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# Homogeneous summands of exponentials

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## Introduction

As usual $\mathbb{C}$ denotes the complex numbers, $\mathbb{Z}$ the integers, and $\boldsymbol{T}$ the unit circle. Let $n$ be a positive integer, $\mathbb{Z}_n$ the group of residues modulo $n$, and $\omega$ a primitive $n$'th root of unity.

According to [3, Th. 3.1], any function $f : \mathbb{C} \to \mathbb{C}$ is uniquely the sum of functions, $f = \sum_{j\in\mathbb{Z}_n} f_j$, such that

$$f_j(\omega x) = \omega^j f_j(x) \qquad x \in \mathbb{C} \text{ and } j \in \mathbb{Z}_n\,.$$

Since $\omega^n = 1$, no confusion results from writing $\omega^j$ even if $j \in \mathbb{Z}_n$. The summand $f_j$ is said to be of type $j$. For instance, with $n = 2$, this describes the decomposition of a function into its even and odd parts. If $f$ is entire with series representation $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f_j(x) = \sum_{k\equiv j \,(\mathrm{mod}\; n)} a_k x^k$, where $x \in \mathbb{C}$ and $j \in \mathbb{Z}_n$.

An exponential is a function $f : \mathbb{C} \to \mathbb{C}$ such that

$$(1) \qquad f(x+y) = f(x)f(y) \qquad x, y \in \mathbb{C}\,.$$

The type $j$ summands of an exponential satisfy the system of equations

$$(2) \qquad f_j(x+\omega y) = \sum_{k\in\mathbb{Z}_n} \omega^{j-k} f_k(x) f_{j-k}(y) \qquad x, y \in \mathbb{C},\ j \in \mathbb{Z}_n\,.$$

Furthermore it was demostrated in [3, Th. 3] that any $n$–tuple of functions $(f_j)_{j\in\mathbb{Z}_n}$, satisfying (2), arises from a unique exponential $f$ according to

$$(3) \qquad f_j(x) = \frac{1}{n} \sum_{k\in\mathbb{Z}_n} \omega^{jk} f(\omega^{-k} x) \qquad x \in \mathbb{C},\ j \in \mathbb{Z}_n\,.$$

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In this note we consider a situation which is the analogue of the one above when $n$ is replaced by $\infty$. Here $\mathbb{Z}_n$ gets replaced by $\mathbb{Z}$, and the $n$'th roots of unity by $\boldsymbol{T}$. Finite sums become absolutely convergent series.

## Homogeneous summands

It seems natural to say that a function $f : \mathbb{C} \to \mathbb{C}$ is absolutely decomposable into homogeneous summands provided, for each integer $n$, there is a function $f_n : \mathbb{C} \to \mathbb{C}$ such that

$$f_n(\sigma x) = f_n(x)\sigma^n \qquad \sigma \in \boldsymbol{T},\ x \in \mathbb{C} \tag{4}$$

and

$$f(x) = \sum_{n\in\mathbb{Z}} f_n(x) \ \text{ absolutely} \qquad x \in \mathbb{C} \tag{5}$$

The set of such $f$ will be denoted by $\mathcal{D}$.

**Proposition 1.** *A function $f$ belongs to $\mathcal{D}$ if and only if, for each positive real $r$, the function $f \cdot r : \boldsymbol{T} \to \mathbb{C}$, given by $f \cdot r(\xi) = f(r\xi)$, $\xi \in \boldsymbol{T}$, has absolutely summable Fourier coefficients.*

Proof. If $f$ in $\mathcal{D}$ decomposes as $f = \sum_{n\in\mathbb{Z}} f_n$, then

$$(f \cdot r)(\xi) = \sum_{n\in\mathbb{Z}} f_n(r\xi) = \sum_{n\in\mathbb{Z}} f_n(r)\xi^n \qquad r > 0,\ \xi \in \boldsymbol{T}\,.$$

Since this series converges absolutely, the values $f_n(r)$ give the absolutely summable Fourier coefficients of $f \cdot r$.

Conversely suppose for each $r > 0$ that $f \cdot r$ has absolutely summable Fourier coefficients $(f_n(r))_{n\in\mathbb{Z}}$. If $x \in \mathbb{C}$ and $x \neq 0$, then $x = r\xi$ for a unique $r > 0$ and unique $\xi$ in $\boldsymbol{T}$; and

$$f(x) = f(r\xi) = (f \cdot r)(\xi) = \sum_{n\in\mathbb{Z}} f_n(r)\xi^n$$

with the convergence absolute. For every integer $n$ define $f_n : \mathbb{C} \to \mathbb{C}$ by $f_n(x) = f_n(r)\xi^n$ when $x = r\xi$, $r > 0$, $\xi \in \boldsymbol{T}$; and by $f_n(0) = \delta_{0,n}f(0)$, where $\delta$ is the Knonecker delta. Each $f_n$ satisfies (4), and $f = \sum_{n\in\mathbb{Z}} f_n$ as in (5). Thus $f \in \mathcal{D}$. $\square$

The above proposition gives the recipe for constructing all possible decomposable functions. Because each $f \cdot r$ is continuous, it also implies that each $f$ in $\mathcal{D}$, restricted to any circle centred at the origin, must be continuous.

**Proposition 2.** *If $f \in \mathcal{D}$, then the homogeneous summands of $f$ are unique.*

Proof. It suffices to see that the zero function $\mathbf{0}$ has a unique decomposition. If $\mathbf{0} = \sum_{n\in\mathbb{Z}} f_n$ absolutely with $f_n$ homogeneous, then $\sum_{n\in\mathbb{Z}} f_n(r)\xi^n = 0$ for every $r > 0$ and $\xi$ in $\boldsymbol{T}$. This forces $f_n(r) = 0$ due to [2, Sec. 6.1], and thus $f_n(x) = 0$ when $x \neq 0$. When $x = 0$ the requirement $f_n(0) = f_n(\sigma 0) = f_n(0)\sigma^n$, for each $n$ in $\mathbb{Z}$ and $\sigma$ in $\boldsymbol{T}$, yields $f_n(0) = \delta_{0,n}\mathbf{0}(0) = 0$. Thus the homogeneous summands for the zero function must all be zero. □

The space $\ell^1 = \ell^1(\mathbb{Z})$ of absolutely summable complex bilateral sequences is a commutative algebra under convolution:

$$(x_n)_{n\in\mathbb{Z}} * (y_n)_{n\in\mathbb{Z}} = \left(\sum_{k\in\mathbb{Z}} x_k y_{n-k}\right)_{n\in\mathbb{Z}} \qquad (x_n)_{n\in\mathbb{Z}},\ (y_n)_{n\in\mathbb{Z}} \in \ell^1 .$$

The set $\mathcal{F}$ of all maps $\mathbb{C} \to \ell^1$ inherits this algebra structure under pointwise operations. In particular for $F, G : \mathbb{C} \to \ell^1$ in $\mathcal{F}$ the product $F * G$ is given by

$$(F * G)(x) = F(x) * G(x) \qquad x \in \mathbb{C} .$$

Each $F$ in $\mathcal{F}$ has component maps $f_n : \mathbb{C} \to \mathbb{C}$, i.e. $F(x) = (f_n(x))_{n\in\mathbb{Z}}$, $x \in \mathbb{C}$. The set $\mathcal{A}$ of those $F$ such that their components $f_n$ satisfy (4) constitutes a subalgebra of $\mathcal{F}$. To see this, let $F = (f_n)_{n\in\mathbb{Z}}$, $G = (g_n)_{n\in\mathbb{Z}} \in \mathcal{A}$, and let $H = F * G$. If $x \in \mathbb{C}$ and $H(x) = (h_n(x))_{n\in\mathbb{Z}}$, then $h_n(x) = \sum_{k\in\mathbb{Z}} f_k(x) g_{n-k}(x)$. For $\sigma$ in $\boldsymbol{T}$

$$\begin{aligned} h_n(\sigma x) &= \sum_{k\in\mathbb{Z}} f_k(\sigma x) g_{n-k}(\sigma x) = \sum_{k\in\mathbb{Z}} f_k(x)\sigma^k g_{n-k}(x)\sigma^{n-k} \\ &= \left(\sum_{k\in\mathbb{Z}} f_k(x) g_{n-k}(x)\right)\sigma^n = h_n(x)\sigma^n . \end{aligned}$$

Thus $H \in \mathcal{A}$.

**Proposition 3.** *Under pointwise addition and multiplication the set $\mathcal{D}$ of all decomposable functions $f : \mathbb{C} \to \mathbb{C}$ is an algebra isomorphic to $\mathcal{A}$.*

Proof. The isomorphism $\mathcal{A} \to \mathcal{D}$ is given by $F = (f_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} f_n$. Proposition 2 ensures that this is a bijection. Pointwise convolution in $\mathcal{A}$ corresponds to multiplication in $\mathcal{D}$ because absolutely summable series obey the distributive law, and the order of summation is immaterial. □

**Proposition 4.** *If $f : \mathbb{C} \to \mathbb{C}$ is a continuous exponential, then $f \in \mathcal{D}$.*

Proof. If $f$ is the zero function, then $f \in \mathcal{D}$ for trivial reasons. Otherwise, as is well known (see e.g. [1, Ch. 5, Th. 3]), there exist $a, b$ in $\mathbb{C}$ such that

$$f(x) = e^{ax+b\bar{x}} = e^{ax}e^{b\bar{x}} \qquad x \in \mathbb{C}\,.$$

Let $r > 0$, $\xi \in \boldsymbol{T}$ and $x = r\xi$. Then

$$e^{ax} = e^{ar\xi} = \sum_{n\geq 0} \frac{a^n r^n}{n!}\xi^n, \quad e^{b\bar{x}} = e^{br\xi^{-1}} = \sum_{n\geq 0} \frac{b^n r^n}{n!}\xi^{-n}\,.$$

Since the coefficients $a^n r^n/n!$ and $b^n r^n/n!$ are absolutely summable, Proposition 1 yields that the maps $x \mapsto e^{ax}$, $x \mapsto e^{b\bar{x}}$ are in $\mathcal{D}$. Their product $f$ is in $\mathcal{D}$, from Proposition 3. $\square$

Decompose a continuous exponential $f$ as $\sum_{n\in\mathbb{Z}} f_n$ where $f_n$ satisfy (4). Since $f$ satisfies (3), it follows from Proposition 3 that

$$f_n(x+y) = \sum_{k\in\mathbb{Z}} f_k(x)f_{n-k}(y) \qquad x, y \in \mathbb{C},\ n \in \mathbb{Z}\,. \tag{6}$$

Equations (4) and (6), for the homogeneous summands of a continuous exponential, seem to be the natural extension of (2) to the limiting case of $\infty$. The point of this note is that, as in [3, Th. 3], an exponential $f$, now continuous, can be recovered from (4) and (6).

**Theorem 5.** *Suppose $F = (f_n)_{n\in\mathbb{Z}} \in \mathcal{A}$ and $F(x+y) = F(x) * F(y)$ for $x, y$ in $\mathbb{C}$; that is, $f_n$ satisfy (4) and (6) as well as absolute summability. Then the map $f = \sum_{n\in\mathbb{Z}} f_n$ is a continuous exponential with the $f_n$'s as its homogeneous components.*

Proof. From Proposition 3 it is clear that $f$ is an exponential and that the $f_n$'s give its homogeneous components. There remains the question of the continuity of $f$.

As observed after Proposition 1, the restriction of $f$ to any circle centred at the origin is continuous. By considering $f(x+y) = f(x)f(y)$ where $x, y \in \mathbb{C}$, fixing $y$ and letting $x$ vary, it is seen that the restriction of $f$ to a circle with any centre is continuous. This consideration also shows that, if $f$ is continuous at one point in $\mathbb{C}$, then $f$ is continuous at all points. Thus it suffices to prove the continuity of $f$ at the point $1 + i$, $i = \sqrt{-1}$.

Consider the smooth map $\varphi$, of two real variables, given by

$$\varphi : s + it \mapsto \left(s + 1 - \sqrt{2t - t^2}\right) + i\left(t + 1 - \sqrt{2s - s^2}\right) \quad 0 < s, t < 2\,.$$

A simple verification shows that $\varphi(1+i)=1+i$, and the Jacobian matrix of $\varphi$ at $1+i$ is the $2\times 2$ identity matrix. By the inverse function theorem, the restriction of $\varphi$ to some open neightbourhood $V$ of $1+i$ is an open mapping and injective. Since $\varphi(V)$ is now another open neighbourhood of $1+i$, homeomorphic to $V$ by $\varphi$, $f$ becomes continuous at $1+i$ provided $f\circ\varphi$ restricted to $V$ is continuous at $1+i$.

Suppose $s_n+it_n$ is a sequence in $V$ and $s_n+it_n\to 1$ as $n\to\infty$, i.e. $s_n\to 1,\ t_n\to 1$. Let

$$u_n = s_n + i\left(1-\sqrt{2s_n - s_n^2}\right), \qquad v_n = 1-\sqrt{2t_n-t_n^2}+it_n\,.$$

Clearly $u_n\to 1,\ v_n\to i$ and $u_n+v_n=\varphi(s_n+t_n)$. Now observe that $|1+i-u_n|=|1+i-v_n|=1$, which means that $u_n, v_n$ lie on the circle $K$ of centre $1+i$ and radius 1. The exponential $f$ when restricted to such $K$, is continuous. Thus $f(u_n)\to f(1),\ f(v_n)\to f(i)$ as $n\to\infty$. This leads to the conclusion that

$$f\circ\varphi(s_n+it_n)=f(u_n+v_n)=f(u_n)f(v_n)\to \\ f(1)f(i)=f(1+i)=f\circ\varphi(1+i)\,,$$

as desired. □

The above proof leads one to speculate, out of mere curiosity, whether any function $\mathbb{C}\to\mathbb{C}$, having continuous restrictions to all circles in $\mathbb{C}$, can still possess a point of discontinuity.

What is the analogue of (3) when $n$ is replaced by $\infty$? Suppose $f:\mathbb{C}\to\mathbb{C}$ is a continuous exponential; and thus $f=\sum_{n\in\mathbb{Z}} f_n$ absolutely such that the $f_n$'s satisfy (4) and (6). If $x\in\mathbb{C}$ and $\sigma\in\boldsymbol{T}$, then $f(\sigma x)=\sum_{n\in\mathbb{Z}} f_n(x)\sigma^n$. Since the coefficients of $\sigma^n$ are absolutely summable, this series converges uniformly over all $\sigma$ in $\boldsymbol{T}$. For every integer $j$, term by term integration of $\sigma^{-j}f(\sigma x)$ over $\sigma$ in $\boldsymbol{T}$ gives the summand $f_j$ in terms of the exponential. Namely

$$\frac{1}{2\pi}\int_{\boldsymbol{T}}\sigma^{-j}f(\sigma x)d\sigma=\sum_{n\in\mathbb{Z}} f_n(x)\frac{1}{2\pi}\int_{\boldsymbol{T}}\sigma^{n-j}d\sigma = f_j(x)\,,$$

for every $x$ in $\mathbb{C}$. The analogue of the summation (3) is the familiar computation of Fourier coefficients over $\boldsymbol{T}$, the dual group of $\mathbb{Z}$.

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