Publ. Math. Debrecen
83/3 (2013), 415–433
DOI: 10.5486/PMD.2013.5541

Diophantine equations involving normalized binomial mid-coefficients

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Abstract. For a positive integer n, let μ_n be the normalized binomial mid-coefficients. We discuss the following Diophantine equation involving power means of n variables μ_i,

$$M_k(\mu_{a_1},\dots,\mu_{a_n}) = M_l(\mu_{b_1},\dots,\mu_{b_n}), \quad k,l\in\mathbb{Z}.$$

For $n=2,3$ and other general cases, we get some results on this equation. Moreover, for $k=l=0$ and for every $n\geq 3$, we obtain infinitely many solutions of equation $\mu_{a_1}\mu_{a_2}\cdots\mu_{a_n} = \mu_{b_1}\mu_{b_2}\cdots\mu_{b_n}$.

1. Introduction

Let $\mathbb{Z},\mathbb{N},\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. For any nonnegative integer n, the normalized binomial mid-coefficients is defined by

$$\mu_n = 2^{-2n}\binom{2n}{n} = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}.$$

This coefficient μ_n is closely connected to the Euler's gamma function $\Gamma(x)$, Gauss's hypergeometric function, etc. For more details, see [2], [7], [12]. There are many results for the lower and upper bounds of the estimates of μ_n. The proofs and other inequalities for μ_n can be found in [13], [14].

Mathematics Subject Classification: Primary: 11D09; Secondary: 26D07.
Key words and phrases: normalized binomial mid-coefficients, Diophantine equations, power means, p-adic valuation.

Let t be a real number. The power mean of order t of the positive real numbers $x_1, \dots, x_n$ is defined by

$$M_t(x_1, \dots, x_n) = \left(\frac{1}{n}\sum_{j=1}^{n} x_j^t\right)^{\frac{1}{t}}, \quad \text{if } t \neq 0,$$

and

$$M_0(x_1, \dots, x_n) = \lim_{t\to 0} M_t(x_1, \dots, x_n) = \left(\prod_{j=1}^{n} x_j\right)^{\frac{1}{n}}.$$

The most interesting properties of power means are collected in the monograph [6].

It is very interesting to study the Diophantine equation involving power means of n variables μ_i,

$$M_k(\mu_{a_1}, \dots, \mu_{a_n}) = M_l(\mu_{b_1}, \dots, \mu_{b_n}), \quad k, l \in \mathbb{Z}. \tag{1}$$

In fact, the normalized binomial mid-coefficient has been the subject of an intensive research in number theory and potential theory. See for examples [4] and [11].

In 1990, Bang and Fuglede [3] first studied the Diophantine equation

$$M_0(\mu_p, \mu_q) = M_0(\mu_r, \mu_s). \tag{2}$$

They proved that equation (2) only has the trivial solutions $(p, q) = (r, s), (s, r)$. In 2005, Alzer and Fuglede [1] studied equation (1), for $n = 2$ and $n = 3$. They solved this equation for integers $k, l \neq 0$ when $n = 2$ and for integers $k = l \neq 0, -1$ when $n = 3$. In [1], Alzer and Fuglede set the following problems:

Open problems: (i) Determine all solutions of equation (1) in case $n = 3$ with $k = l = 0$ and $k = l = -1$.

(ii) Study Diophantine equation (1). In particular, determine all solutions of the arithmetic mean-geometric mean equation (1) for $k = 1, l = 0$, i.e.

$$\frac{1}{n}(\mu_{a_1} + \dots + \mu_{a_n}) = (\mu_{b_1} \cdots \mu_{b_n})^{\frac{1}{n}}. \tag{3}$$

In this paper, we discuss Diophantine equation (1) for $n = 2, 3$ and other general cases. First, we solve equation (3) for $n = 2$, and give all solutions of (1), for $n = 2$ and $k = 0$, $l \neq \pm 2$. In Section 3, we study equation (1) for $n = 3$, $k = l = -1$ and $k \neq l$, $k, l \neq 0, \pm 1$. Therefore, we solve a part of the problem (i). We also study equation (1) in the case $k = l \neq 0$, $n \geq 4$. In this case, we give the characteristic of nontrivial solutions of equation (1) or some methods for solving equation (1). See Section 4. In Section 5, for $k = l = 0$, $n \geq 3$, we give an infinite number of solutions of equation (1). In the last section, we use the results obtained to set some conjectures related to equation (1).

2. The equation $M_0(\mu_{a_1}, \mu_{a_2}) = M_k(\mu_{b_1}, \mu_{b_2})$

Let p be a prime and v_p the standard p-adic valuation normalized defined by $v_p(0) = +\infty$ and

$$v_p\Big(\frac{a}{b}\Big) = \begin{cases} r, & \text{if } p^r \mid\mid a, \\ -s, & \text{if } p^s \mid\mid b, \end{cases}$$

where $a, b, s, t \in \mathbb{Z}$, $s, t \geq 0$, $ab \neq 0$ and $\gcd(a, b) = 1$. Let $q, q_1, q_2 \in \mathbb{Q}$. The following properties on $v_2(q)$ are well-known:

- $v_2(-q) = v_2(q)$, $v_2(q_1 q_1) = v_2(q_1) + v_2(q_2)$;
- $v_2(q_1 + q_2) \geq \min\{v_2(q_1), v_2(q_2)\}$;
- if $v_2(q_1) < v_2(q_2)$, then $v_2(q_1 + q_2) = v_2(q_1)$;
- if $n_1 < n_2$, then $v_2(\mu_{n_1}) > v_2(\mu_{n_2})$.

Now, we recall the following result due to Erdős and Selfridge [9] on the product of consecutive integers.

Lemma 2.1. *The equation*

$$n(n+1)\dots(n+k-1) = y^l, \tag{4}$$

in positive integers $n, y, k, l \geq 2$ has no solution.

We prove the following lemma.

Lemma 2.2. *If $x \neq y$, then the equation*

$$\mu_x \mu_y = q^m, \ m \geq 2, \quad m \in \mathbb{N}, \ q \in \mathbb{Q} \tag{5}$$

has no solution.

Proof. As $x \neq y$, without loss of generality, we assume that $x < y$. If $m = 2$, then equation (5) becomes $\frac{\mu_x}{\mu_y} = \big(\frac{q}{\mu_y}\big)^2$. So

$$(2x+1)(2x+2)\cdots(2y-1)(2y) = \left(\frac{(2x+1)\cdots(2y-1)q}{\mu_y}\right)^2. \tag{6}$$

Hence $\frac{(2x+1)\cdots(2y-1)q}{\mu_y} \in \mathbb{N}$ and from Lemma 2.1, equation (6) has no solution.

If $m \geq 3$, from (5) we get

$$\frac{(x+1)(x+2)\cdots(2x)}{1 \cdot 2 \cdot \dots \cdot x} \cdot \frac{(y+1)(y+2)\cdots(2y)}{1 \cdot 2 \cdot \dots \cdot y} = \big(2^{\frac{2x+2y}{m}} q\big)^m. \tag{7}$$

Notice that $\frac{(x+1)(x+2)\cdots(2x)}{1\cdot 2\cdot\dots\cdot x}$ and $\frac{(y+1)(y+2)\cdots(2y)}{1\cdot 2\cdot\dots\cdot y} \in \mathbb{N}$. So $2^{\frac{2x+2y}{m}} q \in \mathbb{N}$. By Erdős's proof of Bertrand's Postulate [8], there exists a prime p such that $\lfloor \frac{n}{2} \rfloor < p < 2\lfloor \frac{n}{2} \rfloor$. Therefore, if $y > 6$, there exists a prime p such that $y + 1 \leq p \leq 2y$.

Thus $p \parallel \frac{(y+1)(y+2)\cdots(2y)}{1\cdot 2\cdot\cdots\cdot y}$ and $v_p\big(\frac{(x+1)(x+2)\cdots(2x)}{1\cdot 2\cdots x}\big) \leq 1$. This is a contradiction to the fact that $m \geq 3$. If $x, y \leq 6$, we directly verify that equation (5) has no solution. $\square$

Using a similar method to that in the proof of Lemma 2.2, it is easy to obtain the following corollary.

Corollary 2.3. *If $n, m \in \mathbb{N}$ with $n < m$, then the equation*

$$\mu_{a_1} \cdots \mu_{a_n} = q^m, \quad q \in \mathbb{Q} \tag{8}$$

has no solution.

We recall here a result obtained by Alzer and Fuglede [1].

Lemma 2.4. *Let $k, l \neq 0$ and $a_1, a_2, b_1, b_2 \geq 0$ be integers. If $k = l$, then the equation*

$$M_k(\mu_{a_1}, \mu_{a_2}) = M_l(\mu_{b_1}, \mu_{b_2}) \tag{9}$$

only has the trivial solutions $(a_1, a_2) = (b_1, b_2), (b_2, b_1)$. And if $k \neq l$, then equation (9) holds if and only if $a_1 = a_2 = b_1 = b_2$.

Now we are ready to prove our main result of this section.

Theorem 2.5. *If $k \in \mathbb{Z}$ with $k \neq 0, \pm 2$, then the equation*

$$\sqrt{\mu_{a_1}\mu_{a_2}} = \Big(\frac{\mu_{b_1}^k + \mu_{b_2}^k}{2}\Big)^{\frac{1}{k}} \tag{10}$$

only has the trivial solutions $a_1 = a_2 = b_1 = b_2$.

Proof. If $a_1 = a_2$ or $b_1 = b_2$, equation (10) becomes $M_k(\mu_{a_1}, \mu_{a_1}) = M_k(\mu_{b_1}, \mu_{b_2})$ or $M_0(\mu_{a_1}, \mu_{a_2}) = M_0(\mu_{b_1}, \mu_{b_2})$. From Lemma 2.4 and the result of Bang and Fuglede [3], equation (10) only has the trivial solutions. Hence, without loss of generality, we assume that $a_1 < a_2$ and $b_1 < b_2$.

If k is odd, we use (10) to deduce the following equation

$$(\mu_{a_1}\mu_{a_2})^{\frac{k-1}{2}}\sqrt{\mu_{a_1}\mu_{a_2}} = \frac{\mu_{b_1}^k + \mu_{b_2}^k}{2}. \tag{11}$$

Then $\sqrt{\mu_{a_1}\mu_{a_2}} = \frac{\mu_{b_1}^k + \mu_{b_2}^k}{2(\mu_{a_1}\mu_{a_2})^{\frac{k-1}{2}}}$, i.e. $\mu_{a_1}\mu_{a_2} = q_1^2$, where $q_1 \in \mathbb{Q}$. From Lemma 2.2, the latter equation has no solution.

If k is even, from (10) we have

$$(\mu_{a_1}\mu_{a_2})^{\frac{k}{2}} = \frac{\mu_{b_1}^k + \mu_{b_2}^k}{2}. \tag{12}$$

Then taking the 2-adic valuation of equation (12), we get

$$\frac{k}{2}v_2(\mu_{a_1}\mu_{a_2}) = v_2\Big(\frac{\mu_{b_1}^k + \mu_{b_2}^k}{2}\Big) = v_2(\mu_{b_1}^k + \mu_{b_2}^k) - 1 = kv_2(\mu_{b_j}) - 1, \tag{13}$$

where $b_j = b_1$ when $k > 0$ and $b_j = b_2$ when $k < 0$. Then from (13), we have $\frac{k}{2} \mid 1$, since $k \neq \pm 2$. This is impossible. Therefore, the proof of Theorem 2.5 is complete. □

Remark 2.6. In Theorem 2.5, we solve equation (3), for $n = 2$. However, we didn't solve equation (10) when $k = \pm 2$. So we set the following problem: find all solutions of the equations

$$2\mu_{a_1}\mu_{a_2} = \mu_{b_1}^2 + \mu_{b_2}^2 \tag{14}$$

and

$$\frac{2}{\mu_{a_1}\mu_{a_2}} = \frac{1}{\mu_{b_1}^2} + \frac{1}{\mu_{b_2}^2}. \tag{15}$$

3. The equation $M_k(\mu_{a_1}, \mu_{a_2}, \mu_{a_3}) = M_l(\mu_{b_1}, \mu_{b_2}, \mu_{b_3})$

In this section, we first consider equation (1), with $n = 3$ and $k = l = -1$. For the proof of the following theorem, we will use ideas of Alzer and Fuglede [1].

Theorem 3.1. *If* $0 \le a_1 \le a_2 \le a_3$, $0 \le b_1 \le b_2 \le b_3$, *then the equation*

$$\frac{1}{\mu_{a_1}} + \frac{1}{\mu_{a_2}} + \frac{1}{\mu_{a_3}} = \frac{1}{\mu_{b_1}} + \frac{1}{\mu_{b_2}} + \frac{1}{\mu_{b_3}} \tag{16}$$

only has the trivial solutions $a_1 = b_1, a_2 = b_2, a_3 = b_3$.

Proof. Without loss of generality, we assume that $a_1 \le b_1$. From (16), we just need to consider two cases: $a_2 \ge b_1$ or $a_2 \le b_1$.

Case 1: $a_2 \ge b_1$. If $a_1 < b_1$, then

$$\begin{aligned} v_2\Big(\frac{1}{\mu_{b_1}}\Big) &= \min\Big\{v_2\Big(\frac{1}{\mu_{a_2}}\Big), v_2\Big(\frac{1}{\mu_{a_3}}\Big), v_2\Big(\frac{1}{\mu_{b_1}}\Big), v_2\Big(\frac{1}{\mu_{b_2}}\Big), v_2\Big(\frac{1}{\mu_{b_3}}\Big)\Big\} \\ &\le v_2\Big(-\frac{1}{\mu_{a_2}} - \frac{1}{\mu_{a_3}} + \frac{1}{\mu_{b_1}} + \frac{1}{\mu_{b_2}} + \frac{1}{\mu_{b_3}}\Big) = v_2\Big(\frac{1}{\mu_{a_1}}\Big) < v_2\Big(\frac{1}{\mu_{b_1}}\Big). \end{aligned} \tag{17}$$

This is a contradiction. Then $a_1 = b_1$. Thus equation (16) becomes $\frac{1}{\mu_{a_2}} + \frac{1}{\mu_{a_3}} = \frac{1}{\mu_{b_2}} + \frac{1}{\mu_{b_3}}$. By Lemma 2.4, equation (16) only has the trivial solutions.

Case 2: $a_2 \le b_1$. From the monotony of μ_x, $a_2 \le b_1$ implies that $a_1 \le a_2 \le b_1 \le b_2 \le b_3 \le a_3$. If $a_1 < a_2$, using the method in Case 1, we get also a contradiction. Therefore, $a_1 = a_2$. Moreover, if $a_2 = b_1$, then equation (16) only has the trivial solutions. So we assume that $a_2 < b_1$. Thus

$$\begin{aligned} v_2\Big(\frac{1}{\mu_{a_2}}\Big) < v_2\Big(\frac{1}{\mu_{b_1}}\Big) &\le v_2\Big(\frac{1}{\mu_{b_1}}+\frac{1}{\mu_{b_2}}+\frac{1}{\mu_{b_3}}-\frac{1}{\mu_{a_3}}\Big) \\ &= v_2\Big(\frac{1}{\mu_{a_1}}+\frac{1}{\mu_{a_2}}\Big) = v_2\Big(\frac{2}{\mu_{a_1}}\Big) = 1 + v_2\Big(\frac{1}{\mu_{a_1}}\Big). \end{aligned} \tag{18}$$

Hence $v_2(\frac{1}{\mu_{b_1}}) = 1 + v_2(\frac{1}{\mu_{a_1}})$. So $b_1 = a_1+1$ and b_1 is odd. In fact, if $b_1 = a_1$, then this is impossible. So $b_1 \ge a_1+1$. If $b_1 \ge a_1+2$, then $v_2(\frac{1}{u_{b_1}}) \ge v_2(\frac{1}{u_{a_1}})+2$, which is also impossible. Moreover, if b_1 is even, then $v_2(\frac{1}{u_{b_1}}) \ge v_2(\frac{1}{u_{a_1}}) + 2$. Therefore, b_1 is odd. Put $b_1 = r$. So we write $a_1 = a_2 = r-1$.

If $b_2 \ge r+1$, then from equation (16) we have

$$\begin{aligned} \frac{1}{\mu_{b_3}} - \frac{1}{\mu_{a_3}} &= \frac{2}{\mu_{r-1}} - \frac{1}{\mu_r} - \frac{1}{\mu_{b_2}} = \frac{2r-1}{r}\cdot\frac{1}{\mu_r} - \frac{1}{\mu_{b_2}} \\ = \Big(\frac{2r-1}{r} - \frac{(2r+2)\cdots(2b_2)}{(2r+1)\cdots(2b_2-1)}\Big)\frac{1}{\mu_r} &= \frac{A}{(2r+1)\cdots(2b_2-1)\,r}\cdot\frac{1}{\mu_r}, \end{aligned} \tag{19}$$

where $2 \nmid A$. Then

$$v_2\Big(\frac{1}{\mu_{b_3}}-\frac{1}{\mu_{a_3}}\Big) = v_2\Big(\frac{1}{\mu_r}\Big) < v_2\Big(\frac{1}{\mu_{b_2}}\Big) \le v_2\Big(\frac{1}{\mu_{b_3}}\Big) \le v_2\Big(\frac{1}{\mu_{b_3}}-\frac{1}{\mu_{a_3}}\Big). \tag{20}$$

This leads to a contradiction. Therefore, $b_2 = r \ge 1$ and

$$\frac{1}{\mu_{b_3}} - \frac{1}{\mu_{a_3}} = \frac{2}{\mu_{r-1}} - \frac{2}{\mu_r} = 2\Big(\frac{2r-1}{2r} - 1\Big)\frac{1}{\mu_r} = -\frac{1}{r}\cdot\frac{1}{\mu_r}. \tag{21}$$

Thus, if $b_3 \ge r+1$, from (21) and as r is odd, we have

$$v_2\Big(\frac{1}{\mu_{b_3}}-\frac{1}{\mu_{a_3}}\Big) = v_2\Big(\frac{1}{r\,\mu_r}\Big) = v_2\Big(\frac{1}{\mu_r}\Big) < v_2\Big(\frac{1}{\mu_{b_3}}\Big) \le v_2\Big(\frac{1}{\mu_{b_3}}-\frac{1}{\mu_{a_3}}\Big). \tag{22}$$

This is also a contradiction. Hence, $b_3 = r$ and

$$\frac{1}{\mu_{a_3}} = \frac{3}{\mu_r} - \frac{2}{\mu_{r-1}} = \Big(3 - \frac{2r-1}{r}\Big)\frac{1}{\mu_r} = \frac{r+1}{r}\cdot\frac{1}{\mu_r}. \tag{23}$$

If $a_3 \ge r+3$, then $\mu_{a_3} \le \mu_{r+3} = \frac{(2r+1)(2r+3)(2r+5)}{(2r+2)(2r+4)(2r+6)}\mu_r$. From (23) we get

$$\frac{r}{r+1} \le \frac{(2r+1)(2r+3)(2r+5)}{(2r+2)(2r+4)(2r+6)}. \tag{24}$$

Therefore, we obtain $r = 1$. Using again equation (23), we get $\mu_{a_3} = \frac{1}{4}$, which is impossible. So $a_3 \le r+2$, i.e. $a_3 = r, r+1, r+2$. We use each of these values of a_3 to verify that equation (23) has no solutions. Therefore, equation (16) only has the trivial solutions. $\square$

Now, we will study the equation

$$M_k(\mu_{a_1},\mu_{a_2},\mu_{a_3}) = M_l(\mu_{b_1},\mu_{b_2},\mu_{b_3}), \quad k\neq l,\ k\,l\neq 0,\ k,l\in\mathbb{Z} \tag{25}$$

and prove the following theorem.

Theorem 3.2. *Let $k,l\in\mathbb{Z}$, and $k\neq\pm1$, $l\neq\pm1$. Assume that $\gcd(k,l)=d$, $k=k_1d$, $l=l_1d$, and*

$$\begin{cases} a_1\le a_2\le a_3, & \text{if } k>1,\\ a_1\ge a_2\ge a_3, & \text{if } k<-1,\end{cases} \quad \text{and} \quad \begin{cases} b_1\le b_2\le b_3, & \text{if } l>1,\\ b_1\ge b_2\ge b_3, & \text{if } l<-1.\end{cases}$$

Then

(1) *When $2\nmid k_1-l_1$, equation (25) only has the trivial solutions $a_1=a_2=a_3=b_1=b_2=b_3$.*

(2) *When $2\mid k_1-l_1$, equation (25) has the trivial solutions and $M_2(\mu_0,\mu_1,\mu_1)=M_{-2}(\mu_1,\mu_0,\mu_0)$. Moreover, if equation (25) has other solutions, these solutions satisfy $a_3=b_3$ and*

$$\min\left\{kv_2\Big(\frac{\mu_{a_2}}{\mu_{a_3}}\Big),\ lv_2\Big(\frac{\mu_{b_2}}{\mu_{b_3}}\Big)\right\} = v_2(|k_1-l_1|)+2 \quad \text{or} \quad v_2(|k_1-l_1|)+1. \tag{26}$$

Proof. From equation (25), we have

$$\big(\mu_{a_1}^k+\mu_{a_2}^k+\mu_{a_3}^k\big)^{l_1}\cdot 3^{k_1-l_1} = \big(\mu_{b_1}^l+\mu_{b_2}^l+\mu_{b_3}^l\big)^{k_1}. \tag{27}$$

If $a_1=a_2=a_3$, then equation (25) becomes

$$M_k(\mu_{a_1},\mu_{a_2},\mu_{a_3}) = M_l(\mu_{a_1},\mu_{a_1},\mu_{a_1}) = M_l(\mu_{b_1},\mu_{b_2},\mu_{b_3}),$$

which is incompatible with the condition $k\neq l$. If $b_1=b_2=b_3$, the same conclusion can be made. Therefore, from the monotony of μ_x, we just need to discuss three cases:

- $a_2\neq a_3$ and $b_2\neq b_3$,
- $a_1\neq a_2=a_3$ and $b_2\neq b_3$,
- $a_1\neq a_2=a_3$ and $b_1\neq b_2=b_3$.

Put $a_3=m$.

Case 3.1: $a_2\neq a_3$ and $b_2\neq b_3$. Without loss of generality, we suppose $k>l$. Since $v_2(\mu_{a_1}^k+\mu_{a_2}^k)>v_2(\mu_{a_3}^k)$, $v_2(\mu_{b_1}^l+\mu_{b_2}^l)>v_2(\mu_{b_3}^l)$, then using equation (27), we get

$$kl_1v_2(\mu_{a_3}) = k_1lv_2(\mu_{b_3}). \tag{28}$$

So $a_3 = b_3 = m$. Thus equation (27) implies

$$\left(\frac{\mu_{a_1}^k}{\mu_m^k} + \frac{\mu_{a_2}^k}{\mu_m^k} + 1\right)^{l_1} 3^{k_1-l_1} = \left(\frac{\mu_{b_1}^l}{\mu_m^l} + \frac{\mu_{b_2}^l}{\mu_m^l} + 1\right)^{k_1}. \tag{29}$$

Write

$$\frac{\mu_{a_1}^k}{\mu_m^k} + \frac{\mu_{a_2}^k}{\mu_m^k} = C, \quad \frac{\mu_{b_1}^l}{\mu_m^l} + \frac{\mu_{b_2}^l}{\mu_m^l} = D.$$

The first, we suppose that $k, l > 1$, then

$$C\left(\sum_{j=0}^{l_1-1}\binom{l_1}{j}C^{l_1-j-1}\right)3^{k_1-l_1} + (3^{k_1-l_1}-1) = D\left(\sum_{j=0}^{k_1-1}\binom{k_1}{j}D^{k_1-j-1}\right). \tag{30}$$

Now we calculate the value of $v_2(3^q-1)$, where $q \in \mathbb{N}$. If $2 \nmid q$, then $3^q - 1 \equiv 2 \pmod 8$. If $q = 2^r q_1$, where $r \in \mathbb{N}$ and $2 \nmid q_1$, since

$$3^q - 1 = 3^{2^r q_1} - 1 = (3^{2^{r-1}q_1} + 1)\cdots(3^{2q_1}+1)(3^{q_1}+1)(3^{q_1}-1),$$

and $3^{2q_1} + 1 \equiv 2 \pmod 8$, $3^{q_1} + 1 \equiv 4 \pmod 8$, $3^{q_1} - 1 \equiv 2 \pmod 8$, then $v_2(3^q - 1) = r + 2$. Therefore, we have

$$v_2(3^q-1) = \begin{cases} 1, & \text{if } 2 \nmid q, \\ v_2(q) + 2, & \text{if } 2 \mid q. \end{cases} \tag{31}$$

If $2 \mid k_1 l_1$ and as $\gcd(k_1, l_1) = 1$, then $k_1 - l_1$ is odd. So $v_2(3^{k_1-l_1} - 1) = 1$. As $a_2 < m$, $b_2 < m$, one can see that $v_2(C) \geq k$ and $v_2(D) \geq l$. Thus, if k_1 is odd and l_1 is even, from equation (30) we have

$$v_2\left(C\left(\sum_{j=0}^{l_1-1}\binom{l_1}{j}C^{l_1-j-1}\right)3^{k_1-l_1}\right) \geq k + v_2\left(\sum_{j=0}^{l_1-1}\binom{l_1}{j}C^{l_1-j-1}\right) \geq k+1,$$

$$v_2\left(D\left(\sum_{j=0}^{k_1-1}\binom{k_1}{j}D^{k_1-j-1}\right)\right) = v_2(D) + v_2\left(\left(\sum_{j=0}^{k_1-1}\binom{k_1}{j}D^{k_1-j-1}\right)\right) \geq 2.$$

This is impossible. Similarly, if k_1 is even and l_1 is odd, we obtain the same contradiction.

If $2 \nmid k_1 l_1$, then $v_2(3^{k_1-l_1} - 1) = v_2(k_1 - l_1) + 2$. Notice that $v_2(C) \geq k \geq 1$, $v_2(D) \geq l \geq 1$, and $2 \nmid k_1$, $2 \nmid l_1$. Hence we get

$$v_2\left(C\left(\sum_{j=0}^{l_1-1}\binom{l_1}{j}C^{l_1-j-1}\right)3^{k_1-l_1}\right) = v_2(C) \geq k,$$

and

$$v_2\left(D\left(\sum_{j=0}^{k_1-1}\binom{k_1}{j}B^{k_1-j-1}\right)\right)=v_2(D).$$

So if $k>l>1$, $k>v_2(k_1-l_1)+2$, then from equation (30) we obtain

$$v_2(D)=v_2(k_1-l_1)+2. \tag{32}$$

Since $v_2(D)=v_2\big(\frac{\mu_{b_1}^l}{\mu_m^l}+\frac{\mu_{b_2}^l}{\mu_m^l}\big)=lv_2\big(\frac{\mu_{b_2}}{\mu_{b_3}}\big)$ or $lv_2\big(\frac{\mu_{b_2}}{\mu_{b_3}}\big)+1$, thus condition (26) holds.

Now, if $k,l<-1$, then from (28) we have

$$C\left(\sum_{j=0}^{-l_1-1}\binom{-l_1}{j}C^{-l_1-j-1}\right)3^{k_1-l_1}+(3^{k_1-l_1}-1)$$
$$=D\left(\sum_{j=0}^{-k_1-1}\binom{-k_1}{j}D^{-k_1-j-1}\right). \tag{33}$$

If $k>1,l<-1$, equation (28) implies $(C+1)^{-l_1}(D+1)^{k_1}=3^{k_1-l_1}$. So

$$\big((C+1)^{-l_1}-1\big)\big((D+1)^{k_1}-1\big)+\big((C+1)^{-l_1}-1\big)+\big((D+1)^{k_1}-1\big)=3^{k_1-l_1}-1.$$

Thus, we obtain

$$\left(\sum_{j=0}^{-l_1-1}\binom{-l_1}{j}C^{-l_1-j-1}\right)D\left(\sum_{j=0}^{-k_1-1}\binom{-k_1}{j}D^{-k_1-j-1}\right)$$
$$=C\left(\sum_{j=0}^{-l_1-1}\binom{-l_1}{j}C^{-l_1-j-1}\right)+D\left(\sum_{j=0}^{-k_1-1}\binom{-k_1}{j}D^{-k_1-j-1}\right)+(3^{-k_1+l_1}-1). \tag{34}$$

Using an approach similar that of (30), one draws the same conclusions, i.e. when $2\mid k_1-l_1$, equation (25) has no other solutions; when $2\nmid k_1-l_1$, the other solutions of equation (25) satisfy $a_3=b_3,\min\{v_2(C),v_2(D)\}=v_2(k_1-l_1)+2$ or $v_2(k_1-l_1)+1$. Therefore, again condition (26) holds.

Case 3.2: $a_2=a_3$ and $b_2\neq b_3$. If $v_2(\mu_{a_1}^k)=v_2(\mu_{a_2}^k+\mu_{a_3}^k)=v_2(2\mu_m^k)$, then $kv_2(\mu_{a_1})=1+kv_2(\mu_m)$. We deduce that $k=\pm1$. This is impossible.

If $v_2(\mu_{a_1}^k)\neq v_2(2\mu_{a_3}^k)$, then $v_2(\mu_{a_1}^k)>v_2(2\mu_{a_3}^k)$. From the monotony of $v_2(\mu_x)$, equation (27) implies

$$l_1\big(kv_2(\mu_{a_3})+1\big)=k_1lv_2(\mu_{b_3}). \tag{35}$$

This yields $k_1d=k=\pm1$, which leads also to a contradiction.

Case 3.3: $a_2 = a_3$ and $b_2 = b_3$. If $v_2(\mu_{a_1}^k) = v_2(2\mu_{a_3}^k)$, then $kv_2(\mu_{a_1}) = kv_2(\mu_{a_3}) + 1$. So we have $k = \pm 1$. If $v_2(\mu_{b_1}^l) = v_2(2\mu_{b_3}^l)$, we get a similar conclusion. Therefore, we suppose that $v_2(\mu_{a_1}^k) \neq v_2(2\mu_{a_3}^k)$ and $v_2(\mu_{b_1}^l) \neq v_2(2\mu_{b_3}^l)$.

As $v_2(\mu_{a_1}^k) \neq v_2(2\mu_{a_3}^k)$ and $v_2(\mu_{b_1}^l) \neq v_2(2\mu_{b_3}^l)$, we deduce that $v_2(\mu_{a_1}^k) > v_2(2\mu_{a_3}^k)$ and $v_2(\mu_{b_1}^l) > v_2(2\mu_{b_3}^l)$. Thus from equation (27), we get

$$l_1\big(kv_2(\mu_{a_3}) + 1\big) = k_1\big(lv_2(\mu_{b_3}) + 1\big). \tag{36}$$

Therefore, $k_1 \mid l_1$, and $l_1 \mid k_1$, i.e. $k = \pm l$. Since $k \neq l$, $\gcd(k_1, l_1) = 1$, then $k = -l$, and from equation (36) we get $k(v_2(\mu_{a_3}) - v_2(\mu_{b_3})) = -2$. As $k \neq \pm 1$, we have $k = \pm 2$ and $v_2(\mu_{a_3}) - v_2(\mu_{b_3}) = \pm 1$.

If $k = 2$ and $v_2(\mu_{a_3}) - v_2(\mu_{b_3}) = -1$, then $l = -2$, and $b_3 = a_3 - 1 = m - 1$, where m is odd. Thus equation (25) becomes

$$(\mu_{a_1}^2 + 2\mu_m^2)\Big(\frac{1}{\mu_{b_1}^2} + \frac{2}{\mu_{m-1}^2}\Big) = 9. \tag{37}$$

Note that $a_1 \leq m - 1$ and $b_1 \geq m$. When $b_1 \geq m + 1$, we have

$$\begin{aligned}(\mu_{a_1}^2 + 2\mu_m^2)\Big(\frac{1}{\mu_{b_1}^2} + \frac{2}{\mu_{m-1}^2}\Big) &\geq (\mu_{m-1}^2 + 2\mu_m^2)\Big(\frac{1}{\mu_{m+1}^2} + \frac{2}{\mu_{m-1}^2}\Big)\\ &= \frac{288m^6 - 80m^4 + 32m^3 + 12m^2 - 8m + 2}{32m^6 - 16m^4 + 2m^2} > 9.\end{aligned}$$

This contradicts (37). Therefore, $b_1 = m$. If $a_1 \leq m - 2$, similar calculations imply

$$(\mu_{a_1}^2 + 2\mu_m^2)\left(\frac{1}{\mu_{b_1}^2} + \frac{2}{\mu_{m-1}^2}\right) \geq (\mu_{m-2}^2 + 2\mu_m^2)\left(\frac{1}{\mu_m^2} + \frac{2}{\mu_{m-1}^2}\right) > 9.$$

This means that $a_1 = m - 1$. From (37), we get

$$(\mu_{m-1}^2 + 2\mu_m^2)\left(\frac{1}{\mu_m^2} + \frac{2}{\mu_{m-1}^2}\right) = 9.$$

We deduce that $m = 1$. Thus equation (25) has the solution $k = 2, l = -2$, $(a_1, a_2, a_3, b_1, b_2, b_3) = (0, 1, 1, 1, 0, 0)$.

Similarly, if $k = -2, l = 2$, one obtains a similar conclusion, i.e. equation (25) has only the solution $(k, l, a_1, a_2, a_3, b_1, b_2, b_3) = (-2, 2, 1, 0, 0, 0, 1, 1)$. This completes the proof of Theorem 3.2. $\square$

Remark 3.3. Using Theorem 3.2, we can find the solutions of equation (25) for some fixed values of k, l. However, we cannot completely solve it. This is the case for examples when $k = 1$, $l = -1$, or $k = 3$, $l = 1$. We believe that it may be difficult to completely solve this equation.

4. The equation $M_k(\mu_{a_1}, \mu_{a_2}, \dots, \mu_{a_n}) = M_k(\mu_{b_1}, \mu_{b_2}, \dots, \mu_{b_n})$

Before proving the main theorem of this section, we give the following lemma.

Lemma 4.1. *Let $k \neq 0$, $0 \leq a_1 \leq a_2 \leq a_3$, $0 \leq b_1 \leq b_2 \leq b_3$, and $a_1 \leq b_1$. The equation*

$$M_k(\mu_{a_1}, \mu_{a_2}, \mu_{a_3}) = M_k(\mu_{b_1}, \mu_{b_2}, \mu_{b_3}), \quad k \in \mathbb{Z} \tag{38}$$

only has the trivial solutions, except for $k = 1$, in which case we have the additional solution $(a_1, a_2, a_3, b_1, b_2, b_3) = (1, 3, 3, 2, 2, 2)$.

Proof. For $k \neq 0, -1$, one can refer to Proposition 2 in [1]. If $k = -1$, from Theorem 3.1, we get the conclusion. $\square$

For $n \geq 4, k = l \neq 0$, we have the main result of this section.

Theorem 4.2. *Let $n \geq 4$, $k \neq 0$. We assume that*

$$\begin{cases} a_1 \leq a_2 \leq \cdots \leq a_n,\ b_1 \leq b_2 \leq \cdots \leq b_n, & \text{if } k \geq 1, \\ a_1 \geq a_2 \geq \cdots \geq a_n,\ b_1 \geq b_2 \geq \cdots \geq b_n, & \text{if } k \leq -1. \end{cases}$$

Then the equation

$$\mu_{a_1}^k + \mu_{a_2}^k + \cdots + \mu_{a_n}^k = \mu_{b_1}^k + \mu_{b_2}^k + \cdots + \mu_{b_n}^k, \quad k \in \mathbb{Z} \tag{39}$$

only has the trivial solutions $a_j = b_j$ $(1 \leq j \leq n)$, except for $k \mid v_2(r)$, where $2 \mid r$ and $2 \leq r \leq n$.

Proof. For the proof, we will use the mathematical induction on n. Put $a_n = m$. First, we will prove the theorem for $n = 4$.

If $a_3 \neq a_4$, and $b_3 \neq b_4$, then we get

$$v_2(\mu_{a_1}^k + \mu_{a_2}^k + \mu_{a_3}^k) > v_2(\mu_{a_4}^k) \quad \text{and} \quad v_2(\mu_{b_1}^k + \mu_{b_2}^k + \mu_{b_3}^k) > v_2(\mu_{b_4}^k).$$

From equation (39), we have $v_2(\mu_{a_4}^k) = v_2(\mu_{b_4}^k)$. So $a_4 = b_4$ and $\mu_{a_1}^k + \mu_{a_2}^k + \mu_{a_3}^k = \mu_{b_1}^k + \mu_{b_2}^k + \mu_{b_3}^k$. By Lemma 4.1, the equation only has the trivial solutions, except for $k = 1$.

If $a_3 = a_4$ and $b_3 \neq b_4$, we consider four cases.

- If $\mu_{a_1}^k = \mu_{a_2}^k = \mu_{a_3}^k = \mu_{a_4}^k$, then $v_2(\mu_{a_4}^k) + 2 = v_2(\mu_{b_4}^k)$. We deduce that $k = \pm 2$ and $b_4 = m \mp 1$ or $k = \pm 1$ and $|b_4 - m| \geq 1$. This contradicts (39).
- If $\mu_{a_1}^k > \mu_{a_2}^k = \mu_{a_3}^k = \mu_{a_4}^k$, then $v_2(3\mu_{a_4}^k) = v_2(\mu_{b_4}^k)$. So $b_4 = m$. Lemma 4.1 implies that the equation only has the trivial solutions.

• If $\mu_{a_1}^k \geq \mu_{a_2}^k > \mu_{a_3}^k = \mu_{a_4}^k$ and $v_2(\mu_{a_2}^k) = v_2(2\mu_{a_4}^k)$, then $k = \pm 1$, $a_2 = m \mp 1$. Therefore, the theorem holds.

• If $\mu_{a_1}^k \geq \mu_{a_2}^k > \mu_{a_3}^k = \mu_{a_4}^k$ and $v_2(\mu_{a_2}^k) \neq v_2(2\mu_{a_4}^k)$, then we get $v_2(\mu_{a_1}^k + \mu_{a_2}^k) > v_2(2\mu_{a_4}^k)$. Thus we have $v_2(2\mu_{a_4}^k) = v_2(\mu_{b_4}^k)$. We deduce that $k = 1$, $b_4 = m - 1$, and m is odd, or $k = -1$, $b_4 = m + 1$, and m is even. If $k = 1$, $b_4 = m - 1$, equation (39) implies

$$\mu_{a_1} + \mu_{a_2} + \frac{m-1}{m} \cdot \mu_{m-1} = \mu_{b_1} + \mu_{b_2} + \mu_{b_3}. \tag{40}$$

Notice that $b_3 \neq b_4$. So we have $b_3 \leq m - 2$. As

$$\begin{aligned} v_2(\mu_{a_2}) \geq v_2(\mu_{a_3 - 1}) = v_2(\mu_{m-2}) &= v_2\Big(\frac{2m-2}{2m-3} \cdot \mu_{m-1}\Big) \\ &= 1 + v_2(m-1) + v_2(\mu_{m-1}) = 1 + v_2\Big(\frac{m-1}{m} \cdot \mu_{m-1}\Big) \end{aligned}$$

and $v_2(\mu_{b_3}) \geq v_2(\mu_{m-2}) = 1 + v_2(\frac{m-1}{m} \cdot \mu_{m-1})$, we get a contradiction to (40). If $k = -1$, $b_4 = m + 1$, using the same method, we come to the same conclusion.

Finally, we suppose that $a_3 = a_4, b_3 = b_4$. If $v_2(\mu_{a_2}^k) \neq v_2(2\mu_{a_4}^k)$ and $v_2(\mu_{b_2}^k) \neq v_2(2\mu_{b_4}^k)$, then by (39) we get $v_2(2\mu_{a_4}^k) = v_2(2\mu_{b_4}^k)$. This implies $a_4 = b_4$. Using Lemma 4.1, one can see that the equation only has the trivial solutions. If $v_2(\mu_{a_2}^k) = v_2(2\mu_{a_4}^k)$ or $v_2(\mu_{b_2}^k) = v_2(2\mu_{b_4}^k)$, then $k = \pm 1$ and $a_2 = m \mp 1$ or $k = \pm 1$ and $b_2 = m \mp 1$. Therefore, the theorem is proved for $n = 4$.

Second, we suppose that Theorem 4.2 holds for $n - 1$ and we will prove that it also holds for n. According to the above discussion, we will also consider three cases: $a_{n-1} \neq a_n, b_{n-1} \neq b_n$ or $a_{n-1} = a_n, b_{n-1} \neq b_n$, or $a_{n-1} = a_n, b_{n-1} = b_n$.

Case 4.1: $a_{n-1} \neq a_n, b_{n-1} \neq b_n$.

Then we obtain $v_2(\mu_{a_1}^k + \cdots + \mu_{a_{n-1}}^k) > v_2(\mu_{a_n}^k)$, $v_2(\mu_{b_1}^k + \cdots + \mu_{b_{n-1}}^k) > v_2(\mu_{b_n}^k)$. From equation (39), we have $v_2(\mu_{a_n}^k) = v_2(\mu_{b_n}^k)$. This implies $a_n = b_n$. Therefore, we obtain $\mu_{a_1}^k + \cdots + \mu_{a_{n-1}}^k = \mu_{b_1}^k + \cdots + \mu_{b_{n-1}}^k$. By the induction hypothesis, the nontrivial solutions of equation (39) satisfy $k \mid v_2(r)$, where $2 \mid r$ and $2 \leq r \leq n - 1$.

Case 4.2: $a_{n-1} = a_n, b_{n-1} \neq b_n$.

Suppose that $\mu_{a_{n-s}}^k > \mu_{a_{n-s+1}}^k = \cdots = \mu_{a_{n-1}}^k = \mu_{a_n}^k$, where $s \geq 2$. If s is odd, then $v_2(s\mu_{a_n}^k) = v_2(\mu_{b_n}^k)$. So we have $a_n = b_n$.

If s is even and $v_2(\mu_{a_{n-s}}^k) \neq v_2(s\mu_{a_n}^k)$, then $v_2(\mu_{a_{n-s}}^k) = v_2(\mu_{b_n}^k)$ or $v_2(s\mu_{a_n}^k) = v_2(\mu_{b_n}^k)$. This implies that $a_{n-s} = b_n$ or $k \mid v_2(s)$, where $2 \leq s \leq n$.

If s is even and $v_2(\mu_{a_{n-s}}^k) = v_2(s\mu_{a_n}^k)$, then $kv_2(\mu_{a_{n-s}}) = v_2(s) + kv_2(\mu_{a_n})$. One obtains the same conclusion.

Case 4.3: $a_{n-1} = a_n, b_{n-1} = b_n$.

Assume that $\mu^k_{a_{n-s}} > \mu^k_{a_{n-s+1}} = \cdots = \mu^k_{a_{n-1}} = \mu^k_{a_n}$, $\mu^k_{b_{n-t}} > \mu^k_{b_{n-t+1}} = \cdots = \mu^k_{b_{n-1}} = \mu^k_{b_n}$, where $s, t \geq 2$.

If $2 \nmid st$, then $v_2(\mu^k_{a_{n-s}}) > v_2(s\mu^k_{a_n})$, $v_2(\mu^k_{b_{n-t}}) > v_2(t\mu^k_{n_n})$. So from (39), we get $v_2(s\mu^k_{a_n}) = v_2(t\mu^k_{b_n})$. Thus, $a_n = b_n$.

If $2 \mid s$, $2 \nmid t$, and $v_2(\mu^k_{a_{n-s}}) \neq v_2(s\mu^k_{a_n})$, then $v_2(\mu^k_{a_{n-s}}) = v_2(t\mu^k_{b_n}) = v_2(\mu^k_{b_n})$ or $v_2(s\mu^k_{a_n}) = v_2(t\mu^k_{b_n}) = v_2(\mu^k_{b_n})$. This yields to $a_{n-s} = b_n$ or $k \mid v_2(s)$. If $v_2(\mu^k_{a_{n-s}}) = v_2(s\mu^k_{a_n})$, we see that $k \mid v_2(s)$.

If $2 \nmid s$ and $2 \mid t$, using a similar method we get the same conclusion.

If $2 \mid s$ and $2 \mid t$, we consider two cases. If $v_2(\mu^k_{a_{n-s}}) \neq v_2(s\mu^k_{a_n})$ and $v_2(\mu^k_{b_{n-t}}) \neq v_2(t\mu^k_{b_n})$, then from (39) we get $v_2(\mu^k_{a_{n-s}}) = v_2(t\mu^k_{b_n})$ or $v_2(s\mu^k_{a_n}) = v_2(t\mu^k_{b_n})$, or $v_2(\mu^k_{b_{n-t}}) = v_2(s\mu^k_{a_n})$. Thus we conclude that $k \mid v_2(s)$ or $k \mid v_2(t)$, or $k \mid v_2(s) - v_2(t)$. It clear that $|v_2(s) - v_2(t)| < \max\{v_2(s), v_2(t)\}$. Therefore, Theorem 4.2 also holds for n and this concludes its proof. $\square$

Using the proof of Theorem 4.2, one can easily deduce the following corollary.

Corollary 4.3. *If $k \neq 0, \pm 1$, then the equation*

$$\mu^k_{a_1} + \mu^k_{a_2} + \mu^k_{a_3} + \mu^k_{a_4} = \mu^k_{b_1} + \mu^k_{b_2} + \mu^k_{b_3} + \mu^k_{b_4}, \quad k \in \mathbb{Z} \tag{41}$$

only has the trivial solutions. For $k = \pm 1$, if equation (41) *has nontrivial solutions, then these solutions satisfy $a_3 = a_4$ and $a_2 = a_4 \mp 1$ or $b_3 = b_4$ and $b_2 = b_4 \mp 1$.*

Again, from Theorem 4.2, we get the following corollary.

Corollary 4.4. *If $k \neq 0, \pm 1, \pm 2$, then the equation*

$$\mu^k_{a_1} + \mu^k_{a_2} + \mu^k_{a_3} + \mu^k_{a_4} + \mu^k_{a_5} = \mu^k_{b_1} + \mu^k_{b_2} + \mu^k_{b_3} + \mu^k_{b_4} + \mu^k_{b_5}, \quad k \in \mathbb{Z} \tag{42}$$

only has the trivial solutions. Moreover, for $k = \pm 2$, if equation (42) *has nontrivial solutions, then these solutions satisfy $a_2 = a_3 = a_4 = a_5$ and $b_5 = a_5 \mp 1$ or $b_2 = b_3 = b_4 = b_5$ and $a_5 = b_5 \mp 1$.*

Proof. By Theorem 4.2, the nontrivial solutions of equation (42) satisfy $k = \pm 1, \pm 2$. If $k = \pm 2$, from the proof of Theorem 4.2, we have $\mu^k_{a_1} > \mu^k_{a_2} = \mu^k_{a_3} = \mu^k_{a_4} = \mu^k_{a_5}$ and $\mu^k_{b_4} > \mu^k_{b_5}$ or $\mu^k_{b_2} > \mu^k_{b_3} = \mu^k_{b_4} = \mu^k_{b_5}$, or $b_1 = b_2 = b_3 = b_4 = b_5$. We write $a_5 = m$.

If $k = 2$, then $a_1 > a_2 = a_3 = a_4 = a_5$. Using equation (42), we get $v_2(\mu^2_{a_1} + 4\mu^2_{a_5}) = v_2(\mu^2_{b_5})$.

If $v_2(\mu_{a_1}^2) = v_2(4\mu_{a_5}^2) = v_2(\mu_{a_5}^2) + 2$, then we obtain $a_1 = a_5 - 1 = m - 1$ and m is odd. Thus from equation (42) we have

$$\begin{aligned} v_2(\mu_{a_1}^2 + 4\mu_{a_5}^2) &= v_2\Big(\frac{4m^2 + 4(2m-1)^2}{(2m-1)^2}\mu_m^2\Big) \\ &= 2 + v_2\big(m^2 + (2m-1)^2\big) + v_2(\mu_m^2) = v_2(\mu_m^2) + 3 = v_2(\mu_{b_5}^2). \end{aligned}$$

Since $v_2(m^2 + (2m-1)^2) = 1$, then $2v_2(\mu_m) + 3 = 2v_2(\mu_{b_5})$. This leads to a contradiction.

If $v_2(\mu_{a_1}^2) \neq v_2(4\mu_{a_5}^2) = v_2(\mu_{a_5}^2) + 2$ and as $a_1 < a_5$, then $v_2(\mu_{a_1}^2) > v_2(4\mu_{a_5}^2)$. Using equation (42), we get $v_2(4\mu_{a_5}^2) = v_2(\mu_{b_5}^2)$. So $1 + v_2(\mu_{a_5}) = v_2(\mu_{b_5})$. Therefore, we have $b_5 = a_5 - 1 = m - 1$ and m is odd.

If $k = -2$, then $a_1 < a_2 = a_3 = a_4 = a_5$. Using the same method, we obtain $b_5 = a_5 + 1$. This completes the proof of Corollary 4.4. □

5. The equation $\mu_{a_1}\mu_{a_2}\cdots\mu_{a_n} = \mu_{b_1}\mu_{b_2}\cdots\mu_{b_n}$

In this section, we will study the equation

$$M_0(\mu_{a_1}, \dots, \mu_{a_n}) = M_0(\mu_{b_1}, \dots, \mu_{b_n}).$$

As Bang and Fuglede [3] have solved the above equation when $n = 2$, we will start with $n = 3$. We obtain the following result.

Theorem 5.1. *Let $a_1 = b_1 - 1$, $a_2 = b_2 - 1$, and $a_3 = b_3 + 1$. Then every solution (b_1, b_2, b_3) of the equation*

$$\mu_{a_1}\mu_{a_2}\mu_{a_3} = \mu_{b_1}\mu_{b_2}\mu_{b_3} \tag{43}$$

can be represented by

$$\begin{cases} b_1 = s(2t + d), \\ b_2 = t\left(2s + \dfrac{4st - 1}{d}\right), \\ b_3 = 2st - 1, \end{cases} \tag{44}$$

where $s, t, d \in \mathbb{N}$ and $d \mid (4st - 1)$.

Proof. We write $b_1 = s_1$, $b_2 = t_1$, $b_3 = u_1 - 1$. Then equation (43) becomes

$$\frac{2s_1 - 1}{2s_1} \cdot \frac{2t_1 - 1}{2t_1} = \frac{2u_1 - 1}{2u_1}. \tag{45}$$

So we obtain $2s_1t_1(2u_1-1) = u_1(2s_1-1)(2t_1-1)$. This implies that u_1 is even and $u_1 \mid (2s_1t_1)$. Put $u_1 = 2v_1$, $v_1 \in \mathbb{N}$. Thus, one can see that $v_1 \mid s_1t_1$. Therefore, there exist $s, t \in \mathbb{N}$ such that $v_1 = st$ with $s \mid s_1$, and $t \mid t_1$. Set $s_1 = sx$, $t_1 = ty$, where $x, y \in \mathbb{N}$. Hence, from (45), we get

$$2xs + 2yt = xy + 1, \tag{46}$$

from which we conclude that

$$y = \frac{2xs - 1}{x - 2t} = 2s + \frac{4st - 1}{x - 2t}.$$

As y is an integer, it is clear that $(x - 2t) \mid (4st - 1)$. So we put $d = x - 2t$ and we see that $x = 2t + d$, $y = 2s + \frac{4st-1}{d}$. Therefore, one obtains (44). This completes the proof of Theorem 5.1 $\square$

Examples 5.2. (1) Let $s = k$, $t = 1$, and $d = 1$. From (44), we get

$$b_1 = 3k, b_2 = 6k - 1, b_3 = 2k - 1.$$

This is another example of Remarks (1) in [1].

(2) Let $s = 3k + 1$, $d = 3$, then $t = 1$. From (44), we get

$$b_1 = 15k + 5, b_2 = 10k + 3, b_3 = 6k + 1.$$

Therefore, equation (43) has the solution given by

$$\mu_{15k+4}\,\mu_{10k+2}\,\mu_{6k+2} = \mu_{15k+5}\,\mu_{10k+3}\,\mu_{6k+1}.$$

For $n \geq 4$ and $k = l = 0$, we will prove the following result.

Theorem 5.3. *Let $n \geq 4$, $0 \leq a_1 < a_2 < \cdots < a_n$, and $0 \leq b_1 < b_2 < \cdots < b_n$. Then the equation*

$$\mu_{a_1}\mu_{a_2}\cdots\mu_{a_n} = \mu_{b_1}\mu_{b_2}\cdots\mu_{b_n} \tag{47}$$

has infinitely many solutions satisfying $a_i \neq b_j$, for any $1 \leq i, j \leq n$.

Proof. We use the solutions of equation (43) to construct an infinite number of solutions of equation (47).

Take $t = 1$ in (44), the equation

$$\mu_p\mu_q\mu_r = \mu_u\mu_v\mu_w, \quad p < q < r,\ u < v < w$$

has the solution

$$\begin{aligned}(p, q, r; u, v, w)\\ &= \left(2s, 2s + \frac{4s-1}{d} - 1, s(2+d) - 1; 2s-1, 2s + \frac{4s-1}{d}, s(2+d)\right). \end{aligned} \tag{48}$$

As the order of the indexes is not important, we slightly change the order the indexes. This doesn't affect the result. First, we set $d = d_3 = 1$ in (48). Taking $s = k_1$, we see that

$$a_1 = 2k_1,\ a_2 = 3k_1 - 1,\ a_3 = 6k_1 - 2,\ b_1 = 2k_1 - 1,\ b_2 = 3k_1,\ b_3 = 6k_1 - 1$$

is a solution of equation (47) when $n = 3$. Second, we put $d = d_4 = 5$ in (48). In order to have $5 \mid 4s - 1$, we take $s = 5k_1' + 4$. Therefore,

$$\begin{aligned} a_{41} &= 10k_1' + 8, \quad a_{42} = 14k_1' + 10, \quad a_{43} = 35k_1' + 27,\\ b_{41} &= 10k_1' + 7, \quad b_{42} = 14k_1' + 11, \quad b_{43} = 35k_1' + 28 \end{aligned}$$

is a solution of the equation $\mu_{a_{41}}\mu_{a_{42}}\mu_{a_{43}} = \mu_{b_{41}}\mu_{b_{42}}\mu_{b_{43}}$. Put $a_3 = a_{43}$. Since $\gcd(6, 35) = 1$, then the solutions of the equation $6k_1 - 2 = 35k_1' + 27$ are $k_1 = 35k_2 - 1, k_1' = 6k_2 - 1$, where $k_2 \in \mathbb{N}$. As

$$\frac{\mu_{a_2}}{\mu_{b_2}} \cdot \frac{\mu_{a_3}}{\mu_{b_3}} = \frac{\mu_{b_1}}{\mu_{a_1}}, \quad \frac{\mu_{a_{42}}}{\mu_{b_{42}}} \cdot \frac{\mu_{a_{43}}}{\mu_{b_{43}}} = \frac{\mu_{b_{41}}}{\mu_{a_{41}}}, \tag{49}$$

$a_3 = a_{43}$, and $b_3 = b_{43}$, then $\frac{\mu_{a_3}}{\mu_{b_3}} = \frac{\mu_{a_{43}}}{\mu_{b_{43}}}$. Hence we get

$$\frac{\mu_{b_{41}}}{\mu_{a_{41}}} \cdot \frac{\mu_{b_{42}}}{\mu_{a_{42}}} \cdot \frac{\mu_{a_2}}{\mu_{b_2}} = \frac{\mu_{b_1}}{\mu_{a_1}}. \tag{50}$$

Therefore, when $n = 4$, equation (47) has the solution

$$(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) = (a_1, a_2, b_{41}, b_{42}, b_1, b_2, a_{41}, a_{42}).$$

Now let us give two particular solutions of equation (47), for $n = 4$. The first example consists in taking $k_1 = 35k_2-1$, $k_1' = 6k_2-1$. Thus one gets the following solution of equation (47)

$$a_1 = 70k_2 - 2, \quad a_2 = 105k_2 - 4, \quad a_3 = 60k_2 - 3, \quad a_4 = 84k_2 - 3,$$

$$b_1 = 70k_2 - 3, \quad b_2 = 105k_2 - 3, \quad b_3 = 60k_2 - 2, \quad b_4 = 84k_2 - 4.$$

For the second example, we take $d = d_5 = 11$ in (48). To satisfy the condition $11 \mid 4s - 1$, we consider $s = 11k_3 + 3$. Therefore,

$$a_{51} = 22k_3 + 6, \quad a_{52} = 26k_3 + 6, \quad a_{53} = 143k_3 + 38,$$

$$b_{51} = 22k_3 + 5, \quad b_{52} = 26k_3 + 7, \quad b_{53} = 143k_3 + 39$$

is a solution of equation (43). Again here, we slightly change the order of the indexes. In the equation

$$\mu_{a_1}\mu_{a_2}\mu_{a_3}\mu_{a_4} = \mu_{b_1}\mu_{b_2}\mu_{b_3}\mu_{b_4}, \tag{51}$$

we take $a_4 = a_{53}$, where $a_4 = b_4 - 1$, i.e. $a_4 = 84k_2 - 4$ and $b_4 = 84k_2 - 3$. Since $\gcd(84, 143) = 1$, then the equation $84k_2 - 4 = 143k_3 + 38$ has a solution of the form $k_2 = 143u+72$, $k_3 = 84u+42$. Therefore, from $\frac{\mu_{a_4}}{\mu_{b_4}} = \frac{\mu_{a_{53}}}{\mu_{b_{53}}}$, $\frac{\mu_{a_{53}}}{\mu_{b_{53}}} = \frac{\mu_{b_{51}}\mu_{b_{52}}}{\mu_{a_{51}}\mu_{a_{52}}}$ and (51), one can see that equation (47) has the solution

$$(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5) = (a_1, a_2, a_3, b_{51}, b_{52}, b_1, b_2, b_3, a_{51}, a_{52}),$$

for $n = 5$.

In general, we use a similar method. Put $d = d_n$ in (48). Then there exist k_n, r_n such that $d_n \mid 4s - 1$, where $s = d_nk_n + r_n$. So (a_{n1}, a_{n2}, a_{n3}, b_{n1}, b_{n2}, b_{n3}) is a solution of equation (48). Now, we suppose that equation (47) has a solution $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$, with $a_n = b_n - 1$. The goal is to show that equation (47) has a solution for $n + 1$. Set $d = d_{n+1}$ in (48), where d_{n+1} satisfies

$$\gcd(d_{n+1},\ 2\prod_{j=3}^{n} d_j(d_j + 2)) = 1. \tag{52}$$

In order to have $d_{n+1} \mid 4s-1$, we can determine k_{n+1}, r_{n+1} with $s = d_{n+1}k_{n+1} + r_{n+1}$ so that $(a_{n+1,1}, a_{n+1,2}, a_{n+1,3}, b_{n+1,1}, b_{n+1,2}, b_{n+1,3})$ is a solution of equation (48). Taking $a_n = a_{n+1,3}$, from (52), we see that the equation $a_n = a_{n+1,3}$ has a solution. Hence, equation (47) has the solution

$$\begin{aligned}(a_1, &a_2, \ldots, a_n, a_{n+1}, b_1, b_2, \ldots, b_n, b_{n+1})\\ &= (a_1, a_2, \ldots, a_{n-1}, b_{n+1,1}, b_{n+1,2}, b_1, b_2, \ldots, b_{n-1}, a_{n+1,1}, a_{n+1,2}),\end{aligned}$$

for $n+1$. Therefore, equation (47) has infinitely many solutions satisfying $a_i \neq b_j$, for any $1 \leq i, j \leq n$. So the proof of Theorem 5.3 is complete. $\square$

6. Conjectures

Using Corollary 2.3 and Theorem 2.5, we believe that the product of n normalized binomial mid-coefficients $\mu_{a_1}\mu_{a_2}\cdots\mu_{a_n}$ cannot be an n-power of rational number, expect when $a_1 = a_2 = \cdots = a_n$. So we make the following conjecture.

Conjecture 6.1. *The equation*

$$\mu_{a_1}\mu_{a_2}\cdots\mu_{a_n} = q^n, \quad n \geq 3,\ n \in \mathbb{N},\ q \in \mathbb{Q} \tag{53}$$

only has the trivial solution $a_1 = a_2 = \cdots = a_n$.

For $n = 4$, we have the following conjecture.

Conjecture 6.2. *Equation* (39) *only has the trivial solution, except for* $k = 1$, *in which case* $(\{a_i, a_j, a_n, a_l\}, \{b_i, b_j, b_n, b_l\}) \in (\{1, 3, 3, m\}, \{2, 2, 2, m\}))$, *where* $m \geq 0$.

In [1], Alzer and Fuglede proved the following result.

Proposition 6.3. *Let* $k \neq 0$, $p, q, r \geq 0$ *be integers, then the equation* $\mu_p^k + \mu_q^k = \mu_r^k$ *has only solutions* $(k, p, q, r) = (1, 1, 1, 0)$, $(-1, 0, 0, 1)$.

Let $a_1, a_2, \ldots a_n, k$ be integers. We consider the general equation

$$\mu_{a_1}^k + \cdots + \mu_{a_n}^k = \mu_b^k, \quad k \neq 0,\ n \geq 3, \tag{54}$$

and we set the following conjecture.

Conjecture 6.4. *Equation* (54) *has only the solutions given by*

- $\mu_2 + \mu_3 + \mu_3 = \mu_0$, *when* $n = 3$,
- $2^k \mu_1^k = \mu_0^k$, $2^k \mu_0^{-k} = \mu_1^{-k}$, *where* $n = 2^k$, $k \geq 2$, $k \in \mathbb{N}$.

Acknowledgements The first author was supported by the fund of the key disciplines in the general colleges and universities of Xin Jiang Uygur Autonomous Region (No. 2012ZDXK21). The first and third authors were supported by the Applied Basic Research Foundation of Sichuan Provincial Science and Technology Department (No. 2011JYZ032, No. 12ZB002). The work on this paper was prepared during a very enjoyable visit of the second author at *l'Institut de Mathématiques de Bordeaux*. He thanks the people of this institution for their hospitality. He was also supported in part by Purdue University North Central.

References

[1] H. Alzer and B. Fuglede, Normalized binomial mid-coefficients and power means, *J. Number Theorey* **115** (2005), 284–294.

[2] H. Alzer, S. Gerhold, M. Kauers and A Lupaş, On Turán's inequality for Legendre polynomials, *Expos. Math.* **25**(2) (2007), 181–186.

[3] T. Bang and B. Fuglede, No two quotients of normalized binomial mid-coefficients are equal, *J. Number Theorey* **35** (1990), 345–349.

[4] C. Berg and B. Fuglede, Liouville's operator for a disc in space, *Manuscripta Math.* **67** (1990), 165–185.

[5] C. Berg and J. Lützen, J. Liouville's unpublished work on on integral operator in potential theory, A historical and mathematical analysis, *Exposition. Math.* **8** (1990), 97–136.

[6] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, Means and Their Inequalities, *Reidel, Dordrecht*, 1988.

[7] J. Dutku, On some gamma function inequalities, *SIAM J. Math. Anal.* **16** (1985), 180–185.

[8] P. Erdős, A theorem of Sylvester and Schur, *J. London Math. Soc.* **9** (1934), 282–288.

[9] P. Erdős and J. L. Selfridge, The product of consecutive integers is never a power, *Illinois J. Math.* **19** (1975), 292–301.

[10] D. S. Mitrinović, J. Sándor and B. Crstici, Handbook of Number Theory, *Kluwer, Dordrecht*, 1996.

[11] L. J. Mordell, Diophantine Equations, *London, Academic Press*, 1969.

[12] F. M. J. Olver, Asymptotics and Special Functions, *Academic Press, New Youk*, 1974.

[13] Z. Sasvári, Inequalities for binomial coefficients, *J. Math. Anal. Appl.* **236** (1999), 223–226.

[14] P. Stănică, Good lower and upper bounds on binomial coefficients, *J. Inequal. Pure Appl. Math.* **2**(3) (2001), Aricle 30.

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(Received May 9, 2012)