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On a theorem of Erdős and Sárközy

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Abstract. Let $A = \{a_1, a_2, \dots\}(a_1 \leqslant a_2 \leqslant \cdots)$ be an infinite sequence of nonnegative integers, $k \geq 2$ be a fixed integer and denote by $R_k(n)$ the number of solutions of $a_{i_1} + a_{i_2} + \cdots + a_{i_k} = n$. In this paper, we prove that if $g(n)$ is a monotonically increasing arithmetic function with $g(n) \to +\infty$ and $g(n) = o\big(n(\log n)^{-2}\big)$, then for any $0 < \varepsilon < 1$, $|R_k(n) - g(n)| > ([k/2]! - \varepsilon)\sqrt{g(n)}$ holds for infinitely many positive integers n. We also prove that for a positive integer d, if $R_k(n) \geq d$ for all sufficiently large integers n, then $R_k(n) \geq d + 2[k/2]!\sqrt{d} + ([k/2]!)^2$ for infinitely many positive integers n.

1. Introduction

Let $A = \{a_1, a_2, \dots\}(0 \leqslant a_1 \leqslant a_2 \leqslant \cdots)$ be an infinite sequence of nonnegative integers, $k \geq 2$ be a fixed integer and denote by $R_k(n)$ the number of solutions of

$$a_{i_1} + a_{i_2} + \cdots + a_{i_k} = n.$$

In 1985, P. Erdős and A. Sárközy [2] proved that if $g(n)$ is an arithmetic function satisfying $g(n) \to +\infty$, $g(n+1) \geq g(n)$ for $n \geq n_0$ and $g(n) = o\big(n(\log n)^{-2}\big)$, then $R_2(n) - g(n) = o((g(n))^{1/2})$ cannot hold. In 2002, G. Horváth [3] extend this result to all $k > 2$. In 2007, G. Horváth [4] improved the above result, he proved that if $0 < \varepsilon < 1$ and $g(n)$ is a real arithmetic function such that

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$g(n) \to +\infty$, $g(n+1) \ge g(n)$ for $n \ge n_0$ and $g(n) = o\big(n(\log n)^{-2}\big)$, then there does not exist n_1 such that $|R_k(n) - g(n)| \le (1-\varepsilon)\sqrt{g(n)}$ for all $n \ge n_1$.

In this paper, we obtain a stronger version of the above results:

Theorem 1. *If $g(n)$ is an arithmetic function such that*

$$g(n) \to +\infty, \qquad g(n+1) \ge g(n) \text{ for } n \ge n_0 \tag{1}$$

and

$$g(n) = o\big(n(\log n)^{-2}\big), \tag{2}$$

then for any $0 < \varepsilon < 1$,

$$|R_k(n) - g(n)| > ([k/2]! - \varepsilon)\sqrt{g(n)}$$

holds for infinitely many positive integers n.

Remark 1. In [1], the authors proved a similar result for $g(n) = cn$, where c is a positive constant.

In [5], the authors proved that for a positive integer d, if $R_k(n) \ge d$ for all sufficiently large integers n, then $R_k(n) \ge d + 2\sqrt{d} + 1$ for infinitely many positive integers n. It happens that the method in this paper can be used to improve this result. The other results in [5] can be improved similarly.

Theorem 2. *Let d be a positive integer. If $R_k(n) \ge d$ for all sufficiently large integers n, then $R_k(n) \ge d + 2[k/2]!\sqrt{d} + ([k/2]!)^2$ for infinitely many positive integers n.*

2. Preliminary lemmas

Let $z = re(\alpha)$, where $r = 1 - 1/N$, N is a large integer and $e(\alpha) = e^{2\pi i\alpha}$, α is a real variable. We write

$$F(z) = \sum_{a\in A} z^a.$$

Suppose that $1 \le R_k(n) \le n$ for all sufficiently large integers n. Then, by

$$F^k(r^2) = \sum_{n=0}^{\infty} R_k(n) r^{2n},$$

the infinite series $F(z) = \sum_{a\in A} z^a$ is absolutely convergent for $|z| < 1$ and $F(r^2) \to +\infty$ as $N \to +\infty$.

Lemma 1. *Let ℓ be a positive integer. Then*

$$\int_0^1 \left|F^{2\ell}(z)\right| d\alpha \ge (\ell! + o(1)) F^{\ell}(r^2).$$

Proof. We have

$$\begin{aligned}
\int_0^1 \left|F^{2\ell}(z)\right| d\alpha &= \int_0^1 F^{\ell}(z)\overline{F^{\ell}(z)} d\alpha \\
&= \int_0^1 \sum_{a_{i_1},\dots,a_{i_{2\ell}} \in A} r^{a_{i_1}+\dots+a_{i_{2\ell}}} e((a_{i_1}+\dots+a_{i_\ell}-a_{i_{\ell+1}}-\dots-a_{i_{2\ell}})\alpha) d\alpha \\
&= \sum_{a_{i_1}+\dots+a_{i_\ell}-a_{i_{\ell+1}}-\dots-a_{i_{2\ell}}=0} r^{a_{i_1}+\dots+a_{i_{2\ell}}} \ge \ell! \sum_{i_1,\dots,i_\ell \text{ pairwise distinct}} r^{2a_{i_1}+\dots+2a_{i_\ell}} \\
&\ge \ell! \cdot \Big(\sum_{i_1,\dots,i_\ell} r^{2a_{i_1}+\dots+2a_{i_\ell}} - \sum_{1\le u<v\le \ell} \sum_{\substack{i_1,\dots,i_\ell \\ i_u=i_v}} r^{2a_{i_1}+\dots+2a_{i_\ell}} \Big) \\
&= \ell! \cdot \Big(F^{\ell}(r^2) - \frac{1}{2}\ell(\ell-1)F(r^4)F^{\ell-2}(r^2) \Big) \\
&\ge \ell! \cdot \Big(F^{\ell}(r^2) - \frac{1}{2}\ell(\ell-1)F^{\ell-1}(r^2) \Big) = (\ell! + o(1))F^{\ell}(r^2).
\end{aligned}$$

This completes the proof of Lemma 1. □

Lemma 2. *Let k be an integer with $k \ge 2$. Then*

$$\int_0^1 \left|F^{k}(z)\right| d\alpha \ge ([k/2]! + o(1)) F^{k/2}(r^2).$$

Proof. If k is even, then Lemma 2 follows from Lemma 1. If $k = 2\ell + 1$ is odd, then, by Hölder inequality and Lemma 1, we have

$$\begin{aligned}
\left(\int_0^1 \left|F^{k}(z)\right| d\alpha \right)^{\frac{k-1}{k}} &= \left(\int_0^1 |F^{k}(z)| d\alpha \right)^{\frac{k-1}{k}} \cdot \left(\int_0^1 1 d\alpha \right)^{1/k} \\
&\ge \int_0^1 |F(z)|^{k-1} d\alpha = \int_0^1 |F(z)|^{2\ell} d\alpha \ge (\ell! + o(1))F^{\ell}(r^2).
\end{aligned}$$

Thus

$$\int_0^1 \left|F^{k}(z)\right| d\alpha \ge (\ell! + o(1))^{\frac{k}{k-1}} F^{k/2}(r^2) \ge ([k/2]! + o(1)) F^{k/2}(r^2).$$

This completes the proof of Lemma 2. □

3. Proof of Theorem 1

Suppose that there exists an infinite sequence $A = \{a_1 \le a_2 \le \dots\}$ of nonnegative integers, $\varepsilon_0 > 0$ and $n_1 \in \mathbb{N}$ such that

$$|R_k(n) - g(n)| \le ([k/2]! - \varepsilon_0)\sqrt{g(n)} \quad \text{for all} \quad n \ge n_1. \tag{3}$$

By (1), (2) and (3) we have $1 \le R_k(n) \le n$ for all sufficiently large integers n. For $|z| < 1$, we have

$$F^k(z) = \sum_{n=0}^{\infty} R_k(n) z^n = \sum_{n=0}^{\infty} g(n) z^n + \sum_{n=0}^{\infty} (R_k(n) - g(n)) z^n.$$

Let

$$J = \int_0^1 \left|F^k(z)\right| d\alpha, \qquad J_1 = \int_0^1 \left|\sum_{n=0}^{\infty} g(n) z^n\right| d\alpha,$$

$$J_2 = \int_0^1 \left|\sum_{n=0}^{\infty} (R_k(n) - g(n)) z^n\right| d\alpha.$$

Then

$$J \le J_1 + J_2. \tag{4}$$

By Lemma 2 we have

$$J \ge ([k/2]! + o(1)) F^{k/2}(r^2). \tag{5}$$

Similar to the proof of Horváth [4], we have

$$J_1 = o\left(\left(\sum_{n=0}^{\infty} R_k(n) r^{2n}\right)^{1/2}\right) = o(F^{k/2}(r^2)). \tag{6}$$

By (3), for all $n \ge n_2 (\ge n_1)$, we have

$$R_k(n) \ge g(n) - ([k/2]! - \varepsilon_0)\sqrt{g(n)} \ge g(n)(1 + \varepsilon_0/(2[k/2]!))^{-2}. \tag{7}$$

By Cauchy's inequality, Parseval's formula, the assumption and (7) we have

$$\begin{aligned} J_2 &\le \left(\int_0^1 \left|\sum_{n=0}^{\infty} (R_k(n) - g(n)) z^n\right|^2 d\alpha\right)^{\frac{1}{2}} = \left(\sum_{n=0}^{\infty} (R_k(n) - g(n))^2 r^{2n}\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=0}^{n_2 - 1} (R_k(n) - g(n))^2 r^{2n} + \sum_{n=n_2}^{\infty} (R_k(n) - g(n))^2 r^{2n}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n=0}^{n_2 - 1} (R_k(n) - g(n))^2 + ([k/2]! - \varepsilon_0)^2 \sum_{n=n_2}^{\infty} g(n) r^{2n}\right)^{\frac{1}{2}} \end{aligned}$$

$$\leq \left(\sum_{n=0}^{n_2-1} (R_k(n) - g(n))^2 + ([k/2]! - \varepsilon_0)^2 (1 + \varepsilon_0/(2[k/2]!))^2 \sum_{n=n_2}^{\infty} R_k(n) r^{2n} \right)^{\frac{1}{2}}$$

$$\leq \left([k/2]! - \frac{1}{2}\varepsilon_0 \right) \left(\sum_{n=0}^{\infty} R_k(n) r^{2n} \right)^{1/2} + O(1)$$

$$= \left([k/2]! - \frac{1}{2}\varepsilon_0 \right) F^{k/2}(r^2) + O(1). \tag{8}$$

By (4), (5), (6) and (8), dividing by $F^{k/2}(r^2)$ and letting $N \to \infty$, we have $[k/2]! \leq [k/2]! - \varepsilon_0/2$, which is impossible. This completes the proof of Theorem 1.

4. Proof of Theorem 2

Suppose that $d \leq R_k(n) < d + 2[k/2]!\sqrt{d} + ([k/2]!)^2$ for all sufficiently large integers n. Then $1 \leq R_k(n) \leq n$ for all sufficiently large integers n. Let l be the largest integer with

$$l < 2[k/2]!\sqrt{d} + ([k/2]!)^2.$$

Then

$$d + l - [k/2]!\sqrt{d+l} < d + [k/2]!\sqrt{d}.$$

Let c be a real number with

$$\max\{d + l - [k/2]!\sqrt{d+l}, d - [k/2]!\sqrt{d}\} < c < d + [k/2]!\sqrt{d}.$$

Then

$$(d + j - c)^2 < ([k/2]!)^2 (d + j), \quad j = 0, 1, \ldots, l.$$

Choose a real number $0 < \delta < [k/2]!$ such that

$$(d + j - c)^2 < ([k/2]! - \delta)^2 (d + j), \quad j = 0, 1, \ldots, l.$$

Thus

$$(R_k(n) - c)^2 < ([k/2]! - \delta)^2 R_k(n) \tag{9}$$

for all integers $n \geq n_3$. For $|z| < 1$, we have

$$F^k(z) = \sum_{n=0}^{\infty} R_k(n) z^n = \sum_{n=0}^{\infty} c z^n + \sum_{n=0}^{\infty} (R_k(n) - c) z^n.$$

Let

$$J=\int_0^1 \left|F^k(z)\right|d\alpha, \qquad J_1=\int_0^1 \left|\sum_{n=0}^{\infty} cz^n\right|d\alpha,$$

$$J_2=\int_0^1 \left|\sum_{n=0}^{\infty}(R_k(n)-c)z^n\right|d\alpha.$$

By Lemma 2 we have

$$J \geq ([k/2]!+o(1))F^{k/2}(r^2). \tag{10}$$

By [4] (see also [5]) we have

$$J_1=|c|\int_0^1 \frac{1}{|1-z|}d\alpha \ll \log N. \tag{11}$$

By Cauchy's inequality, Parseval's formula, the assumption and (9) we have

$$\begin{aligned} J_2 &\leq \left(\int_0^1 \left|\sum_{n=0}^{\infty}(R_k(n)-c)z^n\right|^2 d\alpha\right)^{\frac{1}{2}} = \left(\sum_{n=0}^{\infty}(R_k(n)-c)^2 r^{2n}\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=0}^{n_3-1}(R_k(n)-c)^2 r^{2n} + \sum_{n=n_3}^{\infty}(R_k(n)-c)^2 r^{2n}\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=0}^{n_3-1}(R_k(n)-c)^2 + ([k/2]!-\delta)^2 \sum_{n=n_3}^{\infty} R_k(n) r^{2n}\right)^{\frac{1}{2}} \\ &\leq ([k/2]!-\delta)F^{k/2}(r^2)+O(1). \end{aligned} \tag{12}$$

By $J \leq J_1 + J_2$, (10), (11) and (12), dividing by $F^{k/2}(r^2)$ and letting $N \to \infty$, we have $[k/2]! \leq [k/2]! - \delta/2$, which is impossible. This completes the proof of Theorem 2.

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