Publ. Math. Debrecen
83/3 (2013), 449–464
DOI: 10.5486/PMD.2013.5587

A note on the Apostol–Bernoulli and Apostol–Euler polynomials

By MIN-SOO KIM (Gyeongsangnam-do) and SU HU (Daejeon)

Abstract. Let $\alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. In this paper, we show provide several relationships between the generalized Apostol–Bernoulli polynomials $B_n^{(\alpha)}(x; \lambda)$ and the generalized Apostol–Euler polynomials $E_n^{(\alpha)}(x; \lambda)$ which involve both the main results of Luo–Srivastava in [Q.-M. Luo and H. M. Srivastava, Some relationships between the Apostol–Bernoulli and Apostol–Euler polynomials, Comput. Math. Appl. **51** (2006), 631–642] and the main results of Srivastava–Pintér in [H. M. Srivastava and Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, Appl. Math. Lett. **17** (4) (2004), 375–380] in the case of $\alpha \in \mathbb{N}_0$.

1. Introduction

The Bernoulli polynomials $B_m(x)$ and Euler polynomials $E_m(x)$ are defined by the following exponential generating functions:

$$\left(\frac{t}{e^t - 1}\right) e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}, \qquad \left(\frac{2}{e^t + 1}\right) e^{xt} = \sum_{m=0}^{\infty} E_m(x) \frac{t^m}{m!}, \tag{1.1}$$

respectively.

The following relationship between the Bernoulli and Euler polynomials is well-known:

$$B_n(x) = \sum_{\substack{k=0 \\ (k \neq 1)}}^{n} \binom{n}{k} B_k E_{n-k}(x) \quad (n \in \mathbb{N}_0), \tag{1.2}$$

Mathematics Subject Classification: Primary: 11M35; Secondary: 11B68.
Key words and phrases: convolution identities, generalized Apostol–Bernoulli numbers and polynomials, generalized Apostol–Euler numbers and polynomials.

(see CHEON's work in [5, p. 368, Theorem 3]).

For a real and complex parameter α, the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$, each of degree n in x as well as in α, are defined by the following generating functions (see Section 2.8 of [11]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt}=\sum_{m=0}^{\infty} B_m^{(\alpha)}(x)\frac{t^m}{m!}, \qquad \left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt}=\sum_{m=0}^{\infty} E_m^{(\alpha)}(x)\frac{t^m}{m!}, \tag{1.3}$$

respectively.

Clearly, we have

$$B_n^{(1)}(x)=B_n(x) \quad \text{and} \quad E_n^{(1)}(x)=E_n(x). \tag{1.4}$$

In [33], SRIVASTAVA and PINTÉR got the following relationships between the generalized Bernoulli polynomials and the classical Euler polynomials which generalized (1.2):

Theorem 1.1 (SRIVASTAVA and PINTÉR, [33, Theorem 1 and 2])**.** *The following relationships:*

$$B_n^{(\alpha)}(x+y)=\sum_{i=0}^{n}\binom{n}{i}\left(B_i^{(\alpha)}(x)+\frac{i}{2}B_{i-1}^{(\alpha-1)}(x)\right)E_{n-i}(y),$$

$$E_n^{(\alpha)}(x+y)=\sum_{i=0}^{n+1}\frac{2}{i+1}\binom{n}{i}\left(E_{i+1}^{(\alpha-1)}(x)-E_{i+1}^{(\alpha)}(x)\right)B_{n-i}(y) \quad (\alpha\in\mathbb{C}; n\in\mathbb{N}_0)$$

hold true between the generalized Bernoulli (Euler) polynomials and the classical Euler (Bernoulli) polynomials.

In this paper, we show that similar relationships also exist between the generalized Apostol–Bernoulli polynomials and the generalized Apostol–Euler polynomials. Since for $\alpha\in\mathbb{N}_0$, the generalized Bernoulli polynomials and the generalized Euler polynomials are special cases of the generalized Apostol–Bernoulli polynomials and the generalized Apostol–Euler polynomials, respectively. Thus we generalize the above Srivastava and Pintér's theorem in the case of $\alpha\in\mathbb{N}_0$.

Our paper is organized as follows.

In Section 2, we will recall the definitions, some background and progresses related to Apostol-type polynomials. In Section 3, we shall apply the umbral equivalence of the generating functions to get several recurrence relations of the generalized Apostol–Bernoulli polynomials and the generalized Apostol–Euler polynomials. In Section 4, we will prove our main results which involve both the main results of LUO–SRIVASTAVA in [22, Theorem 1 and 2] (see Corollary 4.2 below) and the main results of SRIVASTAVA–PINTÉR in [33, p. 379] in the case of $\alpha\in\mathbb{N}_0$ (see Corollary 4.3 below).

2. Apostol–Bernoulli polynomials

The Apostol–Bernoulli polynomials $B_m(x,\lambda)$ are natural generalizations of Bernoulli polynomials, they were first introduced by Apostol [1] in order to study the Lipschitz–Lerch zeta functions. Their definitions are as follows,

$$\left(\frac{t}{\lambda e^t - 1}\right) e^{xt} = \sum_{m=0}^{\infty} B_m(x,\lambda)\frac{t^m}{m!}, \tag{2.1}$$

where $|t| \le 2\pi$ when $\lambda = 1$; $|t| \le |\log\lambda|$ when $\lambda \ne 1$ (see [14], [21]).

In particular, $B_m(\lambda) = B_m(0,\lambda)$ are the Apostol–Bernoulli numbers. Letting $\lambda = 1$ in (2.1), we obtained the classical Bernoulli polynomials $B_m(x)$ and Bernoulli numbers B_m, respectively.

In [21, p. 290–302], Luo and Srivastava generalized the definitions of Apostol–Bernoulli polynomials to the higher order (also called the generalized Apostol–Bernoulli polynomials) case as follows:

Definition 2.1. For $\alpha \in \mathbb{N}_0$, the Apostol–Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order α in the variable x are defined by means of the following generating function

$$e^{B^{(\alpha)}(x;\lambda)z} \equiv \sum_{n=0}^{\infty}\frac{(B^{(\alpha)}(x;\lambda)z)^n}{n!} \equiv \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} = \left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} e^{xz},$$

where $|z| < 2\pi$ when $\lambda = 1$; $|z| < |\log\lambda|$ when $\lambda \ne 1$, the symbol $\equiv$ is used to denote symbolic or umbral equivalences. In particular, $B_n^{(\alpha)}(\lambda) = B_n^{(\alpha)}(0,\lambda)$ denote Apostol–Bernoulli numbers of order α. Clearly, we have

$$B_n^{(1)}(x;1) = B_n(x) \quad \text{and} \quad B_n^{(1)}(0;1) = B_n \tag{2.2}$$

in terms of the classical Bernoulli polynomials $B_n(x)$ and the classical Bernoulli numbers B_n.

In [14], Luo also introduced the concept of higher order Apostol–Euler polynomials as follows:

Definition 2.2. For $\alpha \in \mathbb{N}_0$, the Apostol–Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order α in the variable x are defined by means of the following generating function

$$e^{E^{(\alpha)}(x;\lambda)z} \equiv \sum_{n=0}^{\infty}\frac{(E^{(\alpha)}(x;\lambda)z)^n}{n!} \equiv \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} = \left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} e^{xz},$$

where $\lambda \ne -1, |z| < |\log(-\lambda)|$, the symbol $\equiv$ is used to denote the symbolic or umbral equivalences.

In particular, $E_n^{(\alpha)}(\lambda) = E_n^{(\alpha)}(0,\lambda)$ denotes the Apostol–Euler numbers of order α. Clearly, we have

$$E_n^{(1)}(x;1) = E_n(x) \quad \text{and} \quad 2^n E_n^{(1)}\left(\frac{1}{2};1\right) = 2^n E_n\left(\frac{1}{2}\right) = E_n \tag{2.3}$$

in terms of the classical Euler polynomials $E_n(x)$ and the classical Euler numbers E_n.

By Definition 2.1, we have

$$B_i^{(\ell)}(y;\lambda) = 0 \quad (0 \leq i \leq \ell - 1) \tag{2.4}$$

where $\ell \in \mathbb{N}$.

From the generating functions in Definitions 2.1 and 2.2, it is easily seen that

$$B_n^{(0)}(x) = E_n^{(0)}(x) = x^n \quad (n \in \mathbb{N}_0). \tag{2.5}$$

The properties of Apostol-type polynomials have been studied in detail by many authors.

In [36], Srivastava introduced and investigated some of the principal generalizations and unifications of Bernoulli, Euler and Genocchi polynomials and their corresponding numbers by means of suitable generating functions. He also presented several interesting properties of these general polynomial systems including some explicit series representations in terms of the Hurwitz (or generalized) zeta function and the familiar Gauss hypergeometric function. By introducing the λ-Stirling numbers of the second kind, he derived several properties and formulas and considered some of their interesting applications to the family of the Apostol type polynomials. He also gave a brief expository and historical account of the various basic (or q-) extensions of the classical Bernoulli polynomials and numbers, the classical Euler polynomials and numbers, the classical Genocchi polynomials and numbers, and also of their generalizations such as the above-mentioned families of the Apostol-type polynomials and numbers. Finally, he also indicated relevant connections of the definitions and results presented in his survey with those in earlier as well as forthcoming investigations.

In [37], the revised, enlarged and updated version of the earlier book by Srivastava and Choi entitled "Series Associated with the Zeta and Related Functions" (Kluwer Academic Publishers, Dordrecht, Boston and London, 2001), the authors gave a systematic collection of various families of series associated with the Riemann and Hurwitz Zeta functions, as well as with many other higher transcendental functions, which are closely related to these functions. In this book, the historical account, vast literatures and many fundamental properties for the Apostol-type polynomials and numbers have also been introduced.

In [4], Choi, Jang and Srivastava presented an explicit representation of the generalized Bernoulli polynomials in terms of a generalization for the Hurwitz–Lerch zeta function. Boyadzhiev [3] found some relationships between the Apostol–Bernoulli polynomials, the classical Eulerian polynomials and the derivative polynomials for the cotangent functions. We [10] obtained the sums of products identity for the Apostol–Bernoulli numbers which is an analogue of the classical sums of products identity for Bernoulli numbers dating back to Euler. Luo [12], Bayad [2], Navas, Francisco and Varona [29] investigated Fourier expansions for the Apostol–Bernoulli and Apostol–Euler polynomials. Luo [13] got many formulas for the Apostol–Bernoulli polynomials by using the Gaussian hypergeometric functions. Luo [16] investigated multiplication formulas for Apostol-type polynomials and introduced λ-multiple alternating sums, which are evaluated by Apostol-type polynomials, in particular, he derived some explicit recursive formulas and deduced some interesting special cases that involve the classical Raabe formulas and some earlier results of Carlitz and Howard. Luo and Srivastava [24] systematically studied the Apostol–Genocchi polynomials of higher order, in particular, they established several elementary properties, provided some explicit relationships with the Apostol–Bernoulli polynomials and Apostol–Euler polynomials, and they also derived various explicit series representations in terms of the Gaussian hypergeometric function and the Hurwitz (or generalized) zeta function. Luo in [17], Luo and Zhou in [25] investigated the q-Bernoulli and Euler polynomials, q-Genocchi polynomials, respectively. Luo and Srivastavan [23] obtained a q-analogue of the Srivastava–Pintér addition theorem (Theorem 1.1 above). Luo [18] introduced and investigated the λ-Stirling numbers of the second kind, in particular, he gave an explicit relationship between the generalized Apostol–Bernoulli and Apostol–Euler polynomials in terms of the λ-Stirling numbers of the second kind. Luo [19] extended the definition of the Genocchi polynomials and investigated their Fourier expansions and integral representations, he obtained their formulas at rational arguments in terms of Hurwitz zeta function and showed an explicit relationship with Gaussian hypergeometric functions. Luo [20] gave some explicit relationships between the Apostol–Euler polynomials and the generalized Hurwitz–Lerch zeta function, he also obtained some series representations of the Apostol–Euler polynomials of higher order in terms of the generalized Hurwitz–Lerch zeta function. Luo and Srivastava [26] proved several symmetry identities for the generalized Apostol-type polynomials by using their generating functions. Srivastava, Garg, and Choudhary

[34], [35] introduced and investigated a generalization of the Bernoulli and Euler polynomials by means of a suitable generating function, in particular, they gave explicit series representations for these general polynomials in terms of a certain generalized Hurwitz–Lerch zeta function and the Gaussian hypergeometric function. Tremblay, Gaboury and Fugere [39] introduced and investigated a new class of generalized Apostol–Bernoulli polynomials based on a definition given by Natalini and Bernardini in [27] for the generalized Bernoulli polynomials, in particular, they obtained a generalization of the Srivastava–Pintér addition theorem (Theorem 1.1 above). Özarslan [30] presented and studied a unified family of polynomials which involves the Apostol–Bernoulli, Euler and Genocchi polynomials. Garg, Jain and Srivastava [8] derived an explicit representation of the generalized Apostol–Bernoulli polynomials of higher order in terms of a generalization of the Hurwitz–Lerch Zeta function and established a functional relationship between the generalized Apostol–Bernoulli polynomials of rational arguments and the Hurwitz (or generalized) Zeta function, in particular, their results provided extensions of those given earlier by Apostol in [1] and Srivastava in [32]. Srivastava, Kurt and Simsek [38] constructed the generating functions for several families of Genocchi type polynomials, they defined a function which interpolates these polynomials at negative integers by applying the derivative operator to these generating functions, they proved a multiplication theorem for these polynomials, they also proved several other identities and provided many applications associated with these and related polynomials and their interpolation functions.

Recently, Luo and Srivastava proved the following relationships among the generalized Apostol–Bernoulli and the generalized Apostol–Euler polynomials (see [14, Proposition 3 and Proposition 6]) and [21, (56)]:

$$B_n^{(\alpha+\beta)}(x+y;\lambda)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(x;\lambda)B_{n-k}^{(\beta)}(y;\lambda), \tag{2.6}$$

$$E_n^{(\alpha+\beta)}(x+y;\lambda)=\sum_{k=0}^{n}\binom{n}{k}E_k^{(\alpha)}(x;\lambda)E_{n-k}^{(\beta)}(y;\lambda), \tag{2.7}$$

$$\lambda B_n^{(\alpha)}(x+1;\lambda)-B_n^{(\alpha)}(x;\lambda)=nB_{n-1}^{(\alpha-1)}(x;\lambda), \tag{2.8}$$

$$\lambda E_n^{(\alpha)}(x+1;\lambda)+E_n^{(\alpha)}(x;\lambda)=2E_n^{(\alpha-1)}(x;\lambda). \tag{2.9}$$

3. Recurrence relations

The well-known relations among the generalized Apostol–Bernoulli polynomials and the generalized Apostol–Euler polynomials are as follows:

$$B_n^{(\alpha+1)}(x;\lambda) = \left(1-\frac{n}{\alpha}\right) B_n^{(\alpha)}(x;\lambda) + (x-\alpha)\frac{n}{\alpha} B_{n-1}^{(\alpha)}(x;\lambda), \tag{3.1}$$

$$E_n^{(\alpha+1)}(x;\lambda) = \frac{2}{\alpha} E_{n+1}^{(\alpha)}(x;\lambda) - (x-\alpha)\frac{2}{\alpha} E_n^{(\alpha)}(x;\lambda). \tag{3.2}$$

These have been found by several authors, for references, see, e.g., [14, Proposition 8] and [40, Theorem 1.2 and Theorem 1.3.]. In this section, several new recurrence relations will be given among the generalized Apostol–Bernoulli polynomials and the generalized Apostol–Euler polynomials.

From (3.1) and (3.2), replacing x by $x+y$, and letting $\alpha = 1$, we have the following recurrences:

$$B_n^{(2)}(x+y;\lambda) = (1-n)B_n(x+y;\lambda) + n(x+y-1)B_{n-1}(x+y;\lambda), \tag{3.3}$$

$$E_n^{(2)}(x+y;\lambda) = 2E_{n+1}(x+y;\lambda) - 2(x+y-1)E_n(x+y;\lambda). \tag{3.4}$$

Letting $\alpha = \beta = 1$ in (2.6) and (2.7), we have

$$B_n^{(2)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} B_k(x;\lambda) B_{n-k}(y;\lambda), \tag{3.5}$$

$$E_n^{(2)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} E_k(x;\lambda) E_{n-k}(y;\lambda). \tag{3.6}$$

By (3.3), (3.4), (3.5), and (3.6), we have

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} B_k(x;\lambda) B_{n-k}(y;\lambda) &= (1-n)B_n(x+y;\lambda) \\ &\quad + n(x+y-1)B_{n-1}(x+y;\lambda),\end{aligned} \tag{3.7}$$

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} E_k(x;\lambda) E_{n-k}(y;\lambda) &= 2E_{n+1}(x+y;\lambda) \\ &\quad - 2(x+y-1)E_n(x+y;\lambda).\end{aligned} \tag{3.8}$$

These equations can be viewed as a λ-extensions of the following well-known results (see [6, (3.2) and (4.2)]):

$$\sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(y) = (1-n)B_n(x+y) + n(x+y-1)B_{n-1}(x+y), \tag{3.9}$$

$$\sum_{k=0}^{n}\binom{n}{k}E_k(x)E_{n-k}(y)=2E_{n+1}(x+y)-2(x+y-1)E_n(x+y). \tag{3.10}$$

Setting $x=y=0$ in (3.9), we get the following convolution recurrence:

$$\sum_{k=0}^{n}\binom{n}{k}B_kB_{n-k}=(1-n)B_n-nB_{n-1} \tag{3.11}$$

for the classical Bernoulli numbers (see [6], [9], [10]). Notice that, we can also get much more formulas from (3.10). For example, by setting $x=y=\frac{1}{2}$ in (3.10), we get a way to express $E_n(1)$ in the right hand side in terms of the Euler numbers E_n (see (2.3)).

Now we apply the umbral equivalence for the generating functions to get the following two recurrence relations on the generalized Apostol–Bernoulli polynomials and the generalized Apostol–Euler polynomials:

Lemma 3.1. *For $\alpha\in\mathbb{N}_0$ and $n\in\mathbb{N}$, the following formulas hold true*

$$\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}B_k^{(2\alpha)}(x-y;\lambda)=\frac{1}{\lambda^\alpha}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B_k^{(\alpha)}(x;\lambda)B_{n-k}^{(\alpha)}(y;\lambda^{-1}),$$

$$\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}E_k^{(2\alpha)}(x-y;\lambda)=\frac{1}{\lambda^\alpha}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}E_k^{(\alpha)}(x;\lambda)E_{n-k}^{(\alpha)}(y;\lambda^{-1}).$$

Proof. It is obvious that

$$(\lambda e^z)^\alpha\left(\frac{z}{\lambda e^z-1}\right)^{2\alpha}e^{(x-y)z}=\left(\frac{z}{\lambda e^z-1}\right)^{\alpha}e^{xz}\left(\frac{-z}{\lambda^{-1}e^{-z}-1}\right)^{\alpha}e^{-yz}.$$

Thus

$$\lambda^\alpha e^{z\alpha}e^{B^{(2\alpha)}(x-y;\lambda)z}\equiv e^{B^{(\alpha)}(x;\lambda)z}e^{B^{(\alpha)}(y;\lambda^{-1})(-z)}\equiv e^{(B^{(\alpha)}(x;\lambda)-B^{(\alpha)}(y;\lambda^{-1}))z}.$$

So writing in the non-umbral form we have

$$\lambda^\alpha(\alpha+B^{(2\alpha)}(x-y;\lambda))^n=(B^{(\alpha)}(x;\lambda)-B^{(\alpha)}(x;\lambda^{-1}))^n.$$

By Definition 2.2, we can also obtain the second formula similarly. □

We shall obtain the following relationship which also involves the well-known sums of products identities for Bernoulli numbers dating back to Euler:

Theorem 3.2. *For $n \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B_k(x;\lambda)B_{n-k}(x;\lambda^{-1}) = (1-n)B_n(\lambda).$$

Remark 3.3. Setting $\alpha = \lambda = 1$ in the above Theorem, we can obtain the following convolution recurrence relationship:

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}B_kB_{n-k} = (-1)^{n-1}(n-1)B_n,$$

equivalently, we have the well-known sums of products identities for Bernoulli numbers

$$\sum_{k=1}^{n-1}\binom{2n}{2k}B_{2k}B_{2n-2k} = -(2n+1)B_{2n}, \quad n \geq 2.$$

This has been found by many authors, including Euler (see [6], [7], [9], [31]). In [6], Dilcher remarked that: "it may be of interest to find formulas of the above type for sums of products of generalized Bernoulli numbers."

Proof of Theorem 3.2. The following recurrence relationship is well-known:

$$B_0(\lambda) = 0, \quad \lambda(1+B(\lambda))^n - B_n(\lambda) = \begin{cases} 1, & n = 1 \\ 0, & n > 1, \end{cases} \tag{3.12}$$

where we use the usual convention about replacing $(B(\lambda))^i$ by $B_i(\lambda)$ $(i \geq 0)$.

Setting $\alpha = 1$ in first part of Lemma 3.1, we get

$$\begin{aligned} \frac{1}{\lambda}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B_k(x;\lambda)B_{n-k}(y;\lambda^{-1}) &= \sum_{k=0}^{n}\binom{n}{k}B_k^{(2)}(x-y;\lambda) \\ &= \sum_{k=0}^{n}\binom{n}{k}\left\{(1-k)B_k(x-y;\lambda) + k(x-y-1)B_{k-1}(x-y;\lambda)\right\}, \end{aligned} \tag{3.13}$$

which may be considered as a dual of (3.7).

Letting $y = x$ in (3.13), and using (3.3) and (3.12), we have

$$\begin{aligned} \frac{1}{\lambda}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B_k(x;\lambda)B_{n-k}(x;\lambda^{-1}) &= \sum_{k=0}^{n}\binom{n}{k}B_k^{(2)}(\lambda) \\ &= \sum_{k=0}^{n}\binom{n}{k}((1-k)B_k(\lambda) - kB_{k-1}(\lambda)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n}\binom{n}{k}B_k(\lambda) - \sum_{k=0}^{n}\binom{n}{k}kB_k(x) - \sum_{k=0}^{n}\binom{n}{k}kB_{k-1}(\lambda) \\
&= \sum_{k=0}^{n}\binom{n}{k}B_k(\lambda) - \sum_{k=0}^{n}\left\{\binom{n}{k}k + \binom{n}{k+1}(k+1)\right\}B_k(\lambda) \\
&= \sum_{k=0}^{n}\binom{n}{k}B_k(\lambda) - n\sum_{k=0}^{n}\binom{n}{k}B_k(\lambda) \\
&= (1-n)\sum_{k=0}^{n}\binom{n}{k}B_k(\lambda) = (1-n)\frac{1}{\lambda}B_n(\lambda). \qquad (3.14)
\end{aligned}
$$

Finally, by (3.14), we get the desired result. □

We also have the following recurrence relationship for the Apostol–Euler polynomials:

Theorem 3.4. *For $n \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}E_k(x;\lambda)E_{n-k}(x;\lambda^{-1}) = -2E_{n+1}(\lambda).$$

PROOF. The proof is similar to that of Theorem 3.2. □

The following relationship has already involved the well-known relationship between the classical Bernoulli and the classical Euler polynomials (see [33, p. 376, (10) and (11)]):

Lemma 3.5 (Addition theorem)**.** *For $\alpha \in \mathbb{N}_0$, the following relationship holds true:*

$$2^n B_n^{(\alpha)}\left(\frac{x+y}{2};\lambda^2\right) = \sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(x;\lambda)E_{n-k}^{(\alpha)}(y;\lambda)$$

or, equivalently,

$$B_n^{(\alpha)}\left(x+y;\lambda^2\right) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(2x;\lambda)E_{n-k}^{(\alpha)}(2y;\lambda).$$

PROOF. It is obvious that

$$\left(\frac{2z}{\lambda^2 e^{2z}-1}\right)^{\alpha} e^{\left(\frac{x+y}{2}\right)2z} = \left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} e^{xz}\left(\frac{2}{\lambda e^z+1}\right)^{\alpha} e^{yz}.$$

Thus

$$e^{B^{(\alpha)}\left(\frac{x+y}{2};\lambda^2\right)2z} \equiv e^{B^{(\alpha)}(x;\lambda)z} e^{E^{(\alpha)}(y;\lambda)z} \equiv e^{(B^{(\alpha)}(x;\lambda)+E^{(\alpha)}(y;\lambda))z},$$

So writing in the non-umbral form we have

$$\left(2B^{(\alpha)}\left(\frac{x+y}{2};\lambda^2\right)\right)^n = (B^{(\alpha)}(x;\lambda)+E^{(\alpha)}(y;\lambda))^n,$$

which completes the proof of the lemma. □

Remark 3.6. Setting $\alpha=\lambda=1$ and $x=0$ in Lemma 3.5, we obtain the following well-known relationship between the classical Bernoulli and the classical Euler polynomials (see [33, p. 376, (10) and (11)]):

$$B_n(y) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}E_k(2y)$$

equivalently,

$$2^n B_n\left(\frac{y}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}B_k E_{n-k}(y).$$

4. Main result

Theorem 4.1. *For $\alpha\in\mathbb{N}_0$, each of the following relationships holds true:*

$$B_n^{(\alpha)}(x+y;\lambda) = \sum_{i=0}^{n}\binom{n}{i}\left(\frac{1}{2^\ell}\sum_{k=0}^{\ell}\binom{\ell}{k}\lambda^k B_i^{(\alpha)}(x+k;\lambda)\right)E_{n-i}^{(\ell)}(y;\lambda)$$

and

$$\begin{aligned}E_n^{(\alpha)}(x+y;\lambda) &= \sum_{i=0}^{n+\ell}\binom{n+\ell}{i}\frac{1}{(n+1)\cdots(n+\ell)}\\ &\quad\times\left((-1)^\ell\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\lambda^k E_i^{(\alpha)}(x+k;\lambda)\right)B_{n+\ell-i}^{(\ell)}(y;\lambda)\end{aligned}$$

for the generalized Apostol–Bernoulli polynomials and the generalized Apostol–Euler polynomials, respectively.

Proof. It is obvious that

$$\left(\frac{z}{\lambda e^z-1}\right)^\alpha e^{(x+y)z}=\left(\frac{z}{\lambda e^z-1}\right)^\alpha e^{xz}\left(\frac{2}{\lambda e^z+1}\right)^\ell e^{yz}\frac{(\lambda e^z+1)^\ell}{2^\ell}$$
$$\equiv\frac{1}{2^\ell}e^{B^{(\alpha)}(x;\lambda)z}e^{E^{(\ell)}(y;\lambda)z}\sum_{k=0}^{\ell}\binom{\ell}{k}\lambda^k e^{kz}.$$

Thus

$$e^{B^{(\alpha)}(x+y;\lambda)z}\equiv\frac{1}{2^\ell}\sum_{k=0}^{\ell}\binom{\ell}{k}\lambda^k e^{(B^{(\alpha)}(x+k;\lambda)+E^{(\ell)}(y;\lambda))z}.$$

So writing in the non-umbral form we have

$$B_n^{(\alpha)}(x+y;\lambda)=\frac{1}{2^\ell}\sum_{k=0}^{\ell}\binom{\ell}{k}\lambda^k(B^{(\alpha)}(x+k;\lambda)+E^{(\ell)}(y;\lambda))^n,$$

which proves the first result.

For the second result, we have

$$\left(\frac{2}{\lambda e^z+1}\right)^\alpha e^{(x+y)z}=\left(\frac{2}{\lambda e^z+1}\right)^\alpha e^{xz}\left(\frac{z}{\lambda e^z-1}\right)^\ell e^{yz}\frac{(\lambda e^z-1)^\ell}{z^\ell}$$
$$\equiv\frac{(-1)^\ell}{z^\ell}e^{E^{(\alpha)}(x;\lambda)z}e^{B^{(\ell)}(y;\lambda)z}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\lambda^k e^{kz}.$$

Thus

$$e^{E^{(\alpha)}(x+y;\lambda)z}\equiv\frac{(-1)^\ell}{z^\ell}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\lambda^k e^{(E^{(\alpha)}(x+k;\lambda)+B^{(\ell)}(y;\lambda))z}.$$

So writing in the non-umbral form we have

$$\sum_{n=0}^{\infty}E_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}=\frac{(-1)^\ell}{z^\ell}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\lambda^k\sum_{n=0}^{\infty}(E^{(\alpha)}(x+k;\lambda)+B^{(\ell)}(y;\lambda))^n\frac{z^n}{n!}.$$

By (2.4), if $n<\ell$, we have

$$(E^{(\alpha)}(x+k;\lambda)+B^{(\ell)}(y;\lambda))^n=\sum_{i=0}^{n}\binom{n}{i}E_{n-i}^{(\alpha)}(x+k;\lambda)B_i^{(\ell)}(y;\lambda)=0.$$

Thus

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} = \frac{(-1)^\ell}{z^\ell}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\lambda^k\sum_{n=0}^{\infty}(E^{(\alpha)}(x+k;\lambda)+B^{(\ell)}(y;\lambda))^n\frac{z^n}{n!}$$
$$= \frac{(-1)^\ell}{z^\ell}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\lambda^k\sum_{n=0}^{\infty}(E^{(\alpha)}(x+k;\lambda)+B^{(\ell)}(y;\lambda))^{n+\ell}\frac{z^n}{(n+\ell)!}.$$

Comparing the coefficients in the both sides of the above identity, we obtain the second result. □

Setting $\ell = 1$ in the identities of Theorem 4.1, from (2.8) and (2.9), we have

$$\frac{1}{2}\sum_{k=0}^{1}\binom{1}{k}\lambda^k B_i^{(\alpha)}(x+k;\lambda) = \frac{1}{2}\left(B_i^{(\alpha)}(x;\lambda)+\lambda B_i^{(\alpha)}(x+1;\lambda)\right)$$
$$= B_i^{(\alpha)}(x;\lambda)+\frac{i}{2}B_{i-1}^{(\alpha-1)}(x;\lambda) \tag{4.1}$$

and

$$\sum_{k=0}^{1}\binom{1}{k}(-1)^k\lambda^k E_i^{(\alpha)}(x+k;\lambda) = E_i^{(\alpha)}(x;\lambda)-\lambda E_i^{(\alpha)}(x+1;\lambda)$$
$$= 2\left(E_i^{(\alpha)}(x;\lambda)-E_i^{(\alpha-1)}(x;\lambda)\right), \tag{4.2}$$

respectively.

Setting $\ell = 1$ in Theorem 4.1, from (4.1), and (4.2), we arrive at the following well-known Luo–Srivastava's results [22, Theorem 1, p. 379; Theorem 2, p. 380]).

Corollary 4.2 (Luo and Srivastava, [22, Theorem 1 and 2])**.** *For* $\alpha, n \in \mathbb{N}_0$,

$$B_n^{(\alpha)}(x+y;\lambda) = \sum_{i=0}^{n}\binom{n}{i}\left(B_i^{(\alpha)}(x;\lambda)+\frac{i}{2}B_{i-1}^{(\alpha-1)}(x;\lambda)\right)E_{n-i}(y;\lambda),$$
$$E_n^{(\alpha)}(x+y;\lambda) = \sum_{i=0}^{n+1}\frac{2}{n+1}\binom{n+1}{i}\left(E_i^{(\alpha-1)}(x;\lambda)-E_i^{(\alpha)}(x;\lambda)\right)B_{n-i+1}(y;\lambda).$$

Letting $\lambda = 1$ in Corollary 4.2, we obtain the following well-known Srivastava–Pintér's results in the case of $\alpha \in \mathbb{N}_0$:

Corollary 4.3 (Srivastava and Pintér, [33, Theorem 1 and 2]). *For* $\alpha, n \in \mathbb{N}_0$,

$$B_n^{(\alpha)}(x+y) = \sum_{i=0}^{n} \binom{n}{i} \left(B_i^{(\alpha)}(x) + \frac{i}{2} B_{i-1}^{(\alpha-1)}(x) \right) E_{n-i}(y),$$
$$E_n^{(\alpha)}(x+y) = \sum_{i=0}^{n+1} \frac{2}{i+1} \binom{n}{i} \left(E_{i+1}^{(\alpha-1)}(x) - E_{i+1}^{(\alpha)}(x) \right) B_{n-i}(y).$$

Setting $\alpha = 1$ in the first assertion of Corollary 1.1, letting $y = 0$ and make use of (2.5), we also obtain the main relationship in Cheon's work (cf., [5, p. 368, Theorem 3]):

$$B_n(x) = \sum_{\substack{k=0 \\ (k \neq 1)}}^{n} \binom{n}{k} B_k E_{n-k}(x) \quad (n \in \mathbb{N}_0).$$

Acknowledgment. The authors are enormously grateful to the two anonymous referees whose comments and suggestions lead to a large improvement of the paper.

This work was supported by the Kyungnam University Foundation Grant, 2013.

References

[1] T. M. Apostol, On the Lerch zeta function, *Pacific J. Math.* **1** (1951), 161–167.

[2] A. Bayad, Fourier expansions for Apostol–Bernoulli, Apostol–Euler and Apostol-Genocchi polynomials, *Math. Comp.* **80** (2011), 2219–2221.

[3] K. N. Boyadzhiev, Apostol–Bernoulli functions, derivative polynomials and Eulerian polynomials, *Adv. Appl. Discrete Math.* **12** (2008), 109–122.

[4] J. Choi, D. S. Jang and H. M. Srivastava, A generalization of the Hurwitz–Lerch zeta function, *Integral Transforms Spec. Funct.* **19**, no. 1–2 (2008), 65–79.

[5] G.-S. Cheon, A note on the Bernoulli and Euler polynomials, *Appl. Math. Lett.* **16** (2003), 365–368.

[6] K. Dilcher, Sums of products of Bernoulli numbers, *J. Number Theory* **60** (1996), 23–41.

[7] M. Eie, A note on Bernoulli numbers and Shintani generalized Bernoulli polynomials, *Trans. Amer. Math. Soc.* **348** (1996), 1117–1136.

[8] M. Garg, K. Jain and H. M. Srivastava, Some relationships between the generalized Apostol–Bernoulli polynomials and Hurwitz–Lerch Zeta functions, *Integral Transforms Spec. Funct.* **17** (2006), 803–815.

[9] M.-S. Kim, A note on sums of products of Bernoulli numbers, *Appl. Math. Lett.* **24** (2011), 55–61.

[10] M.-S. Kim and S. Hu, Sums of products of Apostol–Bernoulli numbers, *Ramanujan J.* **28** (2012), 113–123.

[11] Y. L. Luke, The Special Functions and their Applications, Volume I, *Academic Press, New York*, 1969.

[12] Q.-M. Luo, Fourier expansions and integral representations for the Apostol–Bernoulli and Apostol–Euler polynomials, *Math. Comp.* **78** (2009), 2193–2208.

[13] Q.-M. Luo, On the Apostol–Bernoulli polynomials, *Cent. Eur. J. Math.* **2**(4) (2004), 509–515.

[14] Q.-M. Luo, Apostol–Euler polynomials of higher order and Gaussian hypergeometric functions, *Taiwanese J. Math.* **10**(4) (2006), 917–925.

[15] Q.-M. Luo, The multiplication formulas for the Apostol–Bernoulli and Apostol–Euler polynomials of higher order, *Integral Transforms Spec. Funct.* **20**, no. 5 (2009), 377–391.

[16] Q.-M. Luo, The multiplication formulas for the Apostol-type polynomials and the multiple alternating sum, *Math. Notes* **91** (2012), 54–65.

[17] Q.-M. Luo, Some results for the q-Bernoulli and q-Euler polynomials, *J. Math. Anal. Appl.* **363** (2010), 7–18.

[18] Q.-M. Luo, An explicit relationship between the generalized Apostol–Bernoulli and Apostol–Euler polynomials associated with λ-Stirling numbers of the second kind, *Houston J. Math.* **36** (2010), 1159–1171.

[19] Q.-M. Luo, Extension for the Genocchi polynomials and its Fourier expansions and integral representations, *Osaka J. Math.* **48** (2011), 291–309.

[20] Q.-M. Luo, Some formulas for Apostol–Euler polynomials associated with Hurwitz zeta function at rational arguments, *Appl. Anal. Discrete Math.* **3** (2009), 336–346.

[21] Q.-M. Luo and H. M. Srivastava, Some generalizations of the Apostol–Bernoulli and Apostol–Euler polynomials, *J. Math. Anal. Appl.* **308**(1) (2005), 290–302.

[22] Q.-M. Luo and H. M. Srivastava, Some relationships between the Apostol–Bernoulli and Apostol–Euler polynomials, *Comput. Math. Appl.* **51** (2006), 631–642.

[23] Q.-M. Luo and H. M. Srivastava, q-Extensions of some relationships between the Bernoulli and Euler polynomials, *Taiwanese J. Math.* **15** (2011), 241–257.

[24] Q.-M. Luo and H. M. Srivastava, Some generalizations of the Apostol–Genocchi polynomials and the Stirling numbers of the second kind, *Appl. Math. Comput.* **217** (2011), 5702–5728.

[25] Q.-M. Luo and Y. Zhou, Extension of the Genocchi polynomials and its q-analogue, *Utilitas Math.* **85** (2011), 281–297.

[26] D.-Q. Lu and H. M. Srivastava, Some series identities involving the generalized Apostol type and related polynomials, *Comput. Math. Appl.* **62** (2011), 3591– 2602.

[27] P. Natalini and A. Bernardini, A generalization of the Bernoulli polynomials, *J. Appl. Math.* **3** (2003), 155–163.

[28] N. Nielson, Traité Élémentaire des Nombers de Bernoulli, *Gauthier-Villars, 1923.*

[29] L. M. Navas, F. J. Ruiz and J. L. Varona, Asymptotic estimates for Apostol–Bernoulli and Apostol–Euler polynomials, *Math. Comp.* **81** (2012), 1707–1722.

[30] M. Ali Özarslan, Unified Apostol–Bernoulli, Euler and Genocchi polynomials, *Comput. Math. Appl.* **62** (2011), 2452–2462.

[31] A. Petojević, New sums of products of Bernoulli numbers, *Integral Transform Spec. Funct.* **19** (2008), 105–114.

[32] H. M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, *Math. Proc. Cambridge Philos. Soc.* **129** (2000), 77–84.

[33] H. M. Srivastava and Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, *Appl. Math. Lett.* **17**(4) (2004), 375–380.

[34] H. M. Srivastava, M. Garg and S. Choudhary, A new generalization of the Bernoulli and related polynomials, *Russian J. Math. Phys.* **17** (2010), 251–261.

[35] H. M. Srivastava, M. Garg and S. Choudhary, Some new families of generalized Euler and Genocchi polynomials, *Taiwanese J. Math.* **15** (2011), 283–305.

[36] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inf. Sci.* **5** (2011), 390–444.

[37] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, *Elsevier Science Publishers, Amsterdam, London and New York*, 2012.

[38] H. M. Srivastava, B. Kurt and Y. Simsek, Some families of Genocchi type polynomials and their interpolation functions, *Integral Transforms Spec. Funct.* **23** (2012), 919–938, see also Corrigendum, *Integral Transforms Spec. Funct.* **23** (2012), 939–940.

[39] R. Tremblay, S. Gaboury and B.-J. Fugere, A new class of generalized Apostol–Bernoulli polynomials and some analogues of the Srivastava–Pinter addition theorem, *Appl. Math. Lett.* **24** (2011), 1888–1893.

[40] W. Wang, C. Jia and T. Wang, Some results on the Apostol–Bernoulli and Apostol–Euler polynomials, *Comput. Math. Appl.* **55**, no. 6 (2008), 1322–1332.

[41] K. S. Williams and N. Y. Zhang, Special values of the Lerch zeta function and the evaluation of certain integrals, *Proc. Amer. Math. Soc.* **119** (1993), 35–49.

MIN-SOO KIM
DIVISION OF CULTURAL EDUCATION
KYUNGNAM UNIVERSITY
7(WORYEONG-DONG) KYUNGNAMDAEHAK-RO
MASANHAPPO-GU, CHANGWON-SI
GYEONGSANGNAM-DO 631-701
SOUTH KOREA

E-mail: mskim@kyungnam.ac.kr

SU HU
DEPARTMENT OF MATHEMATICAL SCIENCES
KOREA ADVANCED INSTITUTE OF SCIENCE
AND TECHNOLOGY (KAIST)
373-1 GUSEONG-DONG
YUSEONG-GU, DAEJEON 305-701
SOUTH KOREA

CURRENT ADDRESS:

DEPARTMENT OF MATHEMATICS
AND STATISTICS
MCGILL UNIVERSITY
805 SHERBROOKE ST. WEST
MONTRÉAL, QUÉBEC
CANADA H3A 2K6

E-mail: suhu1982@gmail.com, hu@math.mcgill.ca

(Received June 28, 2012; revised March 25, 2013)