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On the structure of the homeomorphism and diffeomorphism groups fixing a point

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Abstract. Let M be a manifold, $p \in M$ and let $\mathcal{H}(M,p)$ be the identity component of the group of all compactly supported homeomorphisms of M fixing p. It is shown that $\mathcal{H}(M,p)$ is a perfect group. Next, we prove that the group $\mathcal{H}(\mathbb{R}^n,0)$ is bounded. In contrast, in the C^∞ category the diffeomorphism group $\mathcal{D}^\infty(\mathbb{R}^n,0)$, analogous to $\mathcal{H}(\mathbb{R}^n,0)$, is neither perfect nor bounded. Finally, the boundedness and uniform perfectness of $\mathcal{H}(M,p)$ is studied.

1. Introduction

Let M be a topological metrizable manifold and let $\mathcal{H}(M)$ be the identity component of the group of all compactly supported homeomorphisms of M. By $\mathcal{H}(M,p)$, where $p \in M$, we denote the identity component of the group of all $h \in \mathcal{H}(M)$ with $h(p) = p$.

Recall that a group G is called *perfect* if it is equal to its own commutator subgroup $[G,G]$, that is $H_1(G) = 0$. Moreover we say that a manifold M admits a *compact exhaustion* iff there is a sequence $\{M_i\}_{i=1}^{\infty}$ of compact submanifolds with boundary such that $M_1 \subset \operatorname{Int} M_2 \subset M_2 \subset \dots$ and $M = \bigcup_{i=1}^{\infty} M_i$.

The following basic fact is probably well-known, see e.g., [13].

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Theorem 1.1. *Assume that either M is compact (possibly with boundary), or M is noncompact boundaryless and admits a compact exhaustion. Then $\mathcal{H}(M)$ is perfect. If M is also connected then $\mathcal{H}(M)$ is simple.*

The proof of the perfectness is a consequence of Mather's paper [16] combined with Edwards and Kirby [5], Corollary 1.3 and Remark 7.2. A special case of Theorem 1.1 was already showed by Fisher [7]. There exist some generalizations of Theorem 1.1 (see, e.g., [10], [22], [20]). The simplicity follows from a theorem of Epstein [6] (see also [20]).

Let M be a smooth manifold of class C^r, $r = 1, \dots, \infty$. The symbol $\mathcal{D}^r(M)$ (resp. $\mathcal{D}^r(M, p)$) will stand for the identity component of the group of all compactly supported C^r-diffeomorphisms of M (resp. fixing $p \in M$). Theorem 1.1 is a C^0 analogue of Thurston's theorem which states that the group $\mathcal{D}^\infty(M)$ is perfect and simple (see [26], [3]). Mather in [17] and [18] proved that $\mathcal{D}^r(M)$ is perfect and simple as well for $r \neq \dim M + 1$. The case $r = \dim M + 1$ is unsolved (see [19], [14]). The simplicity theorems on the classical diffeomorphism groups are also known ([2], [3], [11], [23], [26]). The structure of the C^∞-diffeomorphism group of a manifold with boundary has been studied in [24], [15], [21] and [1].

It is easy to see that $\mathcal{D}^r(M, p)$ is not perfect for $r \geq 1$. Moreover, Fukui calculated in [8] that $H_1(\mathcal{D}^\infty(\mathbb{R}^n, 0)) = \mathbb{R}$. In the topological category the situation is different.

Theorem 1.2. (1) *The groups $\mathcal{H}(\mathbb{R}^n, 0)$ and $\mathcal{H}(\mathbb{R}^n_+, 0)$ are perfect, where $\mathbb{R}^n_+ = [0, \infty) \times \mathbb{R}^{n-1}$.*

(2) *If M fulfils the assumption of Theorem 1.1 then the group $\mathcal{H}(M, p)$ is perfect.*

A similar result was obtained by Tsuboi in [28]. He proved that $\mathcal{H}([0, 1])$ is perfect by using different argument than that for Theorem 1.2 (in particular, he did not apply [5]). Next he generalized the result for Lipschitz homeomorphisms and for C^1-diffeomorphisms (resp. C^∞-diffeomorphisms) tangent (resp. infinitely tangent) to the identity at the endpoints. Observe as well that Theorem 1.2 was proved for M closed by Fukui in [9]. However, our proof is different than that in [9] and it leads to Corollary 3.7 on the uniform perfectness.

Recall that a group is *bounded* if it is bounded with respect to any bi-invariant metric. Our main result is the following

Theorem 1.3. (1) *$\mathcal{H}(\mathbb{R}^n, 0)$ is bounded group.*

(2) *Under the assumption of Theorem 1.1 the group $\mathcal{H}(M)$ is bounded whenever $\mathcal{H}(M, p)$ is bounded.*

Note that this theorem is no longer true in the C^r category for $r \geq 1$. (See Proposition 4.2).

The fact that $\mathcal{D}^\infty(M)$ is bounded for many types of manifolds is known in view of the recent result by Burago, Ivanov and Polterovich [4] (see also [13]). In the proofs of the above theorems we develop some technical ideas from [13].

We will also show some other properties of the group $\mathcal{H}(M,p)$. Namely, in Section 3 we prove that $\mathcal{H}(\mathbb{R}^n,0)$ is uniformly perfect and its commutator length diameter is ≤ 2. In Section 5 we show that $\mathcal{H}(M,p)$ is uniformly perfect provided the fragmentation norm fragd_M is bounded. In the last section some concluding remarks are given.

2. Deformation properties of the space of isotopies

The proofs of Theorems 1.1, 1.2 and 1.3 depend on the deformation properties for the spaces of isotopies obtained by Edwards and Kirby in [5]. See also Siebenmann [25].

Let $I=[0,1]$. For an isotopy $\{h_t\}_{t\in I}$ of M we set $\mathrm{supp}(\{h_t\}_{t\in I})=\bigcup_{t\in I}\mathrm{supp}(h_t)$. In the sequel we will write h_t instead of $\{h_t\}_{t\in I}$. By a ball we mean a relatively compact open ball. For $U\subset M$ we denote by $\mathcal{H}_U(M)$ the identity component of the group of all homeomorphisms compactly supported in U.

We have the following fragmentation property.

Lemma 2.1 ([5]). *Let M be as in Theorem 1.1 and let $h_t: M\to M$, $t\in I$, be a compactly supported isotopy of M with $h_0=\mathrm{Id}$. Then there exist isotopies $h^i_t: M\to M$, $i=1,\dots,k$, such that $h_t=h^1_t\dots h^k_t$, $h^i_0=\mathrm{Id}$ and $\mathrm{supp}(h^i_t)\subset B_i$, $i=1,\dots,k$, where each B_i is a ball or half-ball. Moreover*

(1) *If $\partial M\neq\emptyset$ and $h_t=\mathrm{Id}$ on ∂M for all t, then $h^i_t=\mathrm{Id}$ on ∂M for all i and t.*

(2) *Let $p\in M$. If $h_t\in\mathcal{H}(M,p)$ for all t, then $h^i_t\in\mathcal{H}(M,p)$ for all i and t.*

Proof. For M compact the first assertion coincides with Corollary 1.3 in [5]. Next, (1) and (2) follow from Remark 7.2 (p. 81–83 in [5]).

Now, assume that M admits a compact exhaustion $\{M_j\}_{j=1}^\infty$ and let h_t be an isotopy in $\mathcal{H}(M)$. Then h_t is supported in M_j for some $j\in\mathbb{N}$. Hence $h_t=\mathrm{Id}$ on ∂M_j. From (1) with $M=M_j$ we get isotopies $h^1_t,\dots,h^k_t$ in $\mathcal{H}(M_j)$ such that $h_t=h^1_t\dots h^k_t$ and $h^i_t=\mathrm{Id}$ on ∂M_j, for $i=1,\dots,k$. We extend each h^i_t to M by setting $h^i_t=\mathrm{Id}$ outside M_j. □

Corollary 2.2. *Lemma 2.1 is still valid for elements of $\mathcal{H}(M)$ instead of isotopies.*

In the sequel we will need the following version of Isotopy Extension Theorem.

Theorem 2.3 ([5])**.** *Let M be a metrizable topological manifold. Suppose that f_t is an isotopy in $\mathcal{H}(M)$ with $f_0 = \mathrm{Id}$ and that $K \subset M$ is a compact set. Then for any open neighborhood U of the set $\bigcup_{t\in I} f_t(K)$ there exists an isotopy g_t in $\mathcal{H}(M)$ such that $g_t = f_t$ on K and $\mathrm{supp}(g_t) \subset U$ for $t \in I$.*

3. Basic lemma and the perfectness of $\mathcal{H}(M, p)$

In this section we will prove Theorem 1.2. We begin with the following fact which plays an important role in studies on homeomorphism groups.

Lemma 3.1 (Basic lemma, [16])**.** *Let $U \subset M$ be an open set and $B \subset M$ be a ball such that $\overline{B} \subset U$. Then there exist $u \in \mathcal{H}_U(M)$ and a homomorphism $\varphi : \mathcal{H}_B(M) \to \mathcal{H}_U(M)$ such that $h = [\varphi(h), u]$ for all $h \in \mathcal{H}_B(M)$.*

Proof. Choose a ball B' such that $\overline{B} \subset B' \subset \overline{B'} \subset U$. Next, fix $p \in \partial B'$ and set $B_0 = B$. There exists a sequence of balls $\{B_i\}_{i=1}^{\infty}$ with $\overline{B}_i \subset B'$, $i \geq 1$, where the family $\{B_i\}_{i=0}^{\infty}$ is pairwise disjoint, locally finite in B', and $B_i \to p$ as $i \to \infty$.

Since $\mathcal{H}_U(M)$ acts transitively on the family of balls in B' we can find a homeomorphism $u \in \mathcal{H}_U(M)$ such that $u(B_{i-1}) = B_i$ for $i = 1, 2, \ldots$. Then we define a homomorphism $\varphi : \mathcal{H}_B(M) \to \mathcal{H}_U(M)$ by the formula

$$\varphi(h) = \begin{cases} u^i h u^{-i} & \text{on } B_i,\ i = 0, 1, \ldots \\ \mathrm{Id} & \text{outside } \bigcup_{i=0}^{\infty} B_i\,. \end{cases}$$

It is obvious that $h = [\varphi(h), u]$ as required. □

The above fact appeared in Mather's paper [16]. Actually, Mather proved also the acyclicity of $\mathcal{H}(\mathbb{R}^n)$. Obviously, [16] and Lemma 3.1 are no longer true for C^1-homeomorphisms. However, Tsuboi gave an excellent improvement of this reasoning and adapted it for C^r-diffeomorphisms with small r (see [27]).

Let G be a group. For $g \in [G, G]$ the least k such that g is a product of k commutators is called the *commutator length* of g and is denoted by $\mathrm{cl}_G(g)$.

We will need some results from [4]. A subgroup H of G is called *strongly m-displaceable* if there exists $f \in G$ such that the subgroups $H, fHf^{-1}, \ldots, f^m H f^{-m}$ pairwise commute. Then we say that f *m-displaces* H.

Theorem 3.2 ([4]). *Let G be a group and H a subgroup G. If some $g \in G$ m-displaces H for every $m \geq 1$ then for all $h \in [H, H]$ we get $\mathrm{cl}_G(h) \leq 2$.*

Another useful result is the following

Proposition 3.3. *If U, V are open disjoint subsets of M such that there exists $f \in \mathcal{H}(M)$ with $\overline{f(U \cup V)} \subset V$ then f m-displaces $\mathcal{H}_U(M)$ for all $m \geq 1$.*

PROOF. Indeed, this follows from the relation $f^m(U) \subset f^{m-1}(V) \setminus f^m(V)$ for every $m \geq 1$. □

Let $\mathbb{S}^{n-1}$ be the unit sphere. We denote $S = \mathbb{S}^{n-1} \subset \mathbb{R}^n$ in the case $\mathcal{H}(\mathbb{R}^n, 0)$ and $S = \mathbb{S}^{n-1} \cap \mathbb{R}^n_+$ in the case $\mathcal{H}(\mathbb{R}^n_+, 0)$. Moreover let $A(a, b) = S \times (a, b)$ and $\overline{A}(a, b) = S \times [a, b]$ for $0 < a < b < \infty$.

Corollary 3.4. *Let $M = \mathbb{R}^n$ and $U = A(b, a)$ for some $a > b > 0$. Then every $h \in \mathcal{H}_U(\mathbb{R}^n)$ is expressed as the product of two commutators of elements of $\mathcal{H}_U(\mathbb{R}^n)$.*

PROOF. Let $h \in \mathcal{H}_U(\mathbb{R}^n)$. Since $\mathcal{H}_U(\mathbb{R}^n)$ is perfect from Theorem 1.1 then $h \in [\mathcal{H}_U(\mathbb{R}^n), \mathcal{H}_U(\mathbb{R}^n)]$. Let $h|_U = [h_1, h_2] \dots [h_{2k-1}, h_{2k}]$ for some $k \in \mathbb{N}$ and $h_1, \dots, h_{2k} \in \mathcal{H}(U)$. Each h_i has compact support in U so we may extend h_i to $\mathbb{R}^n$ by $h_i = \mathrm{Id}$ outside U. Moreover there are a', b' such that $a > a' > b' > b$ and $\mathrm{supp}(h_i) \subset V = A(b', a')$ for $i = 1, \dots, 2k$. Especially $h \in [\mathcal{H}_V(\mathbb{R}^n), \mathcal{H}_V(\mathbb{R}^n)]$.

Choose c', d' with $b' > c' > d' > b$ and denote $W = A(d', c')$. There exists homeomorphism $\tilde{u} : [0, \infty) \to [0, \infty)$ with support in (b, a) such that $\overline{\tilde{u}(d', a')} \subset (d', c')$. By setting $u = \mathrm{Id}_S \times \tilde{u}$ we get $u \in \mathcal{H}_U(\mathbb{R}^n)$ and $\overline{u(V \cup W)} \subset W$.

From Proposition 3.3 $u \in \mathcal{H}_U(\mathbb{R}^n)$ m-displaces $\mathcal{H}_V(\mathbb{R}^n)$ for every $m \geq 1$. In view of Theorem 3.2 we get $\mathrm{cl}_{\mathcal{H}_U(\mathbb{R}^n)}(h) \leq 2$ as required. □

Suppose that $\{U_i\}_{i \in \mathbb{N}}$ is a pairwise disjoint, locally finite family of open sets of $M \setminus \{p\}$. Put $U = \bigcup_{i=1}^{\infty} U_i$. By $\mathcal{H}(M, U)$ we denote the group of all homeomorphisms supported in U such that there exists the decomposition $h = h_1 h_2 \dots$ with $\mathrm{supp}(h_i) \subset U_i$ such that $h_i \in \mathcal{H}_{U_i}(M)$, $i \in \mathbb{N}$.

Corollary 3.5. *Let $M = \mathbb{R}^n$ and take a sequence*

$$a_1 > b_1 > a_2 > b_2 > \dots > 0$$

tending to 0. Next, set $U = \bigcup_{i=1}^{\infty} U_i$, where $U_i = A(b_i, a_i)$. Then any element of the group $\mathcal{H}(M, U)$ is expressed as the product of two commutators of elements of $\mathcal{H}(M, U)$.

Indeed, we may use Corollary 3.4 for each i and glue together homeomorphisms obtained in this manner.

Proposition 3.6. *For every $f \in \mathcal{H}(\mathbb{R}^n, 0)$ with $\operatorname{supp}(f) \subset S \times [0, d_0]$, $d_0 > 1$, there exists a sequence*

$$c_0 = d_0 > a_1 > b_1 > c_1 > d_1 > \cdots > a_k > b_k > c_k > d_k > \cdots > 0 \tag{3.1}$$

tending to 0, and $g, h \in \mathcal{H}(\mathbb{R}^n, 0)$ such that $f = gh$,

(1) *$g = f$ on $\bigcup_{j=1}^{\infty} \overline{A}(c_j, b_j)$ and $h = f$ on $\bigcup_{j=1}^{\infty} \overline{A}(a_j, d_{j-1})$,*

(2) *$\operatorname{supp}(g) \subset \bigcup_{j=1}^{\infty} A(d_j, a_j)$ and $\operatorname{supp}(h) \subset \bigcup_{j=1}^{\infty} A(b_j, c_{j-1})$*

(3) *if $g = g_1 g_2 \ldots$ with $\operatorname{supp}(g_j) \subset A(d_j, a_j)$ then $g_j \in \mathcal{H}_{A(d_j,a_j)}(\mathbb{R}^n)$ and analogously for $h = h_1 h_2 \ldots$ with $\operatorname{supp}(h_j) \subset A(b_j, c_{j-1})$ we have $h_j \in \mathcal{H}_{A(b_j,c_{j-1})}(\mathbb{R}^n)$ for $j = 1, 2, \ldots$.*

Proof. In the proof we apply Theorem 2.3 for $M = \mathbb{R}^n \setminus \{0\}$.

Let $f \in \mathcal{H}(\mathbb{R}^n, 0)$ and let f_t be an isotopy from Id to f. Choose $d_0 > a_1 > b_1 > c_1 > d_1 > 0$ such that $\bigcup_{t\in I} f_t(\overline{A}(c_1, b_1)) \subset \overline{A}(d_1, a_1)$. From Theorem 2.3 there is an isotopy g_t^1 in $\mathcal{H}(\mathbb{R}^n)$ such that $g_t^1 = f_t$ on $\overline{A}(c_1, b_1)$ and $\operatorname{supp}(g_t^1) \subset A(d_1, a_1)$. Moreover $g_t^1 \in \mathcal{H}_{A(d_1,a_1)}(\mathbb{R}^n)$ that is g_t^1 is isotopy supported in $A(d_1, a_1)$. Here we put $g_t^1(0) = 0$.

Define $h_t^1 = (g_t^1)^{-1} f_t$ on $A(b_1, c_0)$ and $h_t^1 = \mathrm{Id}$ otherwise. Then $h_t^1 = f_t$ on $\overline{A}(a_1, d_0)$, $\operatorname{supp}(h_t^1) \subset A(b_1, c_0)$ and $h_t^1 \in \mathcal{H}_{A(b_1,c_0)}(\mathbb{R}^n)$. Let $f_t^1 = (g_t^1 h_t^1)^{-1} f_t$. Then $\operatorname{supp}(f_t^1) \subset A(0, c_1)$.

Inductively, suppose we have defined a sequence $d_0 > a_1 > b_1 > c_1 > d_1 > \cdots > a_i > b_i > c_i > d_i$ and isotopy $f_t^i \in \mathcal{H}_{A(0,c_i)}(\mathbb{R}^n)$.

We take $d_i > a_{i+1} > b_{i+1} > c_{i+1} > d_{i+1} > 0$ such that $\bigcup_{t\in I} f_t^i(\overline{A}(c_{i+1}, b_{i+1})) \subset \overline{A}(d_{i+1}, a_{i+1})$. From Theorem 2.3 there exists an isotopy g_t^{i+1} in $\mathcal{H}(\mathbb{R}^n)$ such that $g_t^{i+1} = f_t^i$ on $\overline{A}(c_{i+1}, b_{i+1})$ and $g_t^{i+1} \in \mathcal{H}_{A(d_{i+1},a_{i+1})}(\mathbb{R}^n)$. We define $h_t^{i+1} \in \mathcal{H}(\mathbb{R}^n)$ by $h_t^{i+1} = (g_t^{i+1})^{-1} f_t^i$ on $A(b_{i+1}, c_i)$ and $h_t^{i+1} = \mathrm{Id}$ outside this set. We get $h_t^{i+1} = f_t^i$ on $\overline{A}(a_{i+1}, d_i)$ and $h_t^{i+1} \in \mathcal{H}_{A(b_{i+1},c_i)}(\mathbb{R}^n)$. Let $f_t^{i+1} = (g_t^{i+1} h_t^{i+1})^{-1} f_t^i$.

Products $g = \prod_{i=1}^{\infty} g_1^i$ and $h = \prod_{i=1}^{\infty} h_1^i$ have the required properties. □

Proof of Theorem 1.2 for $\mathcal{H}(\mathbb{R}^n, 0)$. For $f \in \mathcal{H}(\mathbb{R}^n, 0)$ we take g, h as in above proposition. (The proof of the case $\mathcal{H}(\mathbb{R}^n_+, 0)$ is contained in the proof of Corollary 3.7 below).

Denote $U_i = A(d_i, a_i)$ and $U = \bigcup_{i=1}^{\infty} U_i$. In view of Corollary 3.5 with $M = \mathbb{R}^n$ we get $g = [g_1, g_2][g_3, g_4]$ for $g_1, g_2, g_3, g_4 \in \mathcal{H}(M, U)$. It is easily seen that

$g_i \in \mathcal{H}(\mathbb{R}^n, 0)$, $i = 1, 2, 3, 4$. Consequently $g \in [\mathcal{H}(\mathbb{R}^n, 0), \mathcal{H}(\mathbb{R}^n, 0)]$. Analogously for h. Thus $\mathcal{H}(\mathbb{R}^n, 0)$ is a perfect group. It follows (1).

To show (2) we use Corollary 2.2 combined with (1). □

For any perfect group G denote by cld_G the commutator length diameter of G, i.e. $\mathrm{cld}_G := \sup_{g \in G} \mathrm{cl}_G(g)$. Next, G is called *uniformly perfect* if G is perfect and $\mathrm{cld}_G < \infty$.

Note that recently Burago, Ivanov and Polterovich in [4] and, independently, Tsuboi in [29] proved that the groups $\mathcal{D}^\infty(M)$ are uniformly perfect for many types of M and calculated some estimations on the commutator length diameter of these groups. The results and their proofs depend on the topology of M.

Using the above proof we get immediately that $\mathrm{cld}_{\mathcal{H}(\mathbb{R}^n,0)} \leq 4$. But by modification of the construction in Lemma 3.1 we obtain better estimation.

Corollary 3.7. *The group $\mathcal{H}(\mathbb{R}^n, 0)$ is uniformly perfect and $\mathrm{cld}_{\mathcal{H}(\mathbb{R}^n,0)} \leq 2$. The same is true for $\mathcal{H}(\mathbb{R}^n_+, 0)$.*

Proof. For $f \in \mathcal{H}(\mathbb{R}^n, 0)$ let g, h and U be as in the proof of Theorem 1.2. (The case $\mathcal{H}(\mathbb{R}^n_+, 0)$ is analogous).

For each $i \geq 1$ choose $\bar{d}_i$, $\bar{a}_i$ such that $d_{i-1} > \bar{a}_i > a_i > d_i > \bar{d}_i > \bar{a}_{i+1} > a_{i+1}$ and set $V_i = A(\bar{d}_i, \bar{a}_i)$. There exists homeomorphism $\tilde{u} : [0, \infty) \to [0, \infty)$ with compact support such that $\tilde{u}(\bar{d}_i, \bar{a}_i) = (\bar{d}_{i+1}, \bar{a}_{i+1})$ and $\tilde{u}(\bar{d}_i, a_i) \subsetneq (\bar{d}_{i+1}, d_{i+1})$ for every $i \geq 1$.

Take $u = \mathrm{Id}_S \times \tilde{u}$. Then $u \in \mathcal{H}(\mathbb{R}^n, 0)$ and $u(U_i) \subset u(V_i) = V_{i+1}$ for $i \geq 1$. Notice also that sets $\overline{u^j(U_i)}$ are pairwise disjoint for all $i \geq 1$, $j \geq 0$, and $u^j(U_i) \to 0$ as $j \to \infty$.

We define

$$\varphi(g) = \begin{cases} u^j g u^{-j} & \text{on } u^j(U_i),\ i \geq 1,\ j \geq 0 \\ \mathrm{Id} & \text{outside } \bigcup_{i,j} u^j(U_i)\,. \end{cases}$$

From the fact that $g \in \mathcal{H}(\mathbb{R}^n, U)$ we obtain $\varphi(g) \in \mathcal{H}(\mathbb{R}^n, 0)$ and $g = [\varphi(g), u]$. Analogously for h. Hence $\mathrm{cl}_{\mathcal{H}(\mathbb{R}^n,0)}(f) \leq 2$. □

4. Conjugation-invariant norms and the boundedness of $\mathcal{H}(M)$

Let G be a group. A *conjugation-invariant norm* (or *norm* for short) on G is a function $\nu : G \to [0, \infty)$ for every $g, h \in G$ we have

(1) $\nu(g) > 0$ if and only if $g \neq e$,

(2) $\nu(g^{-1}) = \nu(g)$,

(3) $\nu(gh) \leq \nu(g) + \nu(h)$,

(4) $\nu(hgh^{-1}) = \nu(g)$.

It is easy to see that G is bounded if and only if any conjugation-invariant norm on G is bounded.

Observe that the commutator length cl_G is a conjugation-invariant norm on $[G, G]$, or on G if G is a perfect group.

From Corollary 2.2 for any $h \in \mathcal{H}(M)$ there is a decomposition $h = h_1 \dots h_k$ such that $h_i \in \mathcal{H}_{B_i}(M)$, where B_i is a ball or half-ball for $i = 1, \dots, k$. Hence we may introduce the following fragmentation norm frag_M on $\mathcal{H}(M)$. Namely, for $h \in \mathcal{H}(M)$, $h \neq \mathrm{Id}$, we define $\mathrm{frag}_M(h)$ to be the least $k > 0$ such that $h = h_1 \dots h_k$ as above. We take $\mathrm{fragd}_M := \sup_{h \in \mathcal{H}(M)} \mathrm{frag}_M(h)$ as the diameter of $\mathcal{H}(M)$ in frag_M.

Analogously, from Lemma 2.1 (2) we may define another fragmentation norm, $\mathrm{frag}_{M,p}$, for the group $\mathcal{H}(M, p)$ instead of $\mathcal{H}(M)$. However, in view of the proof of Remark 7.2 in [5] we obtain

Proposition 4.1. *For every $h \in \mathcal{H}(M, p)$ one has $\mathrm{frag}_{M,p}(h) = \mathrm{frag}_M(h)$.*

Burago, Ivanov and Polterovich proved in [4] that $\mathcal{D}^\infty(M)$ is bounded (and a fortiori uniformly perfect) for many manifolds. They stated there that they did not know any example of M such that $\mathcal{D}^\infty(M)$ is unbounded.

On the other hand, we have the following

Proposition 4.2. *The groups $\mathcal{D}^r(M, p)$ for $r = 1, \dots, \infty$ are unbounded.*

Proof. Choose a chart at p. Then there is the epimorphism $\mathcal{D}^r(M, p) \ni f \mapsto \mathrm{jac}_p f \in \mathbb{R}_+$, where $\mathrm{jac}_p f$ is the jacobian of f at p in this chart. From Proposition 1.3 in [4] an abelian group is bounded if and only if it is finite. Now Lemma 1.10 in [4] implies that $\mathcal{D}^r(M, p)$ is unbounded. □

In the sequel we need

Theorem 4.3 ([4]). *Let G be a group with a conjugation-invariant norm ν and H a subgroup G. Suppose that some $g \in G$ m-displaces H for every $m \geq 1$. Then $\nu(h) \leq 14\nu(g)$ for all $h \in [H, H]$.*

5. Boundedness of $\mathcal{H}(M, p)$

In this section we will prove Theorem 1.3.

Proposition 5.1. *Let $R > 0$. For any sequence*

$$R > a_1 > b_1 > a_2 > b_2 > \cdots > 0$$

tending to 0*, there exists $h \in \mathcal{H}(\mathbb{R}^n, 0)$ such that for every $i = 1, 2, \ldots$*

$$h(\overline{A}(b_{2i-1}, a_{2i-1}) \cup \overline{A}(b_{2i}, a_{2i})) \subset A(b_{2i}, a_{2i}).$$

Moreover, if we have another sequence

$$R > c_1 > d_1 > c_2 > d_2 > \cdots > 0$$

tending to 0*, then there is $\varphi \in \mathcal{H}(\mathbb{R}^n, 0)$ of the form $\varphi = \mathrm{Id}_S \times \tilde{\varphi}$ with $\tilde{\varphi}(b_i, a_i) = (d_i, c_i)$ for $i = 1, 2, \ldots$.*

Proof of Theorem 1.3. First, we show the boundedness of $\mathcal{H}(\mathbb{R}^n, 0)$.

Fix a conjugation-invariant norm ν on $\mathcal{H}(\mathbb{R}^n, 0)$ and let $f \in \mathcal{H}(\mathbb{R}^n, 0)$ with $\mathrm{supp}(f) \subset S \times [0, d_0]$, $d_0 > 1$. From Proposition 3.6 there exist a sequence

$$c_0 = d_0 > a_1 > b_1 > c_1 > d_1 > \cdots > 0$$

tending to 0, and homeomorphisms $h_1, h_2, h_3, h_4 \in \mathcal{H}(\mathbb{R}^n, 0)$ with $f = h_1 h_2 h_3 h_4$ such that

$$h_1 = f \text{ on } \bigcup_{j=1}^{\infty} \overline{A}(c_{2j-1}, b_{2j-1}), \quad \mathrm{supp}(h_1) \subset U_1 := \bigcup_{j=1}^{\infty} A(d_{2j-1}, a_{2j-1}),$$

$$h_2 = f \text{ on } \bigcup_{j=1}^{\infty} \overline{A}(c_{2j}, b_{2j}), \quad \mathrm{supp}(h_2) \subset U_2 := \bigcup_{j=1}^{\infty} A(d_{2j}, a_{2j}),$$

$$h_3 = f \text{ on } \bigcup_{j=1}^{\infty} \overline{A}(a_{2j-1}, d_{2j-2}), \quad \mathrm{supp}(h_3) \subset U_3 := \bigcup_{j=1}^{\infty} A(b_{2j-1}, c_{2j-2}),$$

$$h_4 = f \text{ on } \bigcup_{j=1}^{\infty} \overline{A}(a_{2j}, d_{2j-1}), \quad \mathrm{supp}(h_4) \subset U_4 := \bigcup_{j=1}^{\infty} A(b_{2j}, c_{2j-1}).$$

Moreover we get $h_i \in \mathcal{H}(M, U_i)$ for $i = 1, 2, 3, 4$.

Next, fix a sequence tending to 0

$$R > \bar{a}_1 > \bar{b}_1 > \bar{a}_2 > \bar{b}_2 > \cdots > 0.$$

From Proposition 5.1 there are $g \in \mathcal{H}(\mathbb{R}^n, 0)$ such that

$$g(\overline{A}(\bar{b}_{2i-1}, \bar{a}_{2i-1}) \cup \overline{A}(\bar{b}_{2i}, \bar{a}_{2i})) \subset A(\bar{b}_{2i}, \bar{a}_{2i})$$

and $\varphi_1 \in \mathcal{H}(\mathbb{R}^n, 0)$ such that $\varphi_1(\overline{A}(\bar{b_i}, \bar{a}_i)) = \overline{A}(d_i, a_i)$. Then $\overline{\varphi_1 g \varphi_1^{-1}(U_1 \cup U_2)} \subset U_2$.

From Proposition 3.3 homeomorphism $\varphi_1 g \varphi_1^{-1}$ m-displaces $\mathcal{H}(\mathbb{R}^n, U_1)$ for every $m \geq 1$. Since $h_1 \in [\mathcal{H}(\mathbb{R}^n, U_1), \mathcal{H}(\mathbb{R}^n, U_1)]$ in view of Corollary 3.5, then from Theorem 4.3 we obtain

$$\nu(h_1) \leq 14\nu(\varphi_1 g \varphi_1^{-1}) = 14\nu(g).$$

Using analogous estimations for h_2, h_3, h_4 with some $\varphi_2, \varphi_3, \varphi_4 \in \mathcal{H}(\mathbb{R}^n, 0)$ we get

$$\nu(f) \leq \nu(h_1) + \nu(h_2) + \nu(h_3) + \nu(h_4) \leq 56\nu(g)$$

as required.

Now we prove the second part of Theorem 1.3.

Assume that $\mathcal{H}(M, p)$ is bounded. Let ν be a conjugation-invariant norm on $\mathcal{H}(M)$ and let $f \in \mathcal{H}(M)$.

If M is noncompact we may choose $\varphi \in \mathcal{H}(M)$ such that $\varphi f \varphi^{-1} \in \mathcal{H}(M, p)$. Then

$$\nu(f) = \nu(\varphi f \varphi^{-1}) = \nu|_{\mathcal{H}(M,p)}(\varphi f \varphi^{-1})$$

is bounded.

For M compact, let f_t be an isotopy from Id to f such that $K = \bigcup_{t \in I} f_t(\{p\}) \neq M$. Fix a neighbourhood U of K and $x \notin U$. Then from Theorem 2.3 there is an isotopy g_t in $\mathcal{H}(M)$ such that $g_t = f_t$ on K and $\operatorname{supp}(g_t) \subset U$. Note that $g_t^{-1} f_t(p) = p$ for every t.

Now take $\varphi \in \mathcal{H}(M)$ such that $\varphi(x) = p$ and $\varphi g \varphi^{-1} \in \mathcal{H}(M, p)$ where $g = g_1$. Hence we get

$$\nu(f) \leq \nu(g) + \nu(g^{-1} f) = \nu|_{\mathcal{H}(M,p)}(\varphi g \varphi^{-1}) + \nu|_{\mathcal{H}(M,p)}(g^{-1} f)$$

which is bounded for every conjugation-invariant norm ν on $\mathcal{H}(M)$. □

Corollary 5.2. *If* fragd_M *is bounded then* $\mathcal{H}(M, p)$ *is uniformly perfect and* $\operatorname{cld}_{\mathcal{H}(M,p)} \leq 2 \operatorname{fragd}_M$.

Proof. Let $f \in \mathcal{H}(M, p)$. From Corollary 2.2 we may write $f = f_1 \ldots f_k$ where $\operatorname{supp}(f_i) \subset B_i$ and $f_i \in \mathcal{H}(M, p)$ for $i = 1, \ldots, k$. Here B_i is a ball or half-ball for each i and we may assume that $k \leq \operatorname{fragd}_M$.

Now fix i. If $p \in \operatorname{supp}(f_i)$ then from Corollary 3.7 we have $\operatorname{cl}_{\mathcal{H}(M,p)}(f_i) \leq 2$. If $p \notin \operatorname{supp}(f_i)$ choose an open set U_i of M with $U_i \cap B_i = \emptyset$ and $p \notin U_i$. There exists a homeomorphism $\varphi_i \in \mathcal{H}(M, p)$ such that $\overline{\varphi_i(B_i \cup U_i)} \subset U_i$. Then Proposition 3.3 implies that φ_i m-displaces $\mathcal{H}_{B_i}(M, p)$ for every $m \geq 1$ and from Theorem 3.2 we get $\operatorname{cl}_{\mathcal{H}(M,p)}(f_i) \leq 2$. Hence

$$\operatorname{cl}_{\mathcal{H}(M,p)}(f) \leq \operatorname{cl}_{\mathcal{H}(M,p)}(f_1) + \cdots + \operatorname{cl}_{\mathcal{H}(M,p)}(f_k) \leq 2 \operatorname{fragd}_M$$

as required. □

6. Final remarks

1. In view of the proofs of Theorems 1.2 and 1.3 we have

Corollary 6.1. *The group* $\mathcal{H}([0,1])$ *is perfect and bounded.*

Note that $\mathcal{H}([0,1])$ coincides with the group of all orientation-preserving homeomorphisms of $[0,1]$.

2. Let $0 < s \leq r \leq \infty$ and let $\mathcal{D}^r_s(\mathbb{R}^n, 0)$ be the subgroup of all elements of $\mathcal{H}(\mathbb{R}^n, 0)$ of class C^r that are s-tangent to the identity at 0. It is easily seen that $\mathcal{D}^r_s(\mathbb{R}^n, 0)$ is not perfect if $s < r$. Indeed, for any diffeomorphisms $f, g \in \mathcal{D}^r_s(\mathbb{R}^n, 0)$ we have

$$D^{s+1}(fg)(0) = D^{s+1}f(0) + D^{s+1}g(0), \quad D^{s+1}f^{-1}(0) = -D^{s+1}f(0).$$

Therefore if we choose $h \in \mathcal{D}^r_s(\mathbb{R}^n, 0)$ such that $D^{s+1}h(0) \neq 0$, the above equalities yield that h cannot be in the commutator subgroup.

On the other hand, it is likely that $\mathcal{D}^r_r(\mathbb{R}^n, 0)$ is perfect for $r = 1, \dots, \infty$. See Sergeraert [24], Masson [15] and Tsuboi [28].

3. Haller and Teichmann introduced in [12] the concept of local smooth perfectness of diffeomorphism groups. They proved that essentially the groups $\mathcal{D}^\infty(M)$ are locally smoothly perfect for all boundaryless manifolds M different than $\mathbb{R}$. It is an interesting problem whether the group $\mathcal{D}^\infty_\infty(M)$ is locally smoothly perfect.

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References

[1] K. Abe and K. Fukui, Commutators of C^∞-diffeomorphisms preserving a submanifold, *J. Math. Soc. Japan* **61** (2009), 427–436.

[2] A. Banyaga, Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique, *Comment. Math. Helv.* **53** (1978), 174–227.

[3] A. Banyaga, The structure of classical diffeomorphism groups, Mathematics and its Applications, Vol. 400, *Kluwer Academic Publishers Group, Dordrecht*, 1997.

[4] D. Burago, S. Ivanov and L. Polterovich, Conjugation invariant norms on groups of geometric origin, Groups of Diffeomorphisms, Vol. 52, Adv. Stud. Pure Math., *Math. Soc. Japan, Tokyo*, 2008, 221–250.

[5] R. D. Edwards and R. C. Kirby, Deformations of spaces of imbeddings, *Ann. Math.* (2) **93** (1971), 63–88.

[6] D. B. A. Epstein, The simplicity of certain groups of homeomorphisms, *Compositio Math.* **22** (1970), 165–173.

[7] G. M. Fisher, On the group of all homeomorphisms of a manifold, *Trans. Amer. Math. Soc.* **97** (1960), 193–212.

[8] K. Fukui, Homologies of the group $\mathrm{Diff}^\infty(\mathbb{R}^n, 0)$ and its subgroups, *J. Math. Kyoto Univ.* **20** (1980), 475–487.

[9] K. Fukui, Commutators of foliation preserving homeomorphisms for certain compact foliations, *Publ. RIMS, Kyoto Univ.* **34-1** (1998), 65–73.

[10] K. Fukui and H. Imanishi, On commutators of foliation preserving homeomorphisms, *J. Math. Soc. Japan* **51** (1999), 227–236.

[11] S. Haller and T. Rybicki, On the group of diffeomorphisms preserving a locally conformal symplectic structure, *Ann. Global Anal. and Geom.* **17** (1999), 475–502.

[12] S. Haller and J. Teichmann, Smooth perfectness through decomposition of diffeomorphisms into fiber preserving ones, *Annals of Global Analysis and Geometry* **23** (2003), 53–63.

[13] A. Kowalik and T. Rybicki, On the homeomorphism groups of manifolds and their universal coverings, *Cent. Eur. J. Math.* **9** (2011), 1217–1231.

[14] J. Lech and T. Rybicki, Groups of $C^{r,s}$-diffeomorphisms related to a foliation, Geometry and Topology of Manifolds, Vol. 76, Banach Center Publications, 2007, 437–450.

[15] A. Masson, Sur la perfection du groupe des difféomorphismes d'une variété à bord infinitement tangents à l'identité sur le bord, *C. R. Acad. Sci. Paris Série A* **285** (1977), 837–839.

[16] J. N. Mather, The vanishing of the homology of certain groups of homeomorphisms, *Topology* **10** (1971), 297–298.

[17] J. N. Mather, Commutators of diffeomorphisms, *Comment. Math. Helv.* **49** (1974), 512–528.

[18] J. N. Mather, Commutators of diffeomorphisms, II, *Comment. Math. Helv.* **50** (1975), 33–40.

[19] J. N. Mather, A curious remark concerning the geometric transfer map, *Comment. Math. Helv.* **59** (1984), 86–110.

[20] I. Michalik and T. Rybicki, On the structure of the commutator subgroup of certain homeomorphism groups, *Topology and its Applications* **158** (2011), 1314–1324.

[21] T. Rybicki, Commutators of diffeomorphisms of a manifold with boundary, *Ann. Pol. Math.* **68** (1998), 199–210.

[22] T. Rybicki, On commutators of equivariant homeomorphisms, *Topology and its Applications* **154** (2007), 1561–1564.

[23] T. Rybicki, Commutators of contactomorphisms, *Adv. Math.* **225** (2010), 3291–3326.

[24] F. Sergeraert, Feuilletages et difféomorphismes infinitement tangents à l'identité, *Invent. Math.* **39** (1977), 253–275.

[25] L. C. Siebenmann, Deformations of homeomorphisms on stratified sets, *Comment. Math. Helv.* **47** (1972), 123–163.

[26] W. Thurston, Foliations and groups of diffeomorphisms, *Bull. Amer. Math. Soc.* **80** (1974), 304–307.

[27] T. Tsuboi, On the homology of classifying spaces for foliated products, Foliations (Tokyo, 1983), *Adv. Stud. Pure Math.* **5** (1985), 37–120.

[28] T. Tsuboi, On the perfectness of groups of diffeomorphisms of the interval tangent to the identity at the endpoints, Foliations: geometry and dynamics, Warsaw 2000, (P. Walczak et al., eds.), *World Scientific*, *Singapore*, 2002, 421–440.

[29] T. Tsuboi, On the uniform perfectness of diffeomorphism groups, Groups of diffeomorphisms, *Adv. Stud. Pure Math.* **52** (2008), 505–524.

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