$, i.e., $\mathcal{T} = t_q$. We get a contradiction.

Now, we define the retraction $\rho_s : r_p^{-1}((-\infty, b]) \to r_p^{-1}((-\infty, a])$ by

$$\rho_s(x) := \begin{cases} x & x \in r_p^{-1}((-\infty, a]), \\ \varphi_{-s\cdot t_x}(x) & x \in r^{-1}([a,b]). \end{cases}$$

□

Remark 2. Given a compact subset $A \subset M$. Let $\gamma_A(\cdot) := d(A, \cdot)$. Similar arguments show that Proposition 5.2, Proposition 5.3 and Lemma 5.4 are still true for $r_A(\cdot)$.

The *contractibility radius* c_p at p is defined by

$$c_p := \inf\{r_p(x) : x \text{ is a critical point of } r_p\},$$

It is easy to see that a critical point of r_p lies in the cut locus of p. Hence, $c_p \geq i_p$. Define the *contractibility radius* c_M of (M, F) by $c_M := \inf_{p \in M} c_p$.

Lemma 5.4 then yields the following theorem, whose proof is trivial.

Theorem 5.5. *Let (M, F) be a forward complete Finsler manifold with $c_M \geq R > 0$. Then (M, F) is* LGC(ρ), *where $\rho : [0, R) \to [0, R)$ is the identity map.*

To prove Theorem 1.1, we need the following lemma ([27, Remark 3.1])

Lemma 5.6 ([27])**.** *Let $(M, F, d\mu)$ be a forward Finsler manifold, where $d\mu$ is an arbitrary volume form on M. Suppose that* **Ric** $\geq (n-1)k$ *and $a \leq \tau \leq b$, where τ is the distortion of $d\mu$. Then for each $p \in M$, we have*

$$\frac{\mu(B_p^+(r))}{\mu(B_p^+(R))} \geq e^{a-b} \frac{\int_0^r \mathfrak{s}_k^{n-1}(t)dt}{\int_0^R \mathfrak{s}_k^{n-1}(t)dt},$$

for any $0 < r \leq R$.

Recall the *capacity* $\mathrm{Cap}(X, \epsilon)$ of a compact metric space X is the maximum number of disjoint forward $\epsilon/2$-balls in X. Then we have the following

Theorem 5.7. *Given n and positive numbers k, D, R, δ, the class of closed Finsler n-manifolds (M, F) with $\mathbf{Ric}_M \geq -(n-1)k^2$, $\mathrm{diam}(M) \leq D$, $c_M \geq R$ and $\Lambda_F \leq \delta$, contains at most finitely many homotopy types.*

Proof. Given a Finsler manifold (M, F) satisfying the above assumptions. By [25], we have

$$\frac{\max_{y \in S_xM} \det g_{ij}(x, y)}{\min_{y \in S_xM} \det g_{ij}(x, y)} \leq \delta^n, \ \delta^{-n/2} \leq \frac{c_{n-1}}{\int_{S_xM} d\nu_x(y)} \leq \delta^{n/2},$$

for each $x \in M$. Let τ_{HT} denote the distortion of Holmes–Thompson volume form $d\mu_{HT}$. Thus, $\delta^{-n} \leq e^{\tau_{HT}(y)} \leq \delta^n$ for all $y \in SM$.

Given any $\epsilon > 0$. Since M is compact, there exists only finitely many disjoint forward ϵ-balls inside M. Suppose $B_{x_1}^+(\epsilon), \dots, B_{x_k}^+(\epsilon)$ are disjoint. Let $B_{x_\alpha}^+(\epsilon)$ be the forward ball with the smallest Holmes–Thompson volume. Clearly, $B_{x_\alpha}^+(D) \supset B_{x_i}^+(\epsilon)$ for all $1 \leq i \leq k$. Hence, it follows from Lemma 5.6 that

$$k \leq \frac{\mu_{HT}(B_{x_\alpha}^+(D))}{\mu_{HT}(B_{x_\alpha}^+(\epsilon))} \leq \delta^{2n} \frac{\int_0^D \sinh^{n-1}(kt)dt}{\int_0^\epsilon \sinh^{n-1}(kt)dt}.$$

This implies that $\mathrm{Cap}(M,2\epsilon)\leq N(\epsilon)$, where $N(\epsilon):=\delta^{2n}\frac{\int_0^D \sinh^{n-1}(kt)dt}{\int_0^\epsilon \sinh^{n-1}(kt)dt}$. Note that $\mathrm{Cap}(M,2\epsilon)\geq \mathrm{Cov}(X,2\sqrt{\delta}\epsilon)$ (see [20, Proposition 3.11]). Now the conclusion follows from Corollary 4.6 and Theorem 5.5. □

Likewise, the proof of [20, Theorem 6.3] together with Theorem 5.5 and Corollary 4.6 furnishes the following theorem.

Theorem 5.8. *Given n and positive numbers k,h,D,R,δ, the class of closed Finsler n-manifolds (M,F) with $\mathbf{Ric}_M\geq -(n-1)k^2$, $\mathbf{S}_M\geq (n-1)h$, $\mathrm{diam}(M)\leq D$, $c_M\geq R$ and $\lambda_F\leq\delta$, contains at most finitely many homotopy types.*

Theorem 4.5 implies that every limit point of the class of Finsler manifolds satisfying the assumptions of Theorem 5.8 (or Theorem 5.7) is a LGC space. The uniform upper bound on reversibility (or uniform constant) is very important to the convergence as the following example shows.

Example 1 ([23], [18]). Consider a sequence of compact Finsler 2-manifolds $\{(M_i,F_i)\}$, where $M_i\equiv\mathbb{S}^2$ and in geodesic polar coordinates $(r,\phi)\in(0,\pi)\times[0,2\pi]$,

$$F_i=\frac{\sqrt{(1-\epsilon_i^2\sin^2 r)dr^2+\sin^2 r d\phi^2}-\epsilon_i\sin^2 r d\phi}{1-\epsilon_i^2\sin^2 r},\ \epsilon_i=e^{-\frac{1}{i}}.$$

Note that F_i is defined on $\mathbb{S}^2$ (see [23]). It follows from [23] that $\mathbf{K}_{M_i}=1$ and $\mathbf{S}_{M_i}=0$, where $\mathbf{S}_{M_i}$ is the S-curvature of the Busemann–Hausdorff volume form. By [18, Theorem 11.1], we have $\pi=\mathrm{diam}(M_i)\geq c_{M_i}\geq \mathfrak{i}_{M_i}=\pi$ and

$$\sqrt{\Lambda_i}\geq\lambda_i=\frac{1+e^{-\frac{1}{i}}}{1-e^{-\frac{1}{i}}}\nearrow\infty,\quad \text{as } i\to+\infty,$$

where Λ_i is the the uniform constant and λ_i is the reversibility of F_i.

Thus, $\{(M_i,F_i)\}$ are LGC spaces and satisfy all the conditions of Theorem 5.8 (or Theorem 5.7) except the reversibility (or uniform constant) condition. But $(\mathbb{S}^2,F_\infty)$ is not a metric space.

In the Riemannian case, Theorem 5.7 and Theorem 5.8 are due to Yamaguchi (cf. [26]).

Let $d\mu$ denote either the Busemann–Hausdorff volume form or the Holmes–Thompson volume form. In [27], we showed the following proposition, which is an extension of a result due to Croke [5].

Proposition 5.9 ([27]). *Let (M, F) be a closed reversible Finsler n-manifold. For any $p \in M$ and $0 < r \le r_p$ (or $r \le \mathfrak{i}_M/2$), we have*

$$\mu(B_p(r)) \ge \frac{C^n(n, \Lambda_F)}{n^n} r^n,$$

where $C(n, \Lambda_F) := \frac{c_{n-1}}{\Lambda_F^{(6n+5)/2}(c_n/2)^{1-1/n}}$ *and* $c_n := \mathrm{Vol}(\mathbb{S}^n)$.

Then we obtain the following finiteness theorem, which is an extension of [26, Corollary 2].

Theorem 5.10. *For any n and positive numbers i, V, δ, the class of closed reversible Finsler n-manifolds (M, F) with injectivity radius $\mathfrak{i}_M \ge i$, $\Lambda_F \le \delta$ and $\mu(M) \le V$, contains at most finitely many homotopy types.*

Proof. Since $c_M \ge \mathfrak{i}_M \ge i$, (M, F) is LGC(ρ), where ρ is the identity map of $[0, i)$. By Proposition 5.9, $\mu(B_p(\epsilon)) \ge C(n, \delta)\epsilon^n$ for all $p \in M$ and $\epsilon \le i/2$. Since $\mathrm{Cov}(M, \epsilon) \le \mathrm{Cap}(M, \epsilon)$,

$$\mathrm{Cov}(M, \epsilon) \le \frac{\mu(M)}{C(n, \delta)(\epsilon/2)^n} = C'(n, \delta, V)\epsilon^{-n}.$$

Define the covering function $N(\epsilon) := C'(n, \delta, V)\epsilon^{-n}$, $\epsilon \in (0, i/2)$. The conclusion now follows from Corollary 4.6. □

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