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Stability of points in mean value theorems

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Abstract. We obtain Hyers–Ulam stability results for points in Cauchy, Pompeiu and Taylor mean value theorems.

1. Introduction

S. M. Ulam, on a talk given at Winsconsin University, posed the following question:

Question (Ulam, 1940, [14]). Let $(G, \cdot, \varrho)$ be a metric group, $\epsilon > 0$ and a mapping $f : G \to G$ which satisfies the inequality

$$\varrho(f(x \cdot y), f(x) \cdot f(y)) \leq \epsilon, \quad \forall x, y, \in G.$$

Does there exists a constant $k > 0$ depending only on G and a homomorphism $a : G \to G$ such that

$$\varrho(f(x), a(x)) \leq k\epsilon, \quad \forall x \in G?$$

If the answer is **yes** we say that the equation of the homomorphism

$$a(x \cdot y) = a(x) \cdot a(y)$$

is stable.

The first answer to the Ulam's question was given by D. H. Hyers for the Cauchy functional equation in Banach spaces. More precisey he proved the following theorem:

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Theorem 1 (Hyers, 1941, [5]). *Let X, Y be Banach spaces, $\epsilon > 0$, and $f : X \to Y$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon, \quad \forall x, y \in X.$$

Then there exists a unique additive mapping $a : X \to X$ with the property that

$$\|f(x) - a(x)\| \leq \epsilon, \quad \forall x \in X.$$

A similar question was formulated in the celebrated book by Pólya and Szegő [17] for functions defined on the set of integers.

Due to the question of Ulam and the answer of Hyers the stability of functional equations is called after their names. Generally we say that a functional equation is stable in Hyers–Ulam sense if for every solution of the perturbed equation, *called approximate* solution, there exists a solution of the equation (*exact solution*) near it. For more details, approaches and results on this topic we refer the reader to [1], [2], [9], [11], [12], [15], [16]. A similar question can be formulated for the points in mean value theorems: "Suppose that a function f satisfies a mean value theorem with a point c. If d is a point close to c, does there exists a function g close to f satisfying the same mean value theorem with the point d?"

It seems that the first result to the previous question was given by D. H. Hyers and S. M. Ulam in the case of differential expressions:

Theorem 2 (Ulam, Hyers, 1954, [6]). *Let $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ be a function having a n-th derivative in a neighborhood N of a point $a \in \operatorname{int} I$. If $f^{(n)}(a) = 0$, and $f^{(n)}$ changes the sign at $x = a$, then corresponding to each $\epsilon > 0$ there exists a $\delta > 0$ such that, for each $g : I \to \mathbb{R}$ having an n-th derivative in N and satisfying the inequality $|g(x) - f(x)| < \delta$ in N there exists a point $b \in N$ such that $g^{(n)}(b) = 0$ and $|b - a| < \epsilon$.*

By a neighborhood N of a point a we mean an open interval centered in a. Recent results on the stability of points in mean value theorems were obtained for the Flett, Sahoo–Riedel and Lagrange points.

Recall in what follows these theorems and the corresponding stability results.

Theorem 3 (Flett, 1958, [4]). *Let $f : [a, b] \to \mathbb{R}$ be a function of class C^1 with the property $f'(a) = f'(b)$. Then there exists a point $c \in (a, b)$ such that*

$$f(c) - f(a) = f'(c)(c - a).$$

Abel, Ivan, Riedel [7] proved a Flett type theorem for divided differences. In 1998, Sahoo and Riedel obtained a generalization of Flett's mean value theorem removing the boundary assumptions on the derivatives.

Theorem 4 (Sahoo, Riedel, 1998, [3]). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on $[a, b]$. Then there exists a point $c \in (a, b)$ such that*

$$f(c) - f(a) = f'(c)(c-a) - \frac{f'(b) - f'(a)}{2(b-a)}(c-a)^2.$$

The Flett's theorem follows from Theorem 4 if $f'(a) = f'(b)$. The stability of Sahoo–Riedel point was obtained by W. Lee, S. Xu and F. Ye. They observed a gap in the proof of stability point in Flett's mean value theorem given in [3] and proved the following result:

Theorem 5 (W. Lee, S. Xu, F. Ye, 2009, [10]). *Let $f, h : [a, b] \to \mathbb{R}$ be differentiable functions and η be a Sahoo–Riedel point of f in (a, b). If f has a 2-nd derivative at η and*

$$f''(\eta)(\eta - a) - 2f'(\eta) + \frac{2(f(\eta) - f(a))}{\eta - a} \neq 0$$

then corresponding to any $\epsilon > 0$ and any neighborhood $N \subset (a, b)$ of η there exists a $\delta > 0$ such that, for every h satisfying

$$|h(x) - h(a) - (f(x) - f(a))| < \delta$$

for all $x \in N$ and

$$h'(b) - h'(a) = f'(b) - f'(a)$$

there exists a point $\xi \in N$ sucht that ξ is a Sahoo–Riedel point of h and $|\xi - \eta| < \epsilon$.

As a consequence follows the stability of Flett's point. P. Găvruţă and S. M. Jung proved, in appropiate conditions, the stability of Lagrange mean value point.

Theorem 6 (Găvruţă, Jung, 2010, [8]). *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^2, $a, b \in \mathbb{R}$, $a < b$ and suppose that $\eta \in (a, b)$ is the unique Lagrange's mean value point of f in (a, b) and $f''(\eta) \neq 0$. Then for $\epsilon > 0$ there exists $\delta > 0$ such that for every differentiable function $g : \mathbb{R} \to \mathbb{R}$ with $|f(x) - g(x)| < \delta$, $\forall x \in [a, b]$ there exists a Lagrange's mean value point $\xi \in (a, b)$ of g with $|\eta - \xi| < \epsilon$.*

The goal of our paper is to give stability results for Cauchy, Pompeiu and Taylor points.

2. Stability of Cauchy, Pompeiu and Taylor points

In what follows let $I \subseteq \mathbb{R}$ be an open interval and $a, b \in I$, $a < b$. Let $f, g : I \to \mathbb{R}$ be differentiable functions. A number $\eta \in (a, b)$ is called a Cauchy point of the pair (f, g) (on the interval $[a, b]$) if

$$(f(b) - f(a))g'(\eta) - (g(b) - g(a))f'(\eta) = 0.$$

Let $f : I \to \mathbb{R}$ be a differentiable function with the property that $0 \notin [a, b]$. A point $\eta \in (a, b)$ is called a Pompeiu point (see [13], p. 83) if

$$\frac{af(b) - bf(a)}{a - b} = f(\eta) - \eta f'(\eta).$$

Theorem 7 (Stability of Cauchy Points)**.** *Assume that $f, g : I \to \mathbb{R}$ are continuously differentiable functions, η is the unique Cauchy point of (f, g) in the open interval (a, b) and f, g are twice continuously differentiable in a neighborhood of η, satisfying*

$$f''(\eta)(g(b) - g(a)) - g''(\eta)(f(b) - f(a)) \neq 0.$$

Then, for every $\epsilon > 0$ there exists $\delta > 0$ such that, if $f_1, g_1 : I \to \mathbb{R}$ are continuously differentiable functions with the property that $|f(x) - f_1(x)| < \delta$ and $|g(x) - g_1(x)| < \delta$ $\forall x \in [a, b]$ there exists a Cauchy point $\xi \in (a, b)$ of (f_1, g_1) with $|\xi - \eta| < \epsilon$.

Proof. Consider the auxiliary function $H_{f,g} : I \to \mathbb{R}$,

$$H_{f,g}(x) = (f(b) - f(a))(g(x) - g(a)) - (g(b) - g(a))(f(x) - f(a)) \quad \forall x \in I.$$

$H_{f,g}$ is twice differentiable on I and $H_{f,g}(a) = H_{f,g}(b) = 0$.
By Rolle theorem $\exists\ \eta^*$ such that

$$H'_{f,g}(\eta^*) = (f(b) - f(a))g'(\eta^*) - (g(b) - g(a))f'(\eta^*) = 0$$

so η^* is a Cauchy point of (f, g) in (a, b) and by the uniqueness of η in (a, b) it follows that $\eta^* = \eta$.

Since

$$H''_{f,g}(\eta) = (f(b) - f(a))g''(\eta) - (g(b) - g(a))f''(\eta) \neq 0$$

and f'', g'' are continuous in a neighborhood of η it follows that there exists $\sigma > 0$ such that $H''_{f,g} > 0$ or $H''_{f,g} < 0$ on $(\eta - \sigma, \eta + \sigma)$, i.e. $H'_{f,g}$ is strictly increasing or $H'_{f,g}$ is strictly decreasing on $(\eta - \sigma, \eta + \sigma)$, that is $H'_{f,g}$ changes the sign at η.

Define $H_{f_1,g_1} : I \to \mathbb{R}$ by

$$H_{f_1,g_1}(x) = (f_1(b) - f_1(a))(g_1(x) - g_1(a)) - (g_1(b) - g_1(a))(f_1(x) - f_1(a)).$$

Take $\epsilon > 0$. By Theorem 2 it follows that there exists $\delta_1 > 0$ such that for every continuously differentiable functions $f_1, g_1 : I \to \mathbb{R}$ with the property that

$$|H_{f,g}(x) - H_{f_1,g_1}(x)| < \delta_1, \quad \forall x \in [a,b]$$

there exists $\xi \in (a,b)$ with $H'_{f_1,g_1}(\xi) = 0$ (that is ξ is a Cauchy point of (f_1, g_1)) and $|\eta - \xi| < \epsilon$.

For a continuous function $f : [a,b] \to \mathbb{R}$ define

$$V_{f,a,b} := \max\{|f(x) - f(y)| : x, y \in [a,b]\},$$

and analogously $V_{g,a,b}$. Define $V_{f,g,a,b} := \max\{V_{f,a,b}, V_{g,a,b}\}$.

Let $\delta = \min\{V_{f,g,a,b}, \frac{\delta_1}{16V_{f,g,a,b}}\}$ and $f_1, g_1 : I \to \mathbb{R}$ satisfying

$$|f(x) - f_1(x)| < \delta \quad \text{and} \quad |g(x) - g_1(x)| < \delta, \quad \forall x \in [a,b].$$

It follows that

$$\begin{aligned}
|f_1(x) - f_1(y)| &= |f_1(x) - f(x) + f(x) - f(y) + f(y) - f_1(y)| \\
&\leq |f_1(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_1(y)| \\
&< 2\delta + V_{f,a,b} \leq 3V_{f,g,a,b}.
\end{aligned}$$

This implies that $V_{f_1,a,b} \leq 3V_{f,g,a,b}$. The same same relation holds for $V_{g_1,a,b}$, so

$$V_{f_1,g_1,a,b} = \max\{V_{f_1,a,b}, V_{g_1,a,b}\} \leq 3V_{f,g,a,b}$$

For the completeness of the proof it remains to show that

$$|H_{f,g}(x) - H_{f_1,g_1}(x)| < \delta_1,$$

$\forall x \in [a,b]$. We have:

$$\begin{aligned}
|H_{f,g}(x) - H_{f_1,g_1}(x)| &= |(f(b) - f(a))(g(x) - g(a)) - (g(b) - g(a))(f(x) - f(a)) \\
&\quad - [(f_1(b) - f_1(a))(g_1(x) - g_1(a)) - (g_1(b) - g_1(a))(f_1(x) - f_1(a))]| \\
&= |(f(b) - f(a))(g(x) - g(a)) - (f_1(b) - f_1(a))(g(x) - g(a)) \\
&\quad + (f_1(b) - f_1(a))(g(x) - g(a)) - (f_1(b) - f_1(a))(g_1(x) - g_1(a)) \\
&\quad - (g(b) - g(a))(f(x) - f(a)) + (g_1(b) - g_1(a))(f(x) - f(a))
\end{aligned}$$

$$
\begin{aligned}
&- (g_1(b) - g_1(a))(f(x) - f(a)) + (g_1(b) - g_1(a))(f_1(x) - f_1(a))| \\
&\le |(g(x) - g(a))| \cdot |f(b) - f(a) - f_1(b) + f_1(a)| \\
&\quad + |f_1(b) - f_1(a)| \cdot |g(x) - g(a) - g_1(x) + g_1(a)| \\
&\quad + |f(x) - f(a)| \cdot |g(b) - g(a) - g_1(b) + g_1(a)| \\
&\quad + |g_1(b) - g_1(a)| \cdot |f(x) - f(a) - f_1(x) + f_1(a)|.
\end{aligned}
$$

$$
\begin{aligned}
|H_{f,g}(x) - H_{f_1,g_1}(x)| &\le |(g(x) - g(a))|(|f(b) - f_1(b)| + |f(a) - f_1(a)|) \\
&\quad + |f_1(b) - f_1(a)(|g(x) - g_1(x)| + |g(a) - g_1(a)| \\
&\quad + |(f(x) - f(a)|(|g(b) - g_1(b)| + |g(a) - g_1(a)|) \\
&\quad + |g_1(b) - g_1(a)|(|f(x) - f_1(x)| + |f(a) - f_1(a)|) \\
&\le V_{f,g,a,b}(|f(b) - f_1(b)| + |f(a) - f_1(a)| \\
&\quad + V_{f_1,g_1,a,b}|g(x) - g_1(x)| + |g(a) - g_1(a)| \\
&\quad + V_{f,g,a,b}|g(b) - g_1(b)| + |g(a) - g_1(a)| \\
&\quad + V_{f_1,g_1,a,b}|f(x) - f_1(x)| + |f(a) - f_1(a)|) \\
&\le 8V_{f,g,a,b}\delta \le V_{f,g,a,b}\frac{\delta_1}{16V_{f,g,a,b}} = \delta_1.
\end{aligned}
$$

The theorem is proved. □

For $g : I \to \mathbb{R}$ the identity function (i.e. $g(x) = x, \forall x \in I$) from Theorem 7 it follows the Lagrange stability theorem proved by Găvruţă–Jung in [8].

Theorem 8 (Stability of Pompeiu Points)**.** *Let a, b be real numbers satisfying $ab \neq 0$ and 0 is not in (a, b). Assume that $f : I \to \mathbb{R}$ is twice continuously differentiable function, η is the unique Pompeiu point of f in the open interval (a, b) and*

$$f''(\eta) \neq 0.$$

Then, for a given $\epsilon > 0$ there exists $\delta > 0$ such that for every function $g : I \to \mathbb{R}$ with $|f(x) - g(x)| < \delta$ for all $x \in [a, b]$, there is a Pompeiu mean value point $\xi \in (a, b)$ of g with $|\xi - \eta| < \epsilon$.

Proof. Consider the function

$$F : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}, \quad F(x) = xf\left(\frac{1}{x}\right).$$

F is twice differentiable on $[\frac{1}{b}, \frac{1}{a}]$ and

$$F'(x) = f\left(\frac{1}{x}\right) - \frac{1}{x}f'\left(\frac{1}{x}\right), \quad F"(x) = \frac{1}{x^3}f"\left(\frac{1}{x}\right).$$

Applying Lagrange mean value theorem on $[\frac{1}{b}, \frac{1}{a}]$ to F it follows that there exists $c \in (\frac{1}{b}, \frac{1}{a})$ such that

$$\frac{F\left(\frac{1}{a}\right) - F\left(\frac{1}{b}\right)}{\frac{1}{a} - \frac{1}{b}} = F'(c)$$

which is equivalent to

$$\frac{bf(a) - af(b)}{b-a} = f\left(\frac{1}{c}\right) - \frac{1}{c}f'\left(\frac{1}{c}\right)$$

and, in view of the uniqueness of η, we have $\eta = \frac{1}{c}$. Generally, observe that if c is a Lagrange point for F then $\frac{1}{c}$ is a Pompeiu point for f. Now let $\epsilon > 0$. Since $F"(c) \neq 0$, according to Theorem 6, it follows that there exists $\delta_1 > 0$ such that for every differentiable function $G : \left[\frac{1}{a}, \frac{1}{b}\right] \to \mathbb{R}$ with

$$|F(x) - G(x)| < \delta_1, \quad \forall x \in \left[\frac{1}{b}, \frac{1}{a}\right]$$

there exists a Lagrange point $d \in \left(\frac{1}{b}, \frac{1}{a}\right)$ of G such that $|d - c| < \epsilon \min\{a^2, b^2\}$. Define

$$g : [a,b] \to \mathbb{R}, \quad g(x) = xG\left(\frac{1}{x}\right).$$

Then $\xi = \frac{1}{d}$ is a Pompeiu point of g. Choose

$$\delta = \delta_1 \max\{|a|, |b|\}.$$

We have

$$|f(x) - g(x)| = |x| \left|F\left(\frac{1}{x}\right) - G\left(\frac{1}{x}\right)\right| \leq |x|\delta_1 \leq \delta, \quad \forall x \in [a,b]$$

and

$$|\xi - \eta| = \left|\frac{1}{c} - \frac{1}{d}\right| = \frac{|c-d|}{|cd|} \leq \frac{\epsilon \min\{a^2, b^2\}}{|cd|} < \epsilon. \qquad \square$$

Let $f : I \to \mathbb{R}$ be a function $(n+1)$- times differentiable at the point $x_0 \in I$. We denote by $T_n^f(x, x_0)$ the Taylor polynomial of n-th degree of the function f at the point x_0, i.e.

$$T_n^f(x, x_0) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \quad x \in I.$$

Theorem 9 (Stability of Taylor Points). *Let $f : I \to \mathbb{R}$ be a function $(n+1)$-times differentiable on the interval I and suppose that η is the unique Taylor point of the function f on the interval $[a,b]$, i.e.,*

$$f(b) = T_n^f(b,a) + \frac{f^{(n+1)}(\eta)}{(n+1)!}(b-a)^{(n+1)}.$$

If f is $(n+2)$-times differentiable in a neighborhood of η and $f^{(n+2)}(\eta) \neq 0$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that if $g : \mathbb{R} \to \mathbb{R}$ is $(n+1)$-times differentiable in I and $|f^{(k)}(x) - g^{(k)}(x)| < \delta$ for every $x \in [a,b]$ and every nonnegative k, $0 \leq k \leq n$, then there exists a Taylor point ξ of g on $[a,b]$ such that $|\xi - \eta| < \epsilon$.

Proof. Let $H_f : I \to \mathbb{R}$ be given by

$$H_f(x) = T_n^f(b,x) + \frac{f(b) - T_n^f(b,a)}{(b-a)^{n+1}}(b-x)^{n+1}, \quad x \in I.$$

Since $H_f(a) = H_f(b) = f(b)$ and $H_f \in C^1[a,b]$, then there exists a point $\theta \in (a,b)$ with $H_f'(\theta) = 0$, according to Rolle's theorem. We get

$$\begin{aligned} H_f'(x) &= \frac{f^{(n+1)}(x)}{n!}(b-x)^n - \frac{f(b) - T_n^f(b,a)}{(b-a)^{n+1}}(n+1)(b-x)^n \\ &= (n+1)(b-x)^n\Big[\frac{f^{(n+1)}(x)}{(n+1)!} - \frac{f(b) - T_n^f(b,a)}{(b-a)^{n+1}}\Big] \end{aligned}$$

The relation $H_f'(\theta) = 0$ leads to

$$f(b) = T_n^f(b,a) + \frac{f^{(n+1)}(\theta)}{(n+1)!}(b-a)^{n+1}$$

and taking account of the uniqueness of η it follows $\theta = \eta$.

Since $f^{(n+2)}(\eta) \neq 0$ there exists a neighborhood of η where $f^{(n+1)}$ is strictly monotone, therefore H_f' changes the sign at η. Then, according to Theorem 2 it follows that for every $\epsilon > 0$ there exists δ_1 such that for every function H_g satisfying

$$|H_f(x) - H_g(x)| < \delta_1, \quad \forall x \in [a,b].$$

there exists a point ξ with $H_g'(\xi) = 0$ and $|\xi - \eta| < \epsilon$.

Now let

$$M := 2\sum_{k=0}^{n} \frac{(b-a)^k}{k!} + 1$$

and δ given by $\delta := \frac{\delta_1}{M}$.

Then for every function $g : I \to \mathbb{R}$ which is $(n+1)$-times differentiable on I and satisfies the relations

$$|f^{(k)}(x) - g^{(k)}(x)| < \delta$$

for all $x \in [a, b]$ and all k, $0 \le k \le n$, we have to show that

$$|H_f(x) - H_g(x)| < \delta_1, \ \forall x \in [a, b].$$

Indeed,

$$\begin{aligned}
|H_f(x) - H_g(x)| &= \Bigg| \sum_{k=0}^{n} \frac{f^{(k)}(x) - g^{(k)}(x)}{k!}(b-x)^k \\
&\quad - \sum_{k=0}^{n} \frac{f^{(k)}(a) - g^{(k)}(a)}{k!}(b-a)^k \left(\frac{b-x}{b-a}\right)^{n+1} + \frac{f(b) - g(b)}{(b-a)^{n+1}}(b-x)^{n+1} \Bigg| \\
&\le \sum_{k=0}^{n} \frac{|f^{(k)}(x) - g^{(k)}(x)|}{k!}(b-a)^k + \sum_{k=0}^{n} \frac{|f^{(k)}(a) - g^{(k)}(a)|}{k!}(b-a)^k \left(\frac{b-x}{b-a}\right)^{n+1} \\
&\quad + |f(b) - g(b)| \frac{(b-a)^{n+1}}{(b-x)^{n+1}} < \sum_{k=0}^{n} \frac{\delta}{k!}(b-a)^k + \sum_{k=0}^{n} \frac{\delta}{k!}(b-a)^k + \delta = M\delta = \delta_1,
\end{aligned}$$

for all $x \in [a, b]$. The relation $H_g^{'}(\xi) = 0$ implies that ξ is a Taylor point of g on $[a, b]$ with $|\xi - \eta| < \epsilon$. The theorem is proved. $\square$

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