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On fixed point of a Ljubomir Ciric quasi-contraction mapping in generalized metric spaces

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Abstract. The aim of this paper is to present a correct proof of the Ciric's theorem in generalized metric spaces presented by B. K. Lahiri and P. Das in [8].

1. Preliminaries

In 2000 Branciari [1] introduced the concept of generalized metric spaces (**gms**) where the triangular inequality of a metric space has been replaced with the tetrahedral inequality:

Definition 1.1 ([1]). Let X be a set and $d : X^2 \to R^+$ a mapping such that for all $x, y \in X$ and for all distinct points $z, w \in X$, each of them different from x and y, one has

(a) $d(x, y) = 0$ if and only if $x = y$,

(b) $d(x, y) = d(y, x)$,

(c) $d(x, y) \leq d(x, z) + d(z, w) + d(w, y)$ (Tetrahedral inequality).

Then d is called *a generalized metric* and (X, d) is a *generalized metric space* (or shortly **gms**).

The following example shows that: in a **gms**, contrary to the case of a metric space, the "open" balls $B(a, r) = \{x \in X : d(x, a) < r\}$ are not always open sets and, moreover, the generalized metric d is not always necessarily continuous

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with respect to its variables. Also, the generalized metric space is not always a Hausdorff space and a convergent sequence $\{x_n\}$ in gms is not always a Cauchy sequence. In these circumstances, not every theorem of fixed points for metric spaces can be extended in gms as well. Even in the cases it may be done, the proof of theorem is more complicated and it may require additional conditions.

Example 1.2. Let $X = \left\{1-\frac{1}{n} : n = 1, 2, \dots\right\} \cup \{1, 2\}$. Define $d : X \times X \to R$ as follows:

$$d(x,y) = \begin{cases} 0 & \text{for } x = y \\ \dfrac{1}{n} & \text{for } x \in \{1,2\} \text{ and } y = 1 - \dfrac{1}{n} \text{ or } y \in \{1,2\} \text{ and } x = 1 - \dfrac{1}{n},\ x \neq y \\ 1 & \text{otherwise} \end{cases}$$

Then it is easy to see that (X, d) is a generalized metric space and is not a metric space because it lacks the triangular *inequality:*

$$1 = d\left(\frac{1}{2}, \frac{2}{3}\right) > d\left(\frac{1}{2}, 1\right) + d\left(1, \frac{2}{3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Note that the sequence $\{x_n\} = \{1 - \frac{1}{n}\}$ converges to both 1 and 2 and it is not a Cauchy sequence: $d(x_n, x_m) = d(1 - \frac{1}{n}, 1 - \frac{1}{m}) = 1$, $\forall n, m \in N$.

Since $B(1, r) \cap B(2, r) \neq \phi$ for all $r > 0$, the (X, d) is not a Hausdorff generalized metric space.

The function d is not continuous distance in a sense presented in [1], since although $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1$, we have $1 = \lim_{n\to\infty} d\left(1 - \frac{1}{n}, \frac{1}{2}\right) \neq d\left(1, \frac{1}{2}\right) = \frac{1}{2}$.

In the papers [1], [3], [4], [8], the properties of metric spaces mentioned above, are considered true for gms too which consequently resulted in incorrect proofs. For example, although the generalized distance d may be not continuous, the proof of the main theorem in [8] is done considering d to be continuous in two moments:

1. At last of page 593, where with $m \to \infty$ in (5), the inequality (7) is obtained and

2. In the beginning of page 594 where with $n \to \infty$ the following inequality is obtained

$$d(Tu, u) \leq qd(Tu, u).$$

In the following section we present a correct proof of the Ciric's quasi-contraction principle in a generalized metric space presented by B. K. Lahiri and P. Das [8].

2. Ciric's quasi-contraction principle in a generalized metric space

Definition 2.1 ([8]). A mapping $T : X \to X$ where X is a gms is said to be a quasi-contraction if and only if there exists a number $q, 0 \le q < 1$ such that

$$d(Tx, Ty) \le q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{1}$$

hold for all $x, y \in X$.

Theorem 2.2 ([8]). *Let $T : X \to X$ a quasi-contraction on $X((X, d)$ is a gms) and let X be T-orbitally complete. Then*

(a) *T has a unique fixed point α in X,*

(b) *$\lim_{n\to\infty} T^n x = \alpha$, for every $x \in X$ and*

(c) *$d(T^n x, \alpha)) \le \frac{q^n}{1-q} \max\{d(x, Tx), d(x, T^2 x)\}$, for all $n \in N$.*

Proof. The proof is the same as in [8] until the following inequality is obtained:

$$d(T^n x, T^m x)) \le \frac{q^n}{1-q} \max\{d(x, Tx), d(x, T^2 x)\}. \tag{2}$$

Then it must be continued as follows:

We divide the proof into two cases:

Case I: Suppose $x_m = x_n$ for some $m, n \in N, m \ne n$. Let $m > n$. Then $T^m x = T^{m-n} T^n x = T^n x$ i.e. $T^k \alpha = \alpha$ where $k = m - n$ and $T^n x = \alpha$. Now, if $k > 1$, then we have $\alpha = T^k \alpha = T^{rk} \alpha$, $r \in N$ and by (2), we get

$$\begin{aligned} d(\alpha, T\alpha) &= d(T^k \alpha, T^{k+1} \alpha) = d(T^{rk} \alpha, T^{rk+1} \alpha) = d(T^{rk+n} x, T^{rk+n+1} x) \\ &\le \frac{q^{rk+n}}{1-q} \max\{d(x, Tx), d(x, T^2 x)\}, \quad \forall r \in N. \end{aligned}$$

Since $\lim_{r\to\infty} q^{rk+n} = 0$, $d(\alpha, T\alpha) = 0$. So $T\alpha = \alpha$ and hence α is a fixed point of T.

Case II: Assume that $x_n \ne x_m$ for all $n \ne m$. Then $\{x_n\} = \{T^n x\}$ is a sequence of distinct points. By (2), we have

$$d(x_n, x_{n+m}) = d(T^n x, T^{n+m} x) \le \frac{q^n}{1-q} \max\{d(x, Tx), d(x, T^2 x)\}.$$

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0. \tag{3}$$

It implies that $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is T-orbitally complete, there exists a $\alpha \in X$ such that

$$\lim_{n\to\infty} x_n = \alpha. \tag{4}$$

We now prove that the limit α is unique. Suppose on the contrary that $\lim\limits_{n\to\infty} x_n = \alpha'$ also where $\alpha' \neq \alpha$.

Since $x_n \neq x_m$ for all $n \neq m$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \neq \alpha$ and $x_{n_k} \neq \alpha'$ for all $k \in N$. Without loss of generality, assume that $\{x_n\}$ is this subsequence. Then, by *tetrahedral inequality,* we obtain

$$d(\alpha, \alpha') \leq d(\alpha, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, \alpha').$$

Letting n tend to infinity we get $d(\alpha, \alpha') = 0$ and so $\alpha = \alpha'$.

Let we prove now that α is a fixed point of T. In contrary, if $\alpha \neq T\alpha$, then there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \neq T\alpha$ and $x_{n_k} \neq \alpha$ for all $k \in N$.

By *tetrahedral inequality*, we obtain

$$d(\alpha, T\alpha) \leq d(\alpha, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, T\alpha).$$

Then, if $k \to \infty$, we get

$$d(\alpha, T\alpha) \leq \overline{\lim_{k\to\infty}}\, d(x_{n_k}, T\alpha). \tag{5}$$

From (1),

$$\begin{aligned} d(x_n, T\alpha) &= d(Tx_{n-1}, T\alpha) \\ &\leq q \max\{(d(x_{n-1}, \alpha), d(x_{n-1}, Tx_{n-1}), d(\alpha, T\alpha), d(x_{n-1}, T\alpha), d(\alpha, Tx_{n-1})\} \\ &= q \max\{(d(x_{n-1}, \alpha), d(x_{n-1}, x_n), d(\alpha, T\alpha), d(x_{n-1}, T\alpha), d(\alpha, x_n)\}. \end{aligned}$$

Letting n tend to infinity, by $\overline{\lim}_{n\to\infty} d(x_n, T\alpha) = \overline{\lim}_{n\to\infty} d(x_{n-1}, T\alpha)$, we get

$$\overline{\lim_{n\to\infty}}\, d(x_n, T\alpha) \leq q \max\{(0, 0, d(\alpha, T\alpha), \overline{\lim_{n\to\infty}}\, d(x_{n-1}, T\alpha), 0\} \leq q d(\alpha, T\alpha). \tag{6}$$

From (5) and (6),

$$d(\alpha, T\alpha) \leq \overline{\lim_{k\to\infty}}\, d(x_{n_k}, T\alpha) \leq \overline{\lim_{n\to\infty}}\, d(x_n, T\alpha) \leq q d(\alpha, T\alpha).$$

Since $0 \leq q < 1$, we have $d(\alpha, T\alpha) = 0$. So α is a fixed point of T.

Let we prove now the uniqueness (for case I and II in the same time). Assume that $\alpha' \neq \alpha$ is also a fixed point of T. From (1) we get

$$d(\alpha, \alpha') = d(T\alpha, T\alpha') \leq q \max\{(d(\alpha, \alpha'), 0, 0, d(\alpha, \alpha'), d(\alpha', \alpha)\} \leq q d(\alpha, \alpha').$$

Since $0 \leq q < 1$, we have $\alpha = \alpha'$. So we have proved (a) and (b). By *tetrahedral inequality* and by (2) we obtain

$$\begin{aligned} d(x_n, \alpha) &\leq d(x_n, x_{n+m}) + d(x_{n+m}, x_{n+m+1}) + d(x_{n+m+1}, \alpha) \\ &\leq \frac{q^n}{1-q} \max\{d(x, Tx), d(x, T^2x)\} + d(x_{n+m}, x_{n+m+1}) + d(x_{n+m+1}, \alpha). \end{aligned}$$

Letting m tend to infinity, we obtain the inequality (c). This completes the proof of the theorem. □

Remark 1. The false properties of generalized metric spaces were first observed by Das and Dey ([5], [6]) where appropriate examples were given and in [5] a general fixed point theorem was proved without the false assumptions. Also these facts were observed independently by Samet [11] and then [9] and also by Sarma, Rao and Rao ([12]) who proved the fixed point theorem by assuming that the generalized metric space is Hausdorff.

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