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On lower bounds of the first eigenvalue of Finsler–Laplacian

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Abstract. By using Bochner technique and gradient estimate, we give the lower bound estimates of the first eigenvalue of Finsler–Laplacian on Finsler manifolds. These results generalize the corresponding famous theorems in the Riemannian geometry.

1. Introduction

The research on the first (nonzero) eigenvalue of Laplacian plays an important role in global differential geometry. In the Riemannian case, Lichnerowicz [10] advocated it for the first time and gave the lower bound estimate of the first eigenvalue via the restriction of the Ricci curvature. Afterwards, Obata [12] further established a rigidity theorem, demonstrating the optimality of Lichnerowicz' estimate. For the nonnegative Ricci curvature, Li–Yau [9] employed the gradient estimates of the eigenfunctions and got the lower bound estimate of the first eigenvalue via the diameter of the manifolds. Then this method was improved further and the optimal result was obtained by Zhong–Yang ([22]). Recently, Hang–Wang [7] proved that S^1 is the only case for the first eigenvalue attaining its lower bound. Precisely, they achieved the following results respectively.

Theorem 1.1 ([10], [12]). *Let (M,g) be an n-dimensional compact Riemannian manifold without boundary. If the Ricci curvature satisfies*

$$\mathrm{Ric}_M \geq (n-1)k$$

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for some given constant $k > 0$, then

$$\lambda_1 \geq nk,$$

where the equality holds if and only if M is isometric to the n-sphere of constant sectional curvature k, so that the diameter of M is $\frac{\pi}{\sqrt{k}}$.

Theorem 1.2 ([7], [9], [22])**.** *Let (M, g) be an n-dimensional compact Riemannian manifold without boundary. If $\mathrm{Ric}_M \geq 0$, then*

$$\lambda_1 \geq \frac{\pi^2}{d^2},$$

where d denotes the diameter of (M, g) and the equality holds if and only if M is isometric to $S^1\left(\frac{d}{\pi}\right)$.

As a natural generalization of Riemannian manifolds, Finsler manifolds are differentiable manifolds of which on each tangent space one endows a Minkowski norm instead of a Euclidean norm. Recent studies on Finsler manifolds have taken on a new look. Up to now, there have been several different definitions of Finsler–Laplacians, introduced respectively by Bao–Lackey [3], Antonelli–Zastawniak [1], Centroe [4], Thomas [19], and Ge–Shen [6]. By using the Finsler–Laplacian, Ge–Shen gave the Faber–Krahn type inequality for the first Dirichlet eigenvalue of the Finsler–Laplacian in [6]. Wu–Xin [20] proved that for a complete noncompact and simply connected Finsler manifold with finite reversibility λ and nonpositive flag curvature, if $\mathrm{Ric} \leq -a^2 (a > 0)$ and $\sup_M \|S\| < a$, then $\lambda_1 \geq \frac{(a - \sup_M \|S\|)^2}{4\lambda^2}$. Another interesting result on this direction, due to Wang–Xia [21], says that for a compact Finsler measure space, if the *weighted Ricci curvature* (see Definition 2.1 below) $\mathrm{Ric}_N \geq K$, $N \in [n, \infty]$, $K \in R$, then $\lambda_1 \geq \lambda_1(K, N, d)$ where $\lambda_1(K, N, d)$ represents the first eigenvalue of the 1-dimensional problem (see [21] for details).

In this paper we focus on lower bound estimates of the first eigenvalue of the Finsler–Laplacian [6] on Finsler manifolds with an arbitrary volume form $d\mu$. The main purpose is to generalize Theorem 1.1 and Theorem 1.2 into the Finsler case. It should be noted that since the Finsler–Laplacian is a nonlinear operator, some methods used in the Riemannian case are not adaptable any more. To overcome these difficulties, we have to utilize the properties of the weighted gradient and the weighted Laplacian in *weighted Riemannian manifold* (M, g_V) [13], [21]. Here the weighted gradient and weighted Laplacian play an important and reasonable role in studying the first eigenvalue of the Finsler–Laplacian. With the help of them, we can convert some nonlinear problems into the linear ones and even calculate something as simple as in the Riemannian case. In addition, we also make use of Bochner technique and some gradient estimates. Then by using the restriction of *weighted-Ricci curvature* [14] (in the Riemannian case it is just the Ricci curvature), we obtain the estimates on the lower bounds for the first eigenvalue λ_1 of Finsler–Laplacian in $(M, F, d\mu)$. Concretely, we get the main results as follows.

Theorem 1.3. *Let (M, F) be an n-dimensional forward geodesically complete connected Finsler manifold without boundary. If the weighted Ricci curvature and S-curvature satisfy*

$$\mathrm{Ric}_N \geq (n-1)k, \quad \dot{S} \leq \frac{(N-n)(n-1)}{N-1}k$$

for some uniform positive constant k and $N \in (n, \infty)$, where $\dot{S}$ denotes the change rate of the S-curvature along geodesics, then

$$\lambda_1 \geq \frac{n-1}{N-1}Nk.$$

Moreover, the diameter of M is $\sqrt{\frac{N-1}{n-1}}\frac{\pi}{\sqrt{k}}$ if the equality holds.

Theorem 1.4. *Let (M, F) be an n-dimensional forward geodesically complete connected Finsler manifold without boundary. If $S = 0$ and Ricci curvature $\mathrm{Ric} \geq (n-1)k$ for some uniform positive constant k, then*

$$\lambda_1 \geq nk.$$

Moreover, if the equality holds, then the diameter of M is $\frac{\pi}{\sqrt{k}}$, and M is homeomorphic to S^n. In particular, if F is reversible and M has Busemann–Hausdorff volume form, then (M, F) is isometric to $S^n\big(\frac{1}{\sqrt{k}}\big)$.

Theorem 1.5. *Let (M, F) be an n-dimensional compact Finsler manifold without boundary. If the weighted Ricci curvature $\mathrm{Ric}_\infty \geq 0$, then*

$$\lambda_1 \geq \frac{\pi^2}{d^2},$$

where d denotes the diameter of (M, F).

Here the term "weighted Ricci curvature" (Definition 2.1) and the notation "$\dot{S}$" (Definition 2.3) will be given in Section 2 below. If F is a Riemannian metric, above Theorems are in accord with Theorem 1.1 and Theorem 1.2.

2. Preliminaries

Throughout this paper, we assume that M is an n-dimensional oriented smooth manifold without boundary. A *Finsler metric* on M is a function $F : TM \longrightarrow [0, \infty)$ satisfying the following properties (i) F is smooth on $TM\backslash 0$; (ii) $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$; (iii) the induced quadratic form g is positive-definite, where

$$g := g_{ij} dx^i \otimes dx^j, \quad g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}.$$

Here and from now on, we will use the following convention of index ranges unless other stated:

$$1 \leq i, j \cdots \leq n; \quad 1 \leq \alpha, \beta \cdots \leq n-1, \quad \bar{\alpha} = n + \alpha.$$

The projection $\pi : TM \longrightarrow M$ gives rise to the pull-back bundle π^*TM and its dual bundle π^*T^*M over $TM\backslash 0$. In π^*T^*M there is a global section $\omega = [F]_{y^i} dx^i$, called the *Hilbert form*, whose dual is $\ell = \ell^i \frac{\partial}{\partial x^i}, \ell^i = \frac{y^i}{F}$, called the *distinguished field*.

Let $\{e_i\}_{i=1}^n$ be a local orthonormal basis on π^*TM such that its dual basis is $\{\omega^i\}_{i=1}^n$ with $\omega^n = \omega$. As is well known that on the pull-back bundle π^*TM there exists uniquely the *Chern connection* ${}^c\nabla$ with ${}^c\nabla e_i = \omega_i^j e_j$ satisfying

$$d\omega^i = -\omega_j^i \wedge \omega^j, \quad \omega_n^\alpha = \omega^{\bar{\alpha}}, \quad \omega_n^n = 0,$$

$$\omega_j^i + \omega_i^j = -2C_{ijk}\omega_n^k, \quad C_{njk} = 0,$$

where $C_{ijk} = \frac{1}{F} A_{ijk}$ is called the *Cartan tensor*.

Let $u : M \longrightarrow R$ be a smooth function. Then we can view u as its lift on the projective sphere bundle SM. Define

$$du := u_i \omega^i, \tag{2.1}$$

$$du_i - u_j \omega_i^j := u_{i|j}\omega^j + u_{i;\alpha}\omega^{\bar{\alpha}}, \tag{2.2}$$

where "$|$" and "$;$" denote the horizontal covariant derivative with respect to ${}^c\nabla$ and vertical derivative, respectively. Taking exterior differentiation of (2.1) and making use of (2.2), the structure equations with respect to the Chern connection, we have

$$u_{i|j} = u_{j|i}, \quad u_{i;\alpha} = 0.$$

The curvature 2-forms of the Chern connection ${}^c\nabla$ are

$$d\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i := \frac{1}{2} R_j{}^i{}_{kl}\omega^k \wedge \omega^l + P_j{}^i{}_{k\alpha}\omega^k \wedge \omega^{\bar{\alpha}},$$

where $R^i_{j\ kl} = -R^i_{j\ lk}$ and $P^i_{j\ k\alpha} = P^i_{k\ j\alpha}$. The Landsberg curvature is defined as ${P^i}_{jk} := P^i_{n\ jk}$, which satisfies

$$P_{ijk} = \delta_{il}{P^l}_{jk} = -\dot{A}_{ijk}, \quad P_{n\alpha\beta} = 0,$$

where "." denotes the covariant derivative along the Hilbert form. The flag curvature tensor can be defined by

$$R_{\alpha\beta} = \delta_{\alpha\gamma}R^{\gamma}_{n\ \beta n}.$$

For a unit vector $V = V^i e_i$, the flag curvature $K(y;V)$ is

$$K(y;V) = R_{\alpha\beta}V^\alpha V^\beta.$$

The Ricci curvature for (M,F) is defined as

$$\mathrm{Ric} := \mathrm{Ric}(y) = \sum_{\alpha=1}^{n-1} K(y;e_\alpha) = \sum_{\alpha=1}^{n-1} R_{\alpha\alpha}.$$

Clearly, Ricci curvature $\mathrm{Ric}(y)$ is positively homogeneous of degree zero. i.e., $\mathrm{Ric}(\lambda y) = \mathrm{Ric}(y)$ for all $\lambda > 0$.

Now we can introduce the weighted Ricci curvature on the Finsler manifolds, which was defined by Ohta in [14], motivated by the work of Lott–Villani [11] and Sturm [18] on metric measure spaces.

Definition 2.1 ([14]). Given a vector $V \in T_xM$, let $\eta : (-\varepsilon, \varepsilon) \longrightarrow M$ be the geodesic such that $\eta'(0) = V$. We set $d\mu = e^{-\Psi}\,\mathrm{vol}_{\eta'}$ along η, where $\mathrm{vol}_{\eta'}$ is the volume form of $g_{\eta'}$. Define weighted Ricci curvature by

- $\mathrm{Ric}_N(V) := \mathrm{Ric}(V) + \frac{(\Psi\circ\eta)''(0)}{F(V)^2} - \frac{(\Psi\circ\eta)'(0)^2}{(N-n)F(V)^2}$ for $N \in (n,\infty)$,
- $\mathrm{Ric}_\infty(V) := \mathrm{Ric}(V) + \frac{(\Psi\circ\eta)''(0)}{F(V)^2}$.

Remark 2.2. The above definition is slightly different from that in [14] where the weighted Ricci curvature is positively homogeneous of degree two.

As is well known that S-curvature is one of the most important non-Riemannian quantities in Finsler geometry. For any $y \in T_xM\backslash 0$, let $\gamma(t)$ be the geodesic with $\gamma(0) = x, \dot\gamma(0) = y$. Then S-*curvature* is defined by

$$S(x,y) = \frac{d}{dt}[\tau(\gamma(t),\dot\gamma(t))]_{t=0}.$$

An n-dimensional Finsler metric F on a manifold is said to have *constant* S-curvature if $S = (n+1)cF$ for some constant c. In order to measure the rate of change of the S-curvature along geodesics, we give the following

Definition 2.3. For any $y \in T_xM\backslash 0$, define

$$\dot{S}(x,y) = \frac{1}{F^2}\frac{d}{dt}[S(\gamma(t),\dot{\gamma}(t))]_{t=0}, \tag{2.3}$$

where $\gamma(t)$ is geodesic satisfying $\gamma(0)=x, \dot{\gamma}(0)=y$.

Remark 2.4. By Definition 2.3, we get $\dot{S}(x,y) = \frac{1}{F^2}S_{|i}y^i = \frac{1}{F^2}\{S_{x^i}y^i - 2S_{y^i}G^i\}$. It follows that $\dot{S}(x,\lambda y) = \dot{S}(x,y),\ \forall \lambda > 0$. In addition, according to Definition 2.1, $d\mu = e^{-\Psi}\,\mathrm{vol}_{\eta'}$ implies $\Psi = \tau$ along geodesic η, here τ denotes the *distortion* of F with respect to $d\mu$. So by definition of S and $\dot{S}$ we have

$$S = (\Psi\circ\eta)'(0), \quad \dot{S} = \frac{(\Psi\circ\eta)''(0)}{F^2}, \tag{2.4}$$

where Ψ, η are defined in Definition 2.1.

Let $X = X^i\frac{\partial}{\partial x^i}$ be a vector field. Then the *covariant derivative* of X by $v \in T_xM$ with reference vector $w \in T_xM\backslash 0$ is defined by

$$D^w_vX(x) := \left\{v^j\frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(w)v^jX^k(x)\right\}\frac{\partial}{\partial x^i}, \tag{2.5}$$

where Γ^i_{jk} denote the coefficients of the Chern connection given by

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\delta g_{lj}}{\delta x^k} + \frac{\delta g_{lk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l}\right),$$

and

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial y^j}, \quad N^j_i = \frac{\partial G^j}{\partial y^i}, \quad G^i = \frac{1}{4}g^{il}\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\}.$$

Now let $L^* : T^*M \longrightarrow TM$ denote the *Legendre transform.* Then L^* is norm-preserving map satisfying $L^*(a\zeta) = aL^*(\zeta)$, for all $a > 0$, $\zeta \in T^*M$. For a smooth function $u : M \longrightarrow R$, the *gradient vector* of u at x is defined as the Legendre transform of the derivative of u, $\nabla u(x) := L^*(du(x)) \in T_xM$. Explicitly, we can write in coordinates

$$\nabla u(x) := \begin{cases} g^{ij}(x,\nabla u)\dfrac{\partial u}{\partial x^j}\dfrac{\partial}{\partial x^i}, & du(x) \neq 0, \\ 0. & du(x) = 0. \end{cases} \tag{2.6}$$

It is C^∞ on the open set $\{du \neq 0\}$ and C^0 at $\{du = 0\}$. Set $M_V := \{x \in M | V(x) \neq 0\}$ for a vector field V on M, and $M_u := M_{\nabla u}$. For a C^∞ vector field V on M and $x \in M_V$, we define $\nabla V(x) \in T_x^*M \otimes T_xM$ by using the covariant derivative as

$$\nabla V(v) := D_v^V V(x) \in T_xM, \quad v \in T_xM. \tag{2.7}$$

We also set $\nabla^2 u(x) := \nabla(\nabla u)(x)$ for the smooth function $u : M \longrightarrow R$ and $x \in M_u$. Let $\{e_a\}_{a=1}^n$ be a local orthonormal basis with respect to $g_{\nabla u}$ on M_u. (In order to distinguish local orthonormal basis with respect to $g_{\nabla u}$ from that with respect to g_y, we use convention of the index range $1 \le a,\ b, \cdots \le n$.) Using (2.5)–(2.7) and noting that $C_{\nabla u}(\nabla u, e_a, e_b) = 0$, we then have

$$\begin{aligned}
\nabla^2 u &= \sum \left(\nabla^2 u(e_b)\right)\omega^b = \sum \left(D_{e_b}^{\nabla u}(\nabla u)\right)\omega^b = \sum g_{\nabla u}\left(D_{e_b}^{\nabla u}(\nabla u), e_a\right) e_a\omega^b \\
&= \sum \{e_b\left(g_{\nabla u}(\nabla u, e_a)\right) - g_{\nabla u}(\nabla u, D_{e_b}^{\nabla u} e_a)\} e_a\omega^b \\
&= \sum \{e_b(e_a(u)) - \left(D_{e_b}^{\nabla u} e_a\right)(u)\} e_a\omega^b = \sum u_{a|b} e_a\omega^b,
\end{aligned}$$

and

$$\begin{aligned}
g_{\nabla u}(\nabla^2 u(e_a), e_b) &= g_{\nabla u}(D_{e_a}^{\nabla u}(\nabla u), e_b) = e_a\left(g_{\nabla u}(\nabla u, e_b)\right) - g_{\nabla u}(\nabla u, D_{e_a}^{\nabla u} e_b) \\
&= e_a(e_b(u)) - g_{\nabla u}(\nabla u, D_{e_b}^{\nabla u} e_a + [e_a, e_b]) \\
&= e_b(e_a(u)) + [e_a, e_b](u) - g_{\nabla u}(\nabla u, D_{e_b}^{\nabla u} e_a) - [e_a, e_b](u) \\
&= e_b\left(g_{\nabla u}(\nabla u, e_a)\right) - g_{\nabla u}(\nabla u, D_{e_b}^{\nabla u} e_a) \\
&= g_{\nabla u}(D_{e_b}^{\nabla u}(\nabla u), e_a) = g_{\nabla u}(\nabla^2 u(e_b), e_a).
\end{aligned}$$

Namely,

$$u_{a|b} = u_{b|a}, \quad \forall a, b.$$

Next we define the *divergence* of a C^∞ vector field V on M with respect to an arbitrary volume form $d\mu$ by

$$\operatorname{div} V := \sum_{i=1}^n \left(\frac{\partial V^i}{\partial x^i} + V^i \frac{\partial \Phi}{\partial x^i}\right), \tag{2.8}$$

where $d\mu = e^\Phi dx^1 dx^2 \cdots dx^n$. Then the *Finsler–Laplacian* of u can be defined by

$$\Delta u := \operatorname{div}(\nabla u). \tag{2.9}$$

Given a vector field V such that $V \neq 0$ on M_u, we define the *weighted gradient vector* [13], [21] and the *weighted Laplacian* [13], [21] on the weighted Riemannian manifold (M, g_V) by

$$\nabla^V u := \begin{cases} g^{ij}(V)\dfrac{\partial u}{\partial x^j}\dfrac{\partial}{\partial x^i}, & \text{on } \ M_u, \\ 0, & \text{on } \ M \backslash M_u, \end{cases} \qquad \Delta^V u := \operatorname{div}(\nabla^V u). \tag{2.10}$$

Clearly, the relation between the two gradients and that between the two Laplacians are

$$\nabla^{\nabla u} u = \nabla u, \quad \Delta^{\nabla u} u = \Delta u.$$

Let $(M, F, d\mu)$ be an n-dimensional Finsler manifold. If there is a constant λ such that

$$\Delta f = -\lambda f$$

for some function $f \in L^{1,2}(M)$, then the constant λ is called the *eigenvalue* of Δ and the function f is called the *eigenfunction* corresponding to λ. The least nonzero eigenvalue λ_1 of Δ is called the *first eigenvalue* on $(M, F, d\mu)$. Let $\Omega \subset M$ be a domain with compact closure and nonempty boundary $\partial\Omega$. The first eigenvalue $\lambda_1(\Omega)$ of Ω is defined by [15]

$$\lambda_1(\Omega) = \inf_{u \in L_0^{1,2}(\Omega)} \frac{\int_\Omega (F^*(du))^2 d\mu}{\int_\Omega u^2 d\mu},$$

where $L_0^{1,2}(\Omega)$ is the completion of C_0^∞ with respect to the norm

$$\|\varphi\|_\Omega^2 = \int_\Omega \varphi^2 d\mu + \int_\Omega (F^*(d\varphi))^2 d\mu.$$

If $\Omega_1 \subset \Omega_2$ are bounded domains, then $\lambda_1(\Omega_1) \geq \lambda_2(\Omega_2) \geq 0$. Thus, if $\Omega_1 \subset \Omega_2 \subset \cdots \subset M$ are bounded domains so that $\bigcup \Omega_i = M$, then the following limit

$$\lambda_1(M) = \lim_{i \to \infty} \lambda_1(\Omega_i) \geq 0$$

exists, and it is independent of the choice of $\{\Omega_i\}$.

At the end of this section, some lemmas are given below.

Lemma 2.5 (Bonnet–Myers)**.** *Let (M, F) be an n-dimensional forward geodesically complete connected Finsler manifold. If its Ricci curvature satisfies*

$$\mathrm{Ric} \geq (n-1)k$$

for some positive constant k, then M is compact and the diameter of (M, F) is at most $\frac{\pi}{\sqrt{k}}$.

Lemma 2.6 ([13])**.** *Let (M, F) be an n-dimensional Finsler manifold. Given $u \in C^\infty(M)$, we have*

$$\Delta^{\nabla u}\left(\frac{F(\nabla u)^2}{2}\right) - D(\Delta u)(\nabla u) = \|\nabla u\|^2 \,\mathrm{Ric}_\infty(\nabla u) + \|\nabla^2 u\|^2_{HS(\nabla u)} \tag{2.11}$$

as well as

$$\Delta^{\nabla u}\left(\frac{F(\nabla u)^2}{2}\right) - D(\Delta u)(\nabla u) \geq \|\nabla u\|^2 \operatorname{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N} \tag{2.12}$$

for $N \in [n, \infty]$, *point-wise on* M_u. *Here* $\|\nabla^2 u\|^2_{HS(\nabla u)}$ *stands for the Hilbert–Schmidt norm with respect to* $g_{\nabla u}$.

According to Lemma 3.3 in [20], Lemma 3.2 in [13] and our discussion on $\nabla^2 u$ above, we can rewrite the result as

Lemma 2.7. *Let* (M, F) *be an* n*-dimensional Finsler manifold and* $u : M \longrightarrow R$ *a smooth function. Then on* M_u *we have*

$$\Delta u = \operatorname{tr}_{g_{\nabla u}}(\nabla^2 u) - S(\nabla u) = \sum_a u_{a|a} - S(\nabla u), \tag{2.13}$$

where $u_{a|a} = g_{\nabla u}\big(\nabla^2 u(e_a), e_a\big)$ *and* $\{e_a\}_{a=1}^n$ *is a local* $g_{\nabla u}$*-orthonormal basis on* M_u.

Lemma 2.8 ([17])**.** *Let* $(M, F, d\mu)$ *be an* n*-dimensional complete connected Finsler manifold. Suppose that*

$$\operatorname{Ric} \geq (n-1)k, \quad \|S\| \leq \Lambda.$$

Then for any $0 < r < R$,

$$\frac{\operatorname{vol}_F^{d\mu}(B(x,R))}{V_{k,\Lambda,n}(R)} \leq \frac{\operatorname{vol}_F^{d\mu}(B(x,r))}{V_{k,\Lambda,n}(r)},$$

where

$$\|S\|_x := \sup_{X \in T_xM \setminus 0} \frac{S(X)}{F(X)}; \quad V_{k,\Lambda,n}(r) := \operatorname{vol}(S^{n-1}(1)) \int_0^r e^{\Lambda t} s_k(t)^{n-1} dt$$

and s_k *denotes the unique solution to* $y'' + ky = 0$ *with* $y(0) = 0$, $y'(0) = 1$.

3. Proofs of the main results

Theorem 3.1. *Let* (M, F) *be an* n*-dimensional forward geodesically complete connected Finsler manifold without boundary. If the weighted Ricci curvature and* S*-curvature satisfy*

$$\operatorname{Ric}_N \geq (n-1)k, \quad \dot{S} \leq \frac{(N-n)(n-1)}{N-1}k$$

for some uniform positive constant k and $N \in (n, \infty)$, where $\dot{S}$ denotes the change rate of the S-curvature along geodesics, then

$$\lambda_1 \geq \frac{n-1}{N-1} N k.$$

Moreover, the diameter of M is $\sqrt{\frac{N-1}{n-1}} \frac{\pi}{\sqrt{k}}$ if the equality holds.

Proof. First of all, from (2.4) we see that, under the hypothesis in Theorem 3.1,

$$\mathrm{Ric} = \mathrm{Ric}_N - \dot{S} + \frac{S^2}{(N-n)F^2} \geq \frac{(n-1)^2}{N-1} k. \tag{3.1}$$

So, M is compact according to Lemma 2.5.

Let u be the first eigenfunction on (M, F) corresponding to the eigenvalue λ_1. This implies that

$$\Delta u = -\lambda_1 u.$$

Furthermore, from the fact

$$\Delta^{\nabla u} u^2 = \mathrm{div}(\nabla^{\nabla u} u^2) = \mathrm{div}(2u \nabla u) = 2u\Delta u + 2\|\nabla u\|^2$$

we get

$$(\Delta u)^2 = -\lambda_1 u \Delta u = \lambda_1 \left(\|\nabla u\|^2 - \frac{1}{2} \Delta^{\nabla u} u^2 \right). \tag{3.2}$$

Integrating (2.12) and using divergence lemma on M, we obtain

$$\int_M \lambda_1 \|\nabla u\|^2 d\mu \geq \int_M \left(\|\nabla u\|^2 \,\mathrm{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N} \right) d\mu.$$

Thus, the assumption of the Theorem 3.1 and (3.2) yield

$$\int_M \left(\frac{N-1}{N} \lambda_1 - (n-1)k \right) \|\nabla u\|^2 d\mu \geq 0,$$

which means that

$$\lambda_1 \geq \frac{n-1}{N-1} N k.$$

If $\lambda_1 = \frac{n-1}{N-1} N k$, then all of the relevant inequalities become the equalities. We recall the formula (2.12), which was derived from (2.11) and the following inequalities.

$$\|\nabla^2 u\|^2_{HS(\nabla u)} = \mathrm{tr}(B(0)^2) = \frac{(\mathrm{tr}\, B(0))^2}{n} + \|B(0) - \frac{\mathrm{tr}(B(0)}{n} I_n\|^2_{HS}$$

$$
\begin{aligned}
&\geq \frac{(\operatorname{tr} B(0))^2}{n} = \frac{\left(\Delta u + D\Psi(\nabla u)\right)^2}{n} \\
&= \frac{(\Delta u)^2}{N} - \frac{\left(D\Psi(\nabla u)\right)^2}{N-n} + \frac{N(N-n)}{n}\left(\frac{\Delta u}{N} + \frac{D\Psi(\nabla u)}{N-n}\right)^2 \\
&\geq \frac{(\Delta u)^2}{N} - \frac{\left(D\Psi(\nabla u)\right)^2}{N-n},
\end{aligned} \tag{3.3}
$$

where $B(0) = (\nabla^2 u) := (u_{a|b})$ in the sense that $\nabla^2 u(e_a) = \sum_{b=1}^n u_{a|b} e_b$ (cf. [13], p. 9, 11–12), $D\Psi(\nabla u) = S(\nabla u)$ by (2.4). So, under the condition $\lambda_1 = \frac{n-1}{N-1} Nk$ we have

$$
B(0) = \frac{tr(B(0)}{n} I_n, \tag{3.4}
$$

$$
\frac{\Delta u}{N} = -\frac{S(\nabla u)}{N-n}. \tag{3.5}
$$

Obviously from (3.4) we can get

$$
u_{a|a} = u_{b|b}, \quad \forall a, b; \qquad u_{a|b} = 0, \quad \text{for } a \neq b. \tag{3.6}
$$

Substituting (3.5) into (3.3), one has

$$
\|\nabla^2 u\|^2_{HS(\nabla u)} = \frac{\left(\Delta u + S(\nabla u)\right)^2}{n} = \frac{n}{N^2}(\Delta u)^2 = \frac{n\lambda_1^2}{N^2} u^2.
$$

Therefore combining (3.6) and the formula above, it holds that

$$
u^2_{a|a} = \frac{\lambda_1^2 u^2}{N^2}, \ \forall a. \tag{3.7}
$$

However, from Lemma 2.7 and (3.6) we also have

$$
-\lambda_1 u = \Delta u = n u_{a|a} - S(\nabla u), \quad \forall a,
$$

which together with (3.5) and (3.7) yields

$$
u_{a|a} = -\frac{\lambda_1 u}{N}, \quad \forall a.
$$

Let $f(x) = \|\nabla u\|^2 + \frac{\lambda_1}{N} u^2$. Then f is C^∞ on the open set M_u and C^0 on $M \backslash M_u$. Its derivative in the direction $e_c, \forall c$ on M_u is

$$
df(e_c) = dg_{\nabla u}(\nabla u, \nabla u)(e_c) + \frac{2\lambda_1}{N} u u_c = 2g_{\nabla u}(\nabla^2 u, \nabla u)(e_c) + \frac{2\lambda_1}{N} u u_c
$$

$$= 2g_{\nabla u}\Big(\sum u_{a|b}e_a\omega^b, \sum u_d e_d\Big)(e_c) + \frac{2\lambda_1}{N}uu_c = 2u_c u_{c|c} + \frac{2\lambda_1}{N}uu_c = 0,$$

which means that f is constant on M_u. On the other hand, we claim that f is also constant on $M\backslash M_u$. In fact, if $M\backslash M_u \ni x$ is an inner point, then $f = \frac{\lambda_1}{N}u^2$ holds on a neighborhood U of x so that $df = 0$ or f is constant on U. If $M\backslash M_u \ni x$ is a boundary point, we choose a sequence $\{x_k\} \subset M_u$ such that $x_k \longrightarrow x, (k \longrightarrow \infty)$. Then $f(x) = f|_{M_u}$ according to the continuity of f. Finally, using the continuity of f again and connectivity of M we obtain that the function $f(x)$ is constant on M.

Suppose that u attains its maximum $u_{\max}$ and minimum $u_{\min}$ at $p \in M$ and $q \in M$ respectively. Since $\|\nabla u\|^2 = 0$ at both p and q, we see that $f(p) = \frac{\lambda_1}{N}(u_{\max})^2 = f(q) = \frac{\lambda_1}{N}(u_{\min})^2$, which implies that $|u_{\max}| = |u_{\min}|$. This also means that all maximum (or minimum) of u are equal. Without loss of generality, we can assume that $u_{\max} = 1$ and $u_{\min} = -1$. Let $\gamma(s)$ be the minimal regular geodesic of (M, F) from p to q with the tangent vector $\dot\gamma(s)$. We can suppose that along $\gamma(s)$ there is not any other extreme point. Otherwise, Since u is continuous, p must not be the cluster point of minimal extreme points of u. Hence we may assume $q' \in \gamma(s)$ is the first minimal extreme point of u from p. Next set off from q' to p along $\overleftarrow{\gamma}(s)$, by the same way we get the maximum extreme point $p' \in \gamma(s)$. Then $\gamma(s)|_{\widehat{p'q'}}$ is the minimal regular geodesic without other extreme point of u. So we might as well assume that $\gamma(s)$ has this property which means $\|\nabla u\|(x) > 0, \forall x \in \gamma(s)\backslash(p,q)$. Consequently $\gamma(s)\backslash\{p,q\} \subset M_u$. Since $\lambda_1 = \frac{n-1}{N-1}Nk$, we have $\frac{\|\nabla u\|}{\sqrt{1-u^2}} = \sqrt{\frac{n-1}{N-1}k}$ along $\gamma(s)$.

Let d_M denote the diameter of (M, F). We then have

$$\sqrt{\frac{n-1}{N-1}k}d_M \geq \sqrt{\frac{n-1}{N-1}k}\int_\gamma F(\dot\gamma)ds = \int_\gamma F(\dot\gamma)\frac{\|\nabla u\|}{\sqrt{1-u^2}}ds. \tag{3.8}$$

From $|\frac{du}{ds}| = |g_{\nabla u}(\nabla u, \dot\gamma)| \leq F(\dot\gamma)\|\nabla u\|$ one gets

$$\int_\gamma F(\dot\gamma)\frac{\|\nabla u\|}{\sqrt{1-u^2}}ds \geq \int_{-1}^{1}\frac{du}{\sqrt{1-u^2}} = \pi. \tag{3.9}$$

It follows from (3.8) and (3.9) that $d_M \geq \sqrt{\frac{N-1}{n-1}}\frac{\pi}{\sqrt{k}}$.

On the other hand, from (3.1) and Lemma 2.5 we can obtain $d_M \leq \sqrt{\frac{N-1}{n-1}}\frac{\pi}{\sqrt{k}}$. So $d_M = \sqrt{\frac{N-1}{n-1}}\frac{\pi}{\sqrt{k}}$. This finishes the proof. $\square$

From the proof of Theorem 3.1, it is not difficult to obtain

Proposition 3.2. *Let (M, F) be an n-dimensional compact connected Finsler manifold without boundary. If the weighted Ricci curvature satisfies* $\mathrm{Ric}_N \geq (n-1)k$ *for some uniform positive constant k and $N \in (n, \infty)$, then*

$$\lambda_1 \geq \frac{n-1}{N-1} Nk.$$

Moreover, the diameter of M is at least $\sqrt{\frac{N-1}{n-1}} \frac{\pi}{\sqrt{k}}$ if the equality holds.

Theorem 3.3. *Let (M, F) be an n-dimensional forward geodesically complete connected Finsler manifold without boundary. If $S = 0$ and Ricci curvature* $\mathrm{Ric} \geq (n-1)k$ *for some uniform positive constant k, then*

$$\lambda_1 \geq nk.$$

Moreover, if the equality holds, then the diameter of M is $\frac{\pi}{\sqrt{k}}$, and M is homeomorphic to S^n. In particular, if F is reversible and M has Busemann–Hausdorff volume form, then (M, F) is isometric to $S^n\big(\frac{1}{\sqrt{k}}\big)$.

Proof. If $S = 0$, then $\mathrm{Ric}_N = \mathrm{Ric}$ from the Definition 2.1. Therefore, by Theorem 3.1 we can easily get the first part of Theorem 3.3. Next we only prove the last part when the equality holds. Under the condition of Theorem 3.3, $f(x) = \|\nabla_u\|^2 + \frac{\lambda_1}{n} u^2$ is constant on M by the proof of Theorem 3.1. Put

$$M^+ = \{x \in M \mid u(x) > 0\},\ M^0 = \{x \in M \mid u(x) = 0\},\ M^- = \{x \in M \mid u(x) < 0\}.$$

Then M^+, M^- are open sets on M, and M^0 is a close set with zero measure. Let p and q are the maximal point and minimal point of u respectively with $u(p) = 1$, $u(q) = -1$. So, if $\lambda_1 = nk$, then $\frac{\|\nabla u\|}{\sqrt{1-u^2}} = \sqrt{k}$. Suppose that γ is the minimal geodesic of (M, F) from p to q with the tangent vector $\dot\gamma(s)$. Denote by $L(\gamma)$ the length of γ. Then

$$\sqrt{k} L(\gamma) = \int_\gamma F(\dot\gamma) \frac{\|\nabla u\|}{\sqrt{1-u^2}} ds \geq \int_{-1}^1 \frac{du}{\sqrt{1-u^2}} = \pi \tag{3.10}$$

which means that $L(\gamma) = d(p, q) = d$. Similarly, we also get $d(q, p) = d$. Furthermore, we claim $B(p, \frac{d}{2}) \subset M^+$. In fact, if there exists a point $x_0 \in M^- \cup M^0$ such that $x_0 \in B(p, \frac{d}{2})$, then we suppose that η is the minimal geodesic of (M, F) from p to x_0 with the tangent vector $\dot\eta(s)$. Thus

$$\sqrt{k} L(\eta) = \int_\eta F(\dot\eta) \frac{\|\nabla u\|}{\sqrt{1-u^2}} ds \geq \int_0^1 \frac{du}{\sqrt{1-u^2}} = \frac{\pi}{2} \tag{3.11}$$

which shows that $L(\eta) = d(p, x_0) \geq \frac{d}{2}$. This contradict the assumption. Similarly, $B(q, \frac{d}{2}) \subset M^-$. So we get

$$B\left(p, \frac{d}{2}\right) \cap B\left(q, \frac{d}{2}\right) = \emptyset. \tag{3.12}$$

Note that if $S = 0$, $k > 0$, then $V_{k,\Lambda,n}(r) = \mathrm{vol}(S^n(k; r))$. Hence from Lemma 2.8 we get

$$\frac{\mathrm{vol}_F^{d\mu}\left(B\left(p, \frac{\pi}{2\sqrt{k}}\right)\right)}{\mathrm{vol}\left(S^n\left(k; \frac{\pi}{2\sqrt{k}}\right)\right)} \geq \frac{\mathrm{vol}_F^{d\mu}\left(B\left(p, \frac{\pi}{\sqrt{k}}\right)\right)}{\mathrm{vol}\left(S^n\left(k; \frac{\pi}{\sqrt{k}}\right)\right)} = \frac{\mathrm{vol}_F^{d\mu} M}{\mathrm{vol}\, S^n\left(\frac{1}{\sqrt{k}}\right)},$$

which implies that

$$\mathrm{vol}_F^{d\mu}\left(B\left(p, \frac{d}{2}\right)\right) \geq \frac{1}{2} \mathrm{vol}_F^{d\mu} M. \tag{3.13}$$

A similar argument yields

$$\mathrm{vol}_F^{d\mu}\left(B\left(q, \frac{d}{2}\right)\right) \geq \frac{1}{2} \mathrm{vol}_F^{d\mu} M. \tag{3.14}$$

From (3.12), (3.13) and (3.14), we have

$$B\left(p, \frac{d}{2}\right) = M^+, \quad B\left(q, \frac{d}{2}\right) = M^-, \tag{3.15}$$

M^0 is the boundary of both $B\left(p, \frac{d}{2}\right)$ and $B\left(q, \frac{d}{2}\right)$. In addition, we can prove that for any point $x \in M^0$, $d(p, x) = \frac{d}{2}$. On one hand, from (3.11), $d(p, x) \geq \frac{d}{2}$. On the other hand, if $d(p, x) > \frac{d}{2}$, then there exists a neighborhood U of x such that $d(p, y) > \frac{d}{2}$ for any $y \in U$. This contradict (3.15). Similarly, for any point $x \in M^0$, $d(q, x) = \frac{d}{2}$.

In the following, we illustrate that u has only one maximal point on M. If not, we assume p_1, p_2 are the two maximal points of u. Let σ_1 be the minimal regular geodesic from p_1 to q. Set $x_1 = \sigma_1 \cap M^0$, then $L(\sigma_1) = d(p_1, q) = d$, $d(p_1, x_1) = L(\sigma_1|_{\widehat{p_1 x_1}}) = d(x_1, q) = L(\sigma_1|_{\widehat{x_1 q}}) = \frac{d}{2}$. Draw a minimal regular geodesic η from p_2 to x_1. Then $d(p_2, x_1) = L(\eta) = \frac{d}{2}$. From (3.10) we have

$$d(p_2, x_1) + d(x_1, q) = d(p_1, q).$$

Let $\sigma_2 := \eta \cup \sigma_1|_{\widehat{x_1 q}}$, then σ_2 is a minimal regular geodesic from p_2 to q with $L(\sigma_2) = d$. Note that the equality in (3.10) holds if and only if $\dot{\gamma}$ is parallel to ∇u and u is monotone decreasing along γ. Hence at x_1, we have $\dot{\sigma}_1(x_1) = \dot{\sigma}_2(x_1) =$

$-\frac{\nabla u}{\|\nabla u\|}(x_1)$. According to the uniqueness of geodesic we have $\sigma_1 = \sigma_2$ so that $p_1 = p_2$. Similarly, u has only one minimal point q on M.

Since $\|\nabla u\|^2 + ku^2 = k$, then we have

$$\begin{aligned}
D^{\nabla u}_{\nabla u}\left(\frac{\nabla u}{\|\nabla u\|}\right) &= D^{\nabla u}_{\nabla u}\left(\frac{\nabla u}{\sqrt{k}\sqrt{1-u^2}}\right)\\
&= \frac{1}{\sqrt{k}\sqrt{1-u^2}} D^{\nabla u}_{\nabla u}\nabla u + D^{\nabla u}_{\nabla u}\left(\frac{1}{\sqrt{1-u^2}}\right)\frac{\nabla u}{\sqrt{k}}\\
&= \frac{1}{\sqrt{k}\sqrt{1-u^2}}\nabla^2 u(\nabla u) + g_{\nabla u}\left(\nabla u, \nabla^{\nabla u}\left(\frac{1}{\sqrt{1-u^2}}\right)\right)\frac{\nabla u}{\sqrt{k}}\\
&= \frac{1}{\sqrt{k}\sqrt{1-u^2}}(\nabla^2 u(\nabla u) + uk\nabla u) = 0,
\end{aligned}$$

which means that $\frac{\nabla u}{\|\nabla u\|}$ is geodesic field. For any $x_0 \in M$, Draw a minimal geodesic γ from q to x_0, then

$$\sqrt{k}L(\gamma) = \int_\gamma F(\dot{\gamma})\frac{\|\nabla u\|}{\sqrt{1-u^2}}ds \geq \int_{-1}^{u(x_0)} \frac{du}{\sqrt{1-u^2}}.$$

Since γ is minimal geodesic, $\dot{\gamma} = \frac{\nabla u}{\|\nabla u\|}$. Further, we have on γ

$$|u'|^2 + ku^2 = k, \quad u(0) = -1, \quad u'(0) = 0,$$

which shows that $u = -\cos\sqrt{k}t, t \in \left[0, \frac{\pi - \arccos u(x_0)}{\sqrt{k}}\right]$. As a geodesic on M, γ is defined in $[0, \infty]$, so we have $u = -\cos\sqrt{k}t, t \in \left[0, \frac{\pi}{\sqrt{k}}\right]$. Particularly, $u\left(\gamma\left(\frac{\pi}{\sqrt{k}}\right)\right) = 1$ which means $p \in \gamma$. Clearly, the point p is the cut locus of q. Thus we conclude that $\exp_q : T_qM \supset B_q\left(\frac{\pi}{\sqrt{k}}\right) \longrightarrow M^n \backslash \{p\}$ is diffeomorphism. On the other hand, $\exp_{\tilde{q}} : T_{\tilde{q}}S^n \supset B_{\tilde{q}}(\pi) \longrightarrow S^n \backslash \{\tilde{p}\}$ is also diffeomorphism where S^n is n-sphere, $\tilde{q}, \tilde{p}$ are the south pole and north pole respectively. Let $(\tilde{r}, \tilde{\theta}^\alpha)$ be the polar coordinate system of $T_{\tilde{q}}S^n$ and (r, θ^α) be the polar coordinate system of T_qM^n. Define $h : T_{\tilde{q}}S^n \longrightarrow T_qM$ by $r = \frac{\tilde{r}}{\sqrt{k}}, \theta^\alpha = \tilde{\theta}^\alpha$, then h is diffeomorphism. Now we define $\psi : M^n \longrightarrow S^n$ by

$$\psi(x) = \begin{cases} \exp_{\tilde{q}} \circ h^{-1} \circ \exp_q^{-1}(x) & x \neq p \\ \tilde{p} & x = p \end{cases}$$

It is not hard to see ψ is homeomorphic. i.e. M is homeomorphic to S^n. At last, if F is reversible, $S_{BH} = 0$ and the diameter of M is $\frac{\pi}{\sqrt{k}}$, then according to the Corollary 1 in [8], (M, F) is isometric to $S^n\left(\frac{1}{\sqrt{k}}\right)$. The theorem has been proved. □

Theorem 3.4. *Let (M, F) be an n-dimensional compact Finsler manifold without boundary. If the weighted Ricci curvature* $\mathrm{Ric}_\infty \geq 0$, *then*

$$\lambda_1 \geq \frac{\pi^2}{d^2},$$

where d denotes the diameter of (M, F).

Proof. Let u be the first eigenfunction on (M, F) corresponding to the first eigenvalue λ_1. Since $\int_M u d\mu = -\frac{1}{\lambda_1}\int_M \Delta u d\mu = 0$ and noting that $-u$ is not necessarily the first eigenfunction on (M, F), we assume that

$$1 = \sup u > \inf u = -k \geq -1 \ (1 \geq k = \sup u > \inf u = -1 \text{ resp.}), \quad 0 < k \leq 1.$$

For small $\varepsilon > 0$, let

$$v = \frac{u - \frac{1}{2}(1-k)}{\frac{1}{2}(1+k)(1+\varepsilon)} \ \left(\text{resp. } v = \frac{u + \frac{1}{2}(1-k)}{\frac{1}{2}(1+k)(1+\varepsilon)}\right).$$

Clearly, $dv = \frac{2}{(1+k)(1+\varepsilon)}du$. Since Legendre transform $L^* : T^*M \longrightarrow TM$ is dimorphic and satisfies $L^*(a\zeta) = aL^*(\zeta), a \in R^+, \zeta \in T^*M$, we have

$$\nabla v = \nabla^{\nabla u} v = \frac{2}{(1+k)(1+\varepsilon)}\nabla u$$

under which

$$\begin{cases} \Delta v = -\lambda_1(v \pm a_\varepsilon), & a_\varepsilon = \dfrac{1-k}{(1+k)(1+\varepsilon)}, \\ \sup v = \frac{1}{1+\varepsilon}, & \inf v = -\frac{1}{1+\varepsilon}. \end{cases}$$

Let $v = \sin\theta$, then

$$-\frac{1}{1+\varepsilon} \leq \sin\theta \leq \frac{1}{1+\varepsilon}, \ \frac{\|\nabla v\|^2}{1-v^2} = \|\nabla^{\nabla u}\theta\|^2.$$

Consider the function

$$f(x) = \frac{\|\nabla v\|^2}{1-v^2}.$$

Since M is compact, we can apply the maximal principle to $f(x)$ on the weighted Riemannian manifold $(M, g_{\nabla u})$. Suppose that $f(x)$ attains its maximum at $x_0 \in M$, then $\nabla^{\nabla u} f(x_0) = 0$, $\Delta^{\nabla u} f(x_0) \leq 0$ and $x_0 \in M_u$.

Let $\{e_a\}_{a=1}^n$ be a local orthonormal basis with respect to $g_{\nabla u}$ on M_u. Write $\nabla v = \sum_a v_a e_a$. Then by simple computations on $\nabla^{\nabla u} f(x_0) = 0$ we have

$$\sum_b v_b v_{b|a} = \frac{\|\nabla v\|^2(-v)v_a}{1-v^2}, \quad \forall a. \tag{3.16}$$

Furthermore, a straightforward calculation yields

$$\begin{aligned}\Delta^{\nabla u} f(x_0) &= \Delta^{\nabla u}\left(\frac{\|\nabla v\|^2}{1-v^2}\right) = \frac{\Delta^{\nabla u}\left(\|\nabla v\|^2\right)}{1-v^2} + \|\nabla v\|^2 \Delta^{\nabla u}\left(\frac{1}{1-v^2}\right)\\ &\quad + 2g_{\nabla u}\left(\nabla^{\nabla u}\left(\|\nabla v\|^2\right), \nabla^{\nabla u}\left(\frac{1}{1-v^2}\right)\right) := A + B + C, \end{aligned}\tag{3.17}$$

where

$$\begin{aligned} A &= \frac{\Delta^{\nabla u}\left(\|\nabla v\|^2\right)}{1-v^2},\\ B &= \|\nabla v\|^2 \Delta^{\nabla u}\left(\frac{1}{1-v^2}\right) = \|\nabla v\|^2 \operatorname{div}\left(\nabla^{\nabla u}\left(\frac{1}{1-v^2}\right)\right)\\ &= \|\nabla v\|^2\left\{\frac{2v}{(1-v^2)^2}\operatorname{div}\left(\nabla^{\nabla u} v\right) + 2g_{\nabla u}\left(\nabla^{\nabla u} v, \nabla^{\nabla u}\left(\frac{v}{(1-v^2)^2}\right)\right)\right\}\\ &= \|\nabla v\|^2\left\{\frac{2v}{(1-v^2)^2}\Delta v + \frac{2\|\nabla v\|^2}{(1-v^2)^2} + \frac{8v^2\|\nabla v\|^2}{(1-v^2)^3}\right\},\\ C &= 2g_{\nabla u}\left(\nabla^{\nabla u}\left(\|\nabla v\|^2\right), \nabla^{\nabla u}\left(\frac{1}{1-v^2}\right)\right) = \frac{8v v_a v_b v_{a|b}}{(1-v^2)^2}. \end{aligned}$$

So, from the formulas above, we can rewrite (3.17) as follows

$$\begin{aligned} 0 \ge \Delta^{\nabla u} f(x_0) &= \frac{\Delta^{\nabla u}(\|\nabla v\|^2)}{1-v^2} + \frac{8v\sum v_a v_b v_{a|b}}{(1-v^2)^2}\\ &\quad - \frac{2\|\nabla v\|^4 + 2v\|\nabla v\|^2\Delta v}{(1-v^2)^2} + \frac{8v^2\|\nabla v\|^4}{(1-v^2)^3}. \end{aligned}$$

Substituting (3.16) into it, one has

$$0 \ge \Delta^{\nabla u}(\|\nabla v\|^2) + \frac{2\|\nabla v\|^4 + 2v\|\nabla v\|^2\Delta v}{1-v^2}. \tag{3.18}$$

From (2.11) and the conditions of Theorem 3.4, we get

$$\begin{aligned}\Delta^{\nabla u}(\|\nabla v\|^2) &= 2\|\nabla v\|^2 \operatorname{Ric}_\infty(\nabla v) + 2D(\Delta v)(\nabla v) + 2\|\nabla^2 v\|^2_{HS(\nabla v)}\\ &\ge 2D(-\lambda_1(v \pm a_\varepsilon))(\nabla v) + 2\sum_{ab} v_{a|b}^2 = -2\lambda_1\|\nabla v\|^2 + 2\sum_{ab} v_{a|b}^2. \end{aligned}\tag{3.19}$$

By the Schwartz inequality and (3.16), we have

$$\sum_{ab} v_{a|b}^2 \sum_b v_b^2 \geq \sum_a \left(\sum_b v_b v_{b|a} \right)^2 = \sum_a \frac{\|\nabla v\|^4 v^2 v_a^2}{(1-v^2)^2},$$

which means that

$$\sum_{ab} v_{a|b}^2 \geq \frac{\|\nabla v\|^4 v^2}{(1-v^2)^2}. \tag{3.20}$$

Utilizing (3.18)-(3.20) above, we obtain at the point x_0 that

$$f(x_0) = \frac{\|\nabla v\|^2}{1-v^2}(x_0) \leq \lambda_1(1+a_\varepsilon).$$

So for any $x \in M$, we have

$$\sqrt{f(x)} = \|\nabla^{\nabla u}\theta\| \leq \sqrt{\lambda_1(1+a_\varepsilon)}. \tag{3.21}$$

Set

$$G(\theta) = \max_{\substack{x\in M\\ \theta(x)=\theta}} \|\nabla^{\nabla u}\theta\|^2 = \max_{\substack{x\in M\\ \theta(x)=\theta}} \frac{\|\nabla v\|^2}{1-v^2}.$$

Clearly, $G(\theta) \in C^0\big(\big[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\big]\big)$, where δ is specified by

$$\sin\left(\frac{\pi}{2}-\delta\right) = \frac{1}{1+\varepsilon}, \quad G\left(-\frac{\pi}{2}+\delta\right) = G\left(\frac{\pi}{2}-\delta\right) = 0.$$

From (3.21) we can write

$$G(\theta) \leq \lambda_1(1+a_\varepsilon),$$

under which we let

$$G(\theta) = \lambda_1(1+a_\varepsilon\varphi(\theta)), \quad \varphi(\theta) \in C^0\left(\left[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right]\right).$$

Since $G(\theta)$ vanishes at the end points of the interval $[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta]$,

$$\varphi\left(\frac{\pi}{2}-\delta\right) = \varphi\left(-\frac{\pi}{2}+\delta\right) < -1.$$

By (3.21) we see that $\varphi(\theta) \leq 1$.

In the following, by the same way in [22], we can get

$$\varphi(\theta) \leq \psi(\theta), \tag{3.22}$$

where $\psi(\theta)$ is defined by

$$\psi(\theta)=\begin{cases}\dfrac{\frac{4}{\pi}(\theta+\cos\theta\sin\theta)-2\sin\theta}{\cos^2\theta}, & \theta\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)\\ \psi\left(\dfrac{\pi}{2}\right)=1, & \psi\left(-\dfrac{\pi}{2}\right)=-1.\end{cases} \tag{3.23}$$

Now we continue to prove Theorem 3.4. From (3.22), we have

$$\|\nabla^{\nabla u}\theta\|\leq\sqrt{\lambda_1}\sqrt{1+a_\varepsilon\psi(\theta)}. \tag{3.24}$$

Let $p,q\in M$ be such points that $\theta(p)=-\frac{\pi}{2}+\delta$, $\theta(q)=\frac{\pi}{2}-\delta$. Let γ be a shortest geodesic joining p and q. Denote by T the tangent vector of γ. Then

$$\begin{aligned}\|\nabla^{\nabla u}\theta\|&=\frac{\|\nabla v\|}{\cos\theta}=\frac{F(\nabla v)}{\cos\theta}\geq\frac{\left|g_{\nabla u}\left(\nabla v,\frac{T}{F(T)}\right)\right|}{\cos\theta}\\&=\frac{|Tv|}{F(T)\cos\theta}=\frac{|\frac{dv}{ds}|}{F(T)\cos\theta}=\frac{|\frac{d\theta}{ds}|}{F(T)}.\end{aligned} \tag{3.25}$$

Therefore from (3.24) and (3.25) one gets

$$\sqrt{\lambda_1}d\geq\int_\gamma\sqrt{\lambda_1}F(T)ds\geq\int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}-\delta}\frac{d\theta}{\sqrt{1+a_\varepsilon\psi(\theta)}}. \tag{3.26}$$

It is easy to see from (3.23) that $\psi(0)=0,\psi(-\theta)=-\psi(\theta),|a_\varepsilon\psi(\theta)|<1$. Hence, we have

$$\begin{aligned}\int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}-\delta}\frac{d\theta}{\sqrt{1+a_\varepsilon\psi(\theta)}}&=\int_0^{\frac{\pi}{2}-\delta}\left(\frac{1}{\sqrt{1+a_\varepsilon\psi(\theta)}}+\frac{1}{\sqrt{1-a_\varepsilon\psi(\theta)}}\right)d\theta\\&=2\int_0^{\frac{\pi}{2}-\delta}\left(1+\sum_{i=1}^{\infty}\frac{1\cdot3\cdots(4i-1)}{2\cdot4\cdots4i}a_\varepsilon^{2i}\psi^{2i}\right)d\theta\\&\geq2\left(\frac{\pi}{2}-\delta\right)=\pi-2\delta.\end{aligned} \tag{3.27}$$

Thus

$$\sqrt{\lambda_1}d\geq\pi-2\delta.$$

Letting $\varepsilon\to0$, so that $\delta\to0$ too, we then obtain

$$\lambda_1\geq\frac{\pi^2}{d^2}. \qquad \square$$

Remark 3.5. The estimate in Theorem 3.4 has been pointed out in [21], where a sharp lower bound for first Neumann eignenvalue of Finsler–Laplacian was given. The conclusion of Theorem 3.4 is not sharp for $n \geq 2$.

If $S = (n+1)cF$ for some constant c, then $\dot{S} = 0$ so that $\mathrm{Ric}_\infty = \mathrm{Ric}$. So we can easily get the following

Corollary 3.6. *Let (M, F) be an n-dimensional compact Finsler manifold without boundary. If M has constant S-curvature and* $\mathrm{Ric} \geq 0$, *then*

$$\lambda_1 \geq \frac{\pi^2}{d^2},$$

where d denotes the diameter of (M, F).

References

[1] P. Antonelli and B. Lackey, The Theory of Finslerian Laplacian and Application, Math. Appl., **459**, *Kluwer Acad. Publ., Dordrecht*, 1998.

[2] D. W. Bao, S. S. Chern and Z. M. Shen, An Introduction to Riemann–Finsler Geometry, Vol. 200, Graduate Texts in Mathematics, *Springer-Verlag, Berlin – Heidelberg – New York*, 2000.

[3] D.W. Bao and B. Lackey, A Hodge decomposition theorem for Finsler spaces, *C. R. Acad. Sci. Paris Sér. I Math.* **323** (1996), 51–56.

[4] P. Centore, Finsler Laplacians and minimal-energy map, *Internat. J. Math.* **11** (2000), 1–13.

[5] I. Chavel, Eigenvalues in Riemannian Geometry, Pure and Applied Mathematics, **115**, *Academic Press, Inc., Orlando, FL*, 1984.

[6] Y. Ge and Z. M. Shen, Eigenvalues and eigenfunctions of metric measure manifolds, *Proc. London Math. Soc.* **82** (2001), 725–746.

[7] F. B. Hang and X. D. Wang, A remark on Zhong-Yang's eigenvalue estimate, *Int. Math. Res. Not.* **18** (2007), Art. ID rnm064, 9pp.

[8] C.-W. Kim and J.-W. Yim, Finsler manifolds with positive constant flag curvature, *Geom. Dedicata* **98** (2003), 47–56.

[9] P. Li and S-Y. Yau, Estimates of eigenvalues of a compact Riemannian manifold, Geometry of the Laplace Operator, (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, *Amer. Math. Soc., Providence, R.I.*, 1980, 205–239.

[10] A. Lichnerowicz, Géometrie des Groupes de Transforamtions, Travaux et Recherches Mathémtiques, III., *Dunod, Paris*, 1958 (in *French*).

[11] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, *Ann. of Math.* **169** (2009), 903–991.

[12] M. Obata, Certain conditions for a Riemannian manifold tobe isometric with a sphere, *J. Math. Soc. Japan.* **14** (1962), 333–340.

[13] S. Ohta and K.-T. Sturm, Bochner–Weitzenboeck formula and Li–Yau estimates on Finsler manifolds, arXiv: 1105.0983[math.AP].

[14] S. Ohta, Finsler interpolation inequalities, *Calc. Var. Partial Differential Equations* **36** (2009), 211–249.

[15] Z. M. Shen, Lectures on Finsler Geometry, *World Scientific Publishing Co., Singapore*, 2001.

[16] Z. M. Shen, The non-linear Laplacian for Finsler manifolds, The Theory of Finslerian Laplacians and Applications, Math. Appl., 459, *Kluwer Acad. Publ., Dordrecht*, 1998.

[17] Z. M. Shen, Volume comparison and its applications in Riemann–Finsler geometry, *Adv. Math.* **128** (1997), 306–328.

[18] K.-T. Sturm, On the geometry of metric measure spaces, I, *Acta Math.* **196** (2006), 65–131.

[19] B. Thomas, A natural Finsler–Laplace operator, arXiv:1104.4326[math.DG].

[20] B. Y. Wu and Y. L. Xin, Comparison theorems in Finsler geometry and their applications, *Math. Ann.* **337** (2007), 177–196.

[21] G. F. Wang and C. Xia, A sharp lower bound for the first eigenvalue on Finsler manifolds, arXiv: 1112.4401 [math.AP].

[22] J. Q. Zhong and H. C. Yang, On the estimates of the first eigenvalue of a compact Riemannian manifold, *Sci. Sinica Ser. A* **27** (1984), 1265–1273.

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