Publl. Math. Debrecen
83/3 (2013), 465–480
DOI: 10.5486/PMD.2013.5592

Characterization of the convergence of weighted averages of double sequences of numbers and functions in two variables

By FERENC MÓRICZ (Szeged) and ULRICH STADTMÜLLER (Ulm)

Dedicated to Professor Zoltán Daróczy on his 75th birthday

Abstract. Let $(p_j \geq 0)$ and $(q_j \geq 0)$ be two sequences of weights such that

$$P_m := \sum_{k=1}^{m} p_k \to \infty \text{ as } m \to \infty \quad \text{and} \quad Q_n := \sum_{k=1}^{n} q_k \to \infty \text{ as } n \to \infty.$$

Weighted averages of a complex-valued double sequence $(c_{j,k} : (j,k) \in \mathbb{N}^2)$ are defined by

$$\sigma_{m,n} := \frac{1}{P_m Q_n} \sum_{j=1}^{m} \sum_{k=1}^{n} p_j \, q_k \, c_{j,k}$$

for large enough m, n where $P_m \cdot Q_n > 0$.

Furthermore, let $p, q : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ be two weight functions such that

$$P(s) := \int_0^s p(u)\, du \to \infty \text{ as } s \to \infty \quad \text{and} \quad Q(t) := \int_0^t q(u)\, du \to \infty \text{ as } t \to \infty.$$

The weighted averages of a measurable function $f : \bar{\mathbb{R}}_+^2 \to \mathbb{C}$ are defined when the product $f\,p\,q$ is locally integrable on $\bar{\mathbb{R}}_+^2$ as follows

$$\sigma(s,t) := \frac{1}{P(s)\,Q(t)} \int_0^s \int_0^t f(u,v)p(u)q(v)\, dudv$$

for large enough s, t where $P(s)Q(t) > 0$.

Under fairly general conditions imposed on the weights, we give necessary and sufficient conditions in order that the finite limits $\sigma_{mn} \to L$ as $m, n \to \infty$ and $\sigma(s,t) \to L$ as $s, t \to \infty$ exist, respectively. These characterizations may find applications in Probability.

Mathematics Subject Classification: Primary: 40B99, 40C05, 40C10; Secondary: 26D15, 60F15, 60G60.

Key words and phrases: convergence in Pringsheim's sense, arithmetic-, harmonic- (also called logarithmic-) weighted averages, moving rectangular averages, moving maximal rectangular averages, strong laws for random fields.

This research was started when the first named author visited University of Ulm in July 2011. He appreciated very much the creative atmosphere and warm hospitality during his visit.

Corresponding author: Ferenc Móricz.

1. Weighted averages of double sequences of complex numbers

Let $\mathbf{c} = (c_{j,k} : (j,k) \in \mathbb{N}^2)$ be a double sequence of complex numbers, in symbols $(c_{j,k}) \subset \mathbb{C}$; and let $\mathbf{p} = (p_{j,k} \geq 0 : (j,k) \in \mathbb{N}^2)$ be a double sequence of weights, that is, a sequence of nonnegative numbers for which

$$P_{m,n} := \sum_{j=1}^{m} \sum_{k=1}^{n} p_{j,k} \to \infty \quad \text{as } \max\{m,n\} \to \infty. \tag{1.1}$$

We recall that the *weighted averages* $\sigma_{m,n} = \sigma_{m,n}(\mathbf{c}, \mathbf{p})$ of the double sequence $(c_{j,k})$ with respect to the weights $(p_{j,k})$ are defined by

$$\sigma_{m,n} := \frac{1}{P_{m,n}} \sum_{j=1}^{m} \sum_{k=1}^{n} p_{j,k}\, c_{j,k} \tag{1.2}$$

for such $(m,n) \in \mathbb{N}^2$, where $P_{m,n} > 0$ (cf. (1.1)).

The following examples frequently occur in the literature (see, e.g., [1], [2, p. 57–59], [4], [5] and [7, p. 74–80]).

(i) $p_{j,k} \equiv 1$ leads to the *arithmetic averages* (also called the Cesàro means of order $(1,1)$), where $P_{m,n} = m\,n$;

(ii) $p_{j,k} = 1/(j\,k)$ leads to the *harmonic averages* (also called the logarithmic means) of the sequence $(c_{j,k})$, where $P_{m,n} \sim \log m \log n$, by which we mean that

$$P_{m,n}/(\log m \log n) \to 1 \quad \text{as } m,n \to \infty,$$

and the logarithm is to the base e;

(iii) $p_{j,k} = 1/((j\log(j+1))(k\log(k+1))$ leads to the *harmonic averages of second order* (it may be called the iterated logarithmic means) of the sequence $(c_{j,k})$, where $P_{m,n} \sim \log\log m \log\log n$;

(iv) $p_{j,k} = 1/(j+k+1)$, this leads to an uncommon average with $P_{m,n} \sim (m+n)\log(m+n)$ as $m,n \to \infty$.

We recall that the double sequence $(c_{j,k})$ is convergent (in the Pringsheim's sense) to the finite limit $L \in \mathbb{C}$, in symbols:

$$c_{j,k} \to L \quad \text{as} \quad j,k \to \infty, \tag{1.3}$$

if for every $\epsilon > 0$ there exists some $\mu_0 = \mu_0(\epsilon) \in \mathbb{N}$ such that

$$|c_{j,k} - L| < \epsilon \quad \text{if } \min\{j,k\} > \mu_0.$$

This definition is attributed to PRINGSHEIM [6]. We note that ZYGMUND [10, Vol. II on p. 303, just below formula (1.18)] uses this convergence without the term "in Pringsheim's sense".

We also note that in contrast to the convergence of single sequences, the convergence of a double sequence in Pringsheim's sense does not imply boundedness of its terms. This is why we consider only bounded double sequences in the sequel.

The following theorem can be easily proved. It is actually an immediate corollary of a theorem of ROBISON [8] (see also in [3]).

Theorem 1.1. *The necessary and sufficient condition that we have*

$$\sigma_{m,n} \to L \quad as \quad m, n \to \infty \tag{1.4}$$

for every bounded double sequence $(c_{j,k})$ *with* (1.3) *are the following ones: for any fixed* $j, k \in \mathbb{N}$*, respectively, we have*

$$\frac{P_{m,k}}{P_{m,n}} \to 0 \quad and \quad \frac{P_{j,n}}{P_{m,n}} \to 0 \quad as\; m, n \to \infty. \tag{1.5}$$

We note that in the case when condition (1.1) and (1.5) are satisfied, the method of weighted averages is called regular. Since the finite limit in (1.4) may exist in cases where the limit in (1.3) fails this fact shows the significance of weighted averages, e.g., in Probability Theory.

One of our goals in this paper is to extend the recent results in [5, Theorems 1.1 and 1.2] from single to bounded double sequences. For notational simplicity we restrict ourselves to double sequences, the generalization to sequences with d-dimensional indices is possible and straight forward. We consider the class of weighted means with multiplicative weights $p_{jk} = p_j\, q_k$ only, the general case is technically more involved, see e.g. [9] for an impression.

2. New results for double sequences of numbers

In the sequel, we assume that the double sequence $(c_{j,k}) \subset \mathbb{C}$ is bounded:

$$\sup\{|c_{j,k}| : (j,k) \in \mathbb{N}^2\} =: C < \infty; \tag{2.1}$$

and the double sequence $(p_{j,k}) \geq 0$ of weights is of the form (cf. examples (i)–(iii))

$$p_{j,k} = p_j\, q_k, \quad (j,k) \in \mathbb{N}^2,$$

where the single sequences $(p_j) \geq 0$ and $(q_k) > 0$ are such that

$$P_m := \sum_{j=1}^{m} p_j \to \infty \quad \text{as } m \to \infty \quad \text{and} \quad Q_n := \sum_{k=1}^{n} q_k \to \infty \quad \text{as } n \to \infty. \tag{2.2}$$

In this case the conditions in (1.5) are clearly satisfied, and the weighted averages $(\sigma_{m,n})$ defined in (1.2) are of the form

$$\sigma_{m,n} := \frac{1}{P_m Q_n} \sum_{j=1}^{m} \sum_{k=1}^{n} p_j q_k c_{j,k} \tag{2.3}$$

for all m, n such that $P_m Q_n > 0$.

In this Section we will characterize the existence of the finite limit in (1.4). In order to avoid technicalities we assume from now on that p_j and q_k are positive for all $j, k \in \mathbb{N}$. Then there exist two sequences

$$1 = m_1 < m_2 < m_3 < \dots \quad \text{and} \quad 1 = n_1 < n_2 < n_3 < \dots$$

of integers such that

$$P_{m_{s+1}-1} < 2P_{m_s} \leq P_{m_{s+1}} \text{ and } Q_{n_{t+1}-1} < 2Q_{n_t} \leq Q_{n_{t+1}} \quad s, t = 1, 2, \dots. \tag{2.4}$$

Hence it follows immediately that

$$\frac{P_{m_{s+1}}}{P_{m_{s+1}} - P_{m_s}} \leq 2 \quad \text{and} \quad \frac{Q_{n_{t+1}}}{Q_{n_{t+1}} - Q_{n_t}} \leq 2, \quad (s,t) \in \mathbb{N}^2. \tag{2.5}$$

First, we characterize the existence of the finite limit of the double subsequence $(\sigma_{m_s,n_t} : (s,t) \in \mathbb{N}^2)$.

Theorem 2.1. *Let $(c_{j,k}) \subset \mathbb{C}$ be a bounded double sequence and let $(p_j) > 0$ and $(q_k) > 0$ be sequences of weights satisfying the conditions in* (2.2). *Then for some $L \in \mathbb{C}$, we have*

$$\sigma_{m_s,n_t} \to L \quad \text{as } s,t \to \infty \tag{2.6}$$

if and only if

$$\frac{1}{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})} \sum_{j=m_s+1}^{m_{s+1}} \sum_{k=n_t+1}^{n_{t+1}} p_j\, q_k\, c_{j,k} \to L \quad \text{as } s,t \to \infty, \tag{2.7}$$

where the sequences (m_s) and (n_t) are defined in (2.4).

We may call the ratio in (2.7) the *moving rectangular average* of $(c_{j,k})$ with respect to the weights $(p_j\, q_k)$. Moving averages along subsequences are helpful in Probability Theory to prove strong laws (see, e.g., [4] and [7, p. 57–59]).

Second, we characterize the existence of the finite limit of the full double sequence $(\sigma_{m,n} : (m,n) \in \mathbb{N}^2)$.

Theorem 2.2. *Let $(c_{j,k}) \subset \mathbb{C}$ be a bounded double sequence and let $(p_j) > 0$ and $(q_k) > 0$ be sequences of weights satisfying the conditions in (2.2) and (2.8), where*

$$\frac{p_m}{P_m} \to 0 \quad \text{as } m \to \infty \quad \text{and} \quad \frac{q_n}{Q_n} \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

Then the finite limit in (1.4) exists if and only if

$$\frac{1}{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})} \times \max_{\substack{m_s < m \le m_{s+1} \\ n_t < n \le n_{t+1}}} \left| \sum_{j=m_s+1}^{m} \sum_{k=n_t+1}^{n} p_j\, q_k (c_{j,k} - L) \right| \to 0 \tag{2.9}$$

as $s, t \to \infty$, where the sequences (m_s) and (n_t) are defined in (2.4).

We may call the ratio in (2.9) the *moving maximal rectangular average* of the difference sequence $(c_{j,k} - L)$ with respect to the weights $(p_j\, q_k)$.

The following example shows that the use of the moving maximal rectangular average in (2.9) is indispensable to ensure the existence of the finite limit in (1.4). In the simplest case of the arithmetic averages when $p_j, q_j \equiv 1$, the inequalities in (2.4) are satisfied if

$$m_s = 2^{s-1} \quad \text{and} \quad n_t = 2^{t-1}, \quad (s,t) \in \mathbb{N}^2.$$

Now, consider the double sequence $(c_{j,k})$ defined by

$$c_{j,k} := \begin{cases} 0 & \text{if } \min\{j,k\} \le 2, \\ 1 & \text{if } (j,k) \in (2^s, 2^{s+1}] \times (2^t, 2^{t+1}] \\ & \qquad - \{(j,k) : j = 2^{s+1} \text{ and } k = 2^{t+1}\}, \\ 1 - 2^{s+t} & \text{if } j = 2^{s+1} \text{ and } k = 2^{t+1}, \end{cases}$$

for $(s,t) \in \mathbb{N}^2$. It is easy to check that $\sigma_{2^{s+1}, 2^{t+1}} \equiv 0$, while

$$\sigma_{2^{s+1}-1, 2^{t+1}} \to \frac{1}{4} \quad \text{and} \quad \sigma_{2^{s+1}, 2^{t+1}-1} \to \frac{1}{4} \quad \text{as } s, t \to \infty.$$

Hence, the full sequence $(\sigma_{m,n})$ cannot converge to any finite limit L, though condition (2.7) is satisfied with $L = 0$.

The next two corollaries of the preceding Theorems 2.1. and 2.2. characterize the validity of the Strong Law of Large Numbers for weighted averages over a random field.

Corollary 2.1. *Let $(X_{j,k},: (j,k) \in \mathbb{N}^2)$ be a random field such that for some constant B,*

$$\mathbb{P}(|X_{j,k}| \le B) = 1 \quad \text{for all } (j,k) \in \mathbb{N}^2, \tag{2.10}$$

and let $(p_j) > 0$ and $(q_k) > 0$ be sequences of weights satisfying the conditions in (2.2). Then for some random variable L we have

$$\mathbb{P}\left(\frac{1}{P_{m_s}\, Q_{n_t}} \sum_{j=1}^{m_s} \sum_{k=1}^{n_t} p_j\, q_k\, X_{j,k} \to L \text{ as } \ s,t \to \infty \right) = 1$$

if and only if

$$\mathbb{P}\left(\frac{1}{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})} \sum_{j=m_s+1}^{m_{s+1}} \sum_{k=n_t+1}^{n_{t+1}} p_j\, q_k\, X_{j,k} \to L \text{ as } s,t \to \infty \right) = 1,$$

where the sequences (m_s) and (n_t) are defined in (2.4).

Corollary 2.2. *Let $(X_{j,k} : (j,k) \in \mathbb{N}^2)$ be a random field such that condition (2.10) is satisfied for some constant B, and let $(p_j) > 0$ and $(q_k) > 0$ be sequences of weights satisfying the conditions in (2.2) and (2.8). Then for some random variable L we have*

$$\mathbb{P}\left(\frac{1}{P_{m,n}} \sum_{j=1}^{m} \sum_{k=1}^{n} p_j q_k X_{j,k} \to L \text{ as } m,n \to \infty \right) = 1$$

if and only if

$$\mathbb{P}\Bigg(\frac{1}{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})} \times \max_{\substack{m_s < m \le m_{s+1} \\ n_t < n \le n_{t+1}}} \left| \sum_{j=m_s+1}^{m} \sum_{k=n_t+1}^{n} p_j\, q_k \big(X_{j,k} - L\big) \right| \to 0 \text{ as } s,t \to \infty \Bigg) = 1,$$

where the sequences (m_s) and (n_t) are defined in (2.4).

We note that in Probability Theory, Corollary 2.2 is usually applied by means of suitable maximal inequalities combined with moment assumptions, and using the Borel–Cantelli lemma.

3. Proofs of Theorems 2.1 and 2.2

In the proof of the Sufficiency part of Theorem 2.1, we will need the following

Lemma 3.1. *If the sequences* $(p_j) > 0$ *and* $(q_k) > 0$, *as well as the sequences* (m_s) *and* (n_t) *are such that the conditions in* (2.4) *are satisfied, then we have for* $1 \le \ell \le s$ *and* $1 \le \ell \le t$ *respectively that*

$$\frac{P_{m_{\ell+1}} - P_{m_\ell}}{P_{m_{s+1}}} \le \left(\frac{1}{2}\right)^{s-\ell} \quad \text{and} \quad \frac{Q_{n_{\ell+1}} - Q_{n_\ell}}{Q_{n_{t+1}}} \le \left(\frac{1}{2}\right)^{t-\ell}. \tag{3.1}$$

Proof. For example, by the first inequality in (2.4), we get for $1 \le \ell \le s$ that

$$\frac{P_{m_{\ell+1}} - P_{m_\ell}}{P_{m_{s+1}}} = \frac{P_{m_s}}{P_{m_{s+1}}} \frac{P_{m_{s-1}}}{P_{m_s}} \cdots \frac{P_{m_{\ell+1}}}{P_{m_{\ell+2}}} \frac{P_{m_{\ell+1}} - P_{m_\ell}}{P_{m_{\ell+1}}} \le \left(\frac{1}{2}\right)^{s-\ell}.$$

This proves the first inequality in (3.1), the second follows analogously. □

Proof of Theorem 2.1. *Necessity*: Assume the finite limit in (2.6). Then for any $\varepsilon > 0$ there exists $\mu_0 = \mu_0(\varepsilon) \in \mathbb{N}$ such that

$$|\sigma_{m_s,n_t} - L| < \varepsilon \quad \text{if } s, t > \mu_0. \tag{3.2}$$

Using notations (2.2)–(2.4), the ratio in (2.7) can be written in the following form:

$$\frac{1}{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})} \big\{ P_{m_{s+1}} Q_{n_{t+1}} (\sigma_{m_{s+1},n_{t+1}} - L)$$
$$- P_{m_s} Q_{n_{t+1}} (\sigma_{m_s,n_{t+1}} - L) - P_{m_{s+1}} Q_{n_t} (\sigma_{m_{s+1},n_t} - L) + P_{m_s} Q_{n_t} (\sigma_{m_s,n_t} - L) \big\}.$$

Taking into account (2.4), (2.5) and (3.2) we obtain

$$\left| \frac{1}{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})} \sum_{j=m_s+1}^{m_{s+1}} \sum_{k=n_t+1}^{n_{t+1}} p_j \, q_k \, (c_{j,k} - L) \right|$$
$$\le \frac{(P_{m_{s+1}} + P_{m_s})(Q_{n_{t+1}} + Q_{n_t})}{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})} \varepsilon \le 9\,\varepsilon \quad \text{if } s, t > \mu_0(\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, this proves (2.7). Observe that in this part we did not use condition (2.1).

Sufficiency: If condition (2.7) is satisfied, then for any $\varepsilon > 0$ there exists some $\mu_0 = \mu_0(\varepsilon)$ such that

$$d_{s,t} := \left| \frac{1}{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})} \sum_{j=m_s+1}^{m_{s+1}} \sum_{k=n_t+1}^{n_{t+1}} p_j \, q_k \, (c_{j,k} - L) \right| < \varepsilon \tag{3.3}$$

if $s,t > \mu_0$. We start with the following decomposition of the double sum in the definition of $\sigma_{m_{s+1},n_{t+1}}$ (cf. (2.3)):

$$\begin{aligned}\sigma_{m_{s+1},n_{t+1}} - L &= \frac{1}{P_{m_{s+1}}Q_{n_{t+1}}} \sum_{j=1}^{m_{s+1}} \sum_{k=1}^{n_{t+1}} p_j\, q_k(c_{j,k} - L)\\ &= \frac{1}{P_{m_{s+1}}Q_{n_{t+1}}} \left\{ \sum_{j=m_s+1}^{m_{s+1}} + \sum_{j=m_{s-1}+1}^{m_s} + \cdots + \sum_{j=m_\mu+1}^{m_{\mu+1}} + \sum_{j=1}^{m_\mu} \right\}\\ &\quad \times \left\{ \sum_{k=n_t+1}^{n_{t+1}} + \sum_{k=n_{t-1}+1}^{n_t} + \cdots + \sum_{k=n_\mu+1}^{n_{\mu+1}} + \sum_{k=1}^{n_\mu} \right\} p_j\, q_k(c_{j,k} - L).\end{aligned}$$

Using (2.1), (2.3), (3.1) and (3.3), we proceed as follows

$$\begin{aligned}\left|\sigma_{m_{s+1},n_{t+1}} - L\right| &\le \frac{1}{P_{m_{s+1}}Q_{n_{t+1}}} \Bigg\{ \left(P_{m_{s+1}} - P_{m_s}\right)\left(Q_{n_{t+1}} - Q_{n_t}\right) d_{s,t} + \ldots\\ + \left(P_{m_{\mu+1}} - P_{m_\mu}\right)&\left(Q_{n_{\mu+1}} - Q_{n_\mu}\right) d_{\mu,\mu} + \left\{ \sum_{j=1}^{m_{s+1}} \sum_{k=1}^{n_\mu} + \sum_{j=1}^{m_\mu} \sum_{k=1}^{n_{t+1}} \right\} p_j\, q_k |c_{j,k} - L| \Bigg\}\\ &\le \varepsilon \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{s-\mu}}\right)\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{t-\mu}}\right)\\ &\quad + \frac{C + |L|}{P_{m_{s+1}}\, Q_{n_{t+1}}} \left(P_{m_{s+1}} Q_{n_\mu} + P_{m_\mu} Q_{n_{t+1}}\right) < 4\varepsilon + \varepsilon = 5\varepsilon,\end{aligned}$$

provided that $\min\{s,t\}$ is large enough. Sine $\varepsilon > 0$ is arbitrary this proves (2.6). The proof of Theorem 2.1 is complete. □

In the proof of the sufficiency part of Theorem 2.2 we will need the follow

Lemma 3.2. *If the sequences $(p_j) > 0$ and $(q_k) > 0$, as well as the sequences (m_s) and (n_t) are such that the conditions in (2.4) and (2.8) are satisfied, then we have for $s \ge 1$ and $t \ge 1$ respectively that*

$$\frac{P_{m_{s+1}} - P_{m_s}}{P_{m_s+1}} < 2(1 + o_s(1)) \quad \text{and} \quad \frac{Q_{n_{t+1}} - Q_{n_t}}{Q_{n_t+1}} < 2(1 + o_t(1)). \tag{3.4}$$

where the terms $o_s(1)$ and $o_t(1)$ denote sequences of positive numbers that converge to zero as $s \to \infty$ or $t \to \infty$, respectively.

Proof. By (2.4), we have

$$\frac{P_{m_{s+1}} - P_{m_s}}{P_{m_s+1}} < \frac{P_{m_{s+1}-1}\left(1 + p_{m_{s+1}}/P_{m_{s+1}-1}\right)}{P_{m_s+1}} < 2\left(1 + \frac{p_{m_{s+1}}}{P_{m_{s+1}-1}}\right) = 2(1 + o_s(1)),$$

due to the fact that by (2.8) we have

$$\frac{p_m}{P_{m-1}} = \frac{p_m}{(P_m - p_m)} = \frac{1}{(P_m/p_m - 1)} \to 0 \quad \text{as } m \to \infty.$$

This proves the first inequality in (3.4) and an analogous argument yields the second. □

Proof of Theorem 2.2. *Necessity*: Assume the existence of the finite limit in (1.4). Then for every $\varepsilon > 0$ there exists some $\mu_0 = \mu_0(\varepsilon) \in \mathbb{N}$ such that

$$|\sigma_{m,n} - L| < \varepsilon \quad \text{if } m, n \geq \mu_0. \tag{3.5}$$

Now, let $(m,n) \in \mathbb{N}^2$ be such that $m_s < m \leq m_{s+1}$ and $n_t < n \leq n_{t+1}$, where $m_s, n_t \geq \mu_0$. Keeping notation (2.4) in mind, by (3.5) we proceed as follows

$$\begin{aligned}
&\left|\sum_{j=m_s+1}^{m} \sum_{k=n_t+1}^{n} p_j q_k (c_{j,k} - L)\right| \\
&= \left|\left\{\sum_{j=1}^{m}\sum_{k=1}^{n} - \sum_{j=1}^{m_s}\sum_{k=1}^{n} - \sum_{j=1}^{m}\sum_{k=1}^{n_t} + \sum_{j=1}^{m_s}\sum_{k=1}^{n_t}\right\} p_j q_k (c_{j,k} - L)\right| \\
&= |P_m Q_n(\sigma_{m,n} - L) - P_{m_s} Q_n(\sigma_{m_s,n} - L) - P_m Q_{n_t}(\sigma_{m,n_t} - L) \\
&\quad + P_{m_s} Q_{n_t}(\sigma_{m_s,n_t} - L)| < (P_m + P_{m_s})(Q_n + Q_{n_t})\varepsilon.
\end{aligned}$$

Using the inequalities in (2.5) we hence get

$$\begin{aligned}
&\frac{1}{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})} \max_{\substack{m_s < m \leq m_{s+1} \\ n_t < n \leq n_{t+1}}} \left|\sum_{j=m_s+1}^{m} \sum_{k=n_t+1}^{n} p_j\, q_k (c_{j,k} - L)\right| \\
&\quad < \frac{(P_{m_{s+1}} + P_{m_s})(Q_{n_{t+1}} + Q_{n_t})}{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})}\varepsilon < 9\,\varepsilon \quad \text{if } m_s, n_t \geq \mu_0.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves (2.9). Observe that in this part we did not use condition (2.1).

Sufficiency: Assume that condition (2.9) is satisfied. Clearly, it follows from (2.9) that the limit in (2.7) also exists with the same L. By virtue of Theorem 2.1 just proved, we conclude the existence of the finite limit in (2.6).

By (2.6) and (2.9), for every $\varepsilon > 0$ there exists another $\mu_0 = \mu_0(\varepsilon) \in \mathbb{N}$ such that

$$|\sigma_{m_s,n_t} - L| \quad \text{if } s,t > \mu_0 \tag{3.6}$$

and with $\Delta_{s,t} = \frac{1}{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})}$

$$D_{s,t} := \Delta_{s,t} \max_{\substack{m_s < m \le m_{s+1} \\ n_t < n \le n_{t+1}}} \left| \sum_{j=m_s+1}^{m} \sum_{k=n_t+1}^{n} p_j\, q_k (c_{j,k} - L) \right| < \varepsilon \quad \text{if } s,t > \mu_0. \tag{3.7}$$

Now, let $(m,n) \in \mathbb{N}^2$ be such that $m_s < m \le m_{s+1}$ and $n_t < n \le n_{t+1}$. Keeping (2.3) in mind, we start with the decomposition

$$\sigma_{m,n} - L = \frac{1}{P_m\, Q_n} \Bigg\{ \sum_{j=1}^{m_s} \sum_{k=1}^{n_t} + \sum_{j=m_s+1}^{m} \sum_{k=n_t+1}^{n} + \sum_{j=1}^{m_s} \sum_{k=n_t+1}^{n} + \sum_{j=m_s+1}^{m} \sum_{k=1}^{n_t} \Bigg\} p_j\, q_k (c_{j,k} - L).$$

Accordingly, we proceed as follows

$$\begin{aligned} \max_{\substack{m_s < m \le m_{s+1} \\ n_t < n \le m_{t+1}}} |\sigma_{m,n} - L| &\le |\sigma_{m_s,n_t} - L| + \frac{(P_{m_{s+1}} - P_{m_s})(Q_{n_{t+1}} - Q_{n_t})}{P_{m_s+1}\, Q_{n_t+1}} D_{s,t} \\ &+ \frac{1}{P_{m_s+1}\, Q_{n_t+1}} \max_{n_t < n < n_{t+1}} \left| \sum_{j=1}^{m_s} \sum_{k=n_t+1}^{n} p_j\, q_k\, (c_{j,k} - L) \right| \\ &+ \frac{1}{P_{m_s+1}\, Q_{n_t+1}} \max_{m_s < m < m_{s+1}} \left| \sum_{j=m_s+1}^{m} \sum_{k=1}^{n_t} p_j\, q_k (c_{j,k} - L) \right| \\ &=: |\sigma_{m_s,n_t} - L| + \sigma_{s,t}^{(1)} + \sigma_{s,t}^{(2)} + \sigma_{s,t}^{(3)}, \end{aligned} \tag{3.8}$$

say, where $D_{s,t}$ is defined in (3.7).

By (3.4) and (3.7), we clearly have that

$$\sigma_{s,t}^{(1)} < 4\varepsilon(1 + o_s(1))(1 + o_t(1)) < 5\,\varepsilon \tag{3.9}$$

provided that $\min\{s,t\}$ is large enough.

In order to estimate $\sigma_{s,t}^{(2)}$, we decompose the double sum with respect to the first one as follows

$$\sum_{j=1}^{m_s}\sum_{k=n_t+1}^{n} p_j\, q_k(c_{j,k}-L)$$
$$= \left\{\sum_{j=m_{s-1}+1}^{m_s} + \sum_{j=m_{s-2}+1}^{m_{s-1}} + \cdots + \sum_{j=m_\mu+1}^{m_{\mu+1}} + \sum_{j=1}^{m_\mu}\right\} \sum_{k=n_t+1}^{n} p_j\, q_k(c_{j,k}-L).$$

Hence, keeping notation (3.7) in mind we obtain

$$\sigma_{s,t}^{(2)} \le \frac{(Q_{n_{t+1}}-Q_{n_t})}{Q_{n_t+1}}\left\{\frac{(P_{m_s}-P_{m_{s-1}})}{P_{m_s+1}}D_{s-1,t} + \frac{(P_{m_{s-1}}-P_{m_{s-2}})}{P_{m_s+1}}D_{s-2,t} + \dots \right.$$
$$\left. + \frac{(P_{m_{\mu+1}}-P_{m_\mu})}{P_{m_s+1}}D_{\mu,t}\right\} + \frac{1}{P_{m_s+1}Q_{n_t+1}}\sum_{j=1}^{m_\mu}\sum_{k=n_t+1}^{n_{t+1}} p_j\, q_k|c_{j,k}-L|,$$

and taking into account the first inequalities in (2.4) and (3.4) as well as (3.7) gives

$$\sigma_{s,t}^{(2)} \le 2(1+o_t(1))\left(\frac{\varepsilon}{1}+\frac{\varepsilon}{2}+\cdots+\frac{\varepsilon}{2^{s-\mu-1}}+\frac{P_{m_\mu}(C+|L|)}{P_{m_s+1}}\right)$$
$$< 2(1+o_t(1))\left(2\varepsilon+\frac{P_{m_\mu}(C+|L|)}{P_{m_s+1}}\right) < 6\varepsilon, \quad (3.10)$$

provided that s, t are large enough and the constant C is from (2.1) and we exploited the first condition in (2.2). An analogous argument gives

$$\sigma_{s,t}^{(3)} \le 2(1+o_s(1))\left(\frac{\varepsilon}{1}+\frac{\varepsilon}{2}+\cdots+\frac{\varepsilon}{2^{t-\mu-1}}+\frac{Q_{n_\mu}(C+|L|)}{Q_{n_t+1}}\right) < 6\,\varepsilon, \qquad (3.11)$$

provided that s,t are large enough, where this time we exploited the second condition in (2.2). Combining (3.6) and (3.8)–(3.11) yields

$$\max_{\substack{m_s<m\le m_{s+1}\\ n_t<n\le n_{t+1}}} |\sigma_{m,n}-L| < \varepsilon+5\varepsilon+6\varepsilon+6\varepsilon = 18\,\varepsilon, \qquad (3.12)$$

provided that $\min\{s,t\}$ is large enough. Since $\varepsilon>0$ is arbitrary we have proved that

$$\max_{\substack{m_s<m\le m_{s+1}\\ n_t<n\le n_{t+1}}} |\sigma_{m,n}-L| \to 0, \quad \text{as } s,t\to\infty$$

which is equivalent to (1.4). The proof of Theorem 2.2 is thus complete. □

4. Double weighted averages of measurable functions in $\mathbb{R}^2_+$

Let $f : \bar{\mathbb{R}}^2_+ \to \mathbb{C}$ be a measurable function in Lebesgue's sense and let $p : \bar{\mathbb{R}}^2_+ \to \bar{\mathbb{R}}_+$ be a weight function such that for every bounded rectangle

$$R := [a_1, b_1] \times [a_2, b_2], \quad 0 \le a_\nu < b_\nu < \infty,\ \nu = 1, 2,$$

$p \in L_1(R)$, in symbols $p \in L^1_{\text{loc}}(\bar{\mathbb{R}}^2_+)$ and possessing the property that

$$P(s,t) := \int_0^s \int_0^t p(u,v)\, dudv \to \infty \quad \text{as } s, t \to \infty. \tag{4.1}$$

We recall that the weighted averages of the function f with respect to the weight function p are defined in case $f\,p \in L^1_{\text{loc}}(\bar{\mathbb{R}}^2_+)$ as

$$\sigma_{s,t}(f,p) := \frac{1}{P(s,t)} \int_0^s \int_0^t f(u,v)\, p(u,v)\, dudv \tag{4.2}$$

for such $(s,t) \in \mathbb{R}^2_+$ where $P(s,t) > 0$ (cf. (4.1)).

The following examples occur frequently in the literature (see, e.g., [2, p. 110–112]).

(i) $p(u,v) \equiv 1$ leads to the *arithmetic average* (also called the Cesàro mean of order (1,1)) of the function f, where $P(s,t) = st$;

(ii) $p(u,v) = 1/(u+1)(v+1)$ leads to the *harmonic average* (also called logarithmic mean) of the function f, where $P(t) = \log(s+1)\log(t+1)$;

(iii) $p(u,v) = 1/((u+1)\log(u+e)(v+1)\log(v+e))$ leads to the *iterated harmonic average* (also called iterated logarithmic mean) of the function f, where $P(s,t) \sim (\log\log s)(\log\log t)$.

The following theorem which is the continuous version of Theorem 1.1 can easily be proved.

Theorem 4.1. *The necessary and sufficient conditions that we have*

$$\sigma_{s,t}(f,p) \to L, \quad \text{as } s,t \to \infty \tag{4.3}$$

for every function $f \in L^\infty(\bar{\mathbb{R}}^2_+)$ *such that*

$$f(u,v) \to L, \quad \text{as } u, v \to \infty,$$

are the following ones: for any fixed $s_0, t_0 \in \mathbb{R}_+$,

$$\frac{P(s,t_0)}{P(s,t)} \to 0 \quad \text{and} \quad \frac{P(s_0,t)}{P(s,t)} \to 0 \quad \text{as } s,t \to \infty. \tag{4.4}$$

Analogously to the discrete case, if conditions (4.1) and (4.4) are satisfied then the weighted averages are called regular.

Another goal in this paper is to extend the recent result in [5, Theorem 3.1] from functions in one variable to those in two variables.

5. New results for functions in $L^\infty(\bar{\mathbb{R}}^2_+)$

From now on, we assume that the weight function p is of form

$$p(u,v) = p(u)\,q(v), \quad (u,v) \in \bar{\mathbb{R}}^2_+,$$

where $p(u) > 0$ and $q(v) > 0$ for almost every $u \in \mathbb{R}_+$ and almost every $v \in \mathbb{R}_+$ respectively; furthermore $p, q \in L^1_{\text{loc}}(\bar{\mathbb{R}}_+)$ and

$$P(s) := \int_0^s p(u)\,du \to \infty \text{ and } Q(s) := \int_0^s q(v)\,dv \to \infty \text{ as } s \to \infty. \tag{5.1}$$

In this case, the conditions in (4.4) are clearly satisfied, and the weighted averages $\sigma_{s,t}$ defined in (4.2) are of the form

$$\sigma_{s,t} := \frac{1}{P(s)\,Q(t)} \int_0^s \int_0^t f(u,v)\,p(u)\,q(v)\,dudv, \;\; s,t > 0. \tag{5.2}$$

Since $p(u) > 0$ and $q(v) > 0$ for almost every u and $v \in \mathbb{R}_+$ the functions $P(s)$ and $Q(t)$ are strictly increasing and being integrals they are continuous on $\mathbb{R}_+$. Therefore, their inverse functions denoted by $P^{-1}(u)$ and $Q^{-1}(v)$ exist and they are continuous and strictly increasing on $\mathbb{R}_+$ as well. In the following Theorem 5.1, we characterize the existence of the finite limit in (4.3).

Theorem 5.1. *Let the function $f : \bar{\mathbb{R}}^2_+ \to \mathbb{C}$ belong to $L^\infty(\bar{\mathbb{R}}^2_+)$, let the weight functions $p(u) > 0$ and $q(v) > 0$ for almost every $u, v \in \mathbb{R}_+$ respectively, and let $p, q \in L^1_{\text{loc}}(\bar{\mathbb{R}}_+)$ be such that conditions* (5.1) *are satisfied. Then the finite limit in* (4.3) *exists if and only if*

$$\frac{1}{st} \int_{P^{-1}(s)}^{P^{-1}(2s)} \int_{Q^{-1}(t)}^{Q^{-1}(2t)} f(u,v)\; p(u)q(v)\,dudv \to L \quad \text{as } s,t \to \infty. \tag{5.3}$$

We may call the ratio in (5.3) the moving rectangular average of the function $f(u,v)$ with respect to the weight function $p(u)\,q(v)$, due to the fact (cf. (2.7) and (5.2))

$$(P^{-1}(2s) - P^{-1}(s))(Q^{-1}(2t) - Q^{-1}(t)) = (2s-s)(2t-t) = st, \;\; s,t > 0. \tag{5.4}$$

It is instructive to compare the conditions in (2.9) and (5.3). For the characterization in Theorem 2.2, we need the notion of moving *maximal* rectangular average. On the other hand, for the characterization in Theorem 5.1, the notion of the moving rectangular average is enough.

We note that in Theorem 5.1, the span size from $P^{-1}(s)$ to $P^{-1}(2s)$ and from $Q^{-1}(t)$ to $Q^{-1}(2t)$ can be replaced by the span size from $P^{-1}(s)$ to $P^{-1}(\kappa s)$ and from $Q^{-1}(t)$ to $Q^{-1}(\lambda t)$ for any real numbers $\kappa, \lambda > 1$.

In the following corollary, we characterize the validity of the Strong Law of Large Numbers in terms of the weighted average of a stochastic process. It is an immediate consequence of Theorem 5.1 .

Corollary 5.1. *Let $p(u) > 0$ and $q(v) > 0$ be weight functions with properties as in Theorem 5.1, and let $(X_{u,v} : (u,v) \in \bar{\mathbb{R}}^2_+)$ be a bounded stochastic process such that all paths are locally integrable with respect to $p(u)\,q(v)$. Then we have for some random variable L,*

$$\mathbb{P}\left(\frac{1}{P(s)\,Q(t)}\int_0^s\int_0^t X_{u,v}\,p(u)\,q(v)\,dudv \to L,\ \text{as } s,t\to\infty\right) = 1$$

if and only if

$$\mathbb{P}\left(\frac{1}{st}\int_{P^{-1}(s)}^{P^{-1}(2s)}\int_{Q^{-1}(t)}^{Q^{-1}(2t)} X_{u,v}\,p(u)\,q(v)\,dudv \to L,\ \text{as } s,t\to\infty\right) = 1,$$

where P^{-1} and Q^{-1} are the inverse functions of P and Q respectively.

6. Proof of Theorem 5.1

Necessity: Assume the existence of the finite limit in (5.1). By (5.2), we may write that

$$\begin{aligned}
&\frac{1}{st}\int_{P^{-1}(s)}^{P^{-1}(2s)}\int_{Q^{-1}(t)}^{Q^{-1}(2t)} f(u,v)\,p(u)\,q(v)\,dudv\\
&= \frac{1}{st}\Bigg\{\int_0^{P^{-1}(2s)}\int_0^{Q^{-1}(2t)} - \int_0^{P^{-1}(s)}\int_0^{Q^{-1}(2t)} - \int_0^{P^{-1}(2s)}\int_0^{Q^{-1}(t)}\\
&\qquad + \int_0^{P^{-1}(s)}\int_0^{Q^{-1}(t)}\Bigg\} f(u,v)\,p(u)\,q(v)\,dudv\\
&= \frac{P(P^{-1}(2s))Q(Q^{-1}(2t))}{st}\sigma_{P^{-1}(2s),Q^{-1}(2t)} - \frac{P(P^{-1}(s))Q(Q^{-1}(2t))}{st}\sigma_{P^{-1}(s),Q^{-1}(2t)}\\
&\quad - \frac{P(P^{-1}(2s))Q(Q^{-1}(t))}{st}\sigma_{P^{-1}(2s),Q^{-1}(t)} + \frac{P(P^{-1}(s))Q(Q^{-1}(t))}{st}\sigma_{P^{-1}(s),Q^{-1}(t)}\\
&\qquad \to 4L - 2L - 2L + L = L,\ \text{as}\ \ s,t\to\infty,
\end{aligned}$$

where we took into account (5.4). This proves (5.3).

Sufficiency: Assume the existence of the finite limit in (5.3). Then for every $\varepsilon > 0$, there exists $\rho_0 = \rho_0(\varepsilon) \in \mathbb{R}_+$ such that

$$\left| \frac{1}{st} \int_{P^{-1}(s)}^{P^{-1}(2s)} \int_{Q^{-1}(t)}^{Q^{-1}(2t)} (f(u,v) - L)\, p(u)\, q(v)\, du dv \right| < \varepsilon \quad \text{if } s, t > \rho_0, \tag{6.1}$$

where we took into account (5.4) again. It is obvious that the existence of the finite limit in (4.3) is equivalent to the existence of the following one:

$$\frac{1}{st} \int_0^{P^{-1}(s)} \int_0^{Q^{-1}(t)} (f(u,v) - L)\, p(u)\, q(v)\, du dv \to 0 \quad \text{as } s, t \to \infty. \tag{6.2}$$

In the sequel we will prove (6.2). To this effect, let an arbitrary $\varepsilon > 0$ be given. For any $s > \rho_0$ and $t > \rho_0$, where ρ_0 occurred in (6.1), there exist integers m and n such that

$$\frac{s}{2^{m+1}} \le \rho_0 < \frac{s}{2^m} \quad \text{and} \quad \frac{t}{2^{n+1}} \le \rho_0 < \frac{t}{2^n}. \tag{6.3}$$

Observe that it follows immediately from (6.3) that

$$\frac{s}{2^m} \le 2\rho_0 \quad \text{and} \quad \frac{t}{2^n} \le 2\rho_0. \tag{6.4}$$

By (6.1),(6.3) and (6.4), we may proceed as follows

$$\begin{aligned} &\left| \frac{1}{st} \int_0^{P^{-1}(s)} \int_0^{Q^{-1}(t)} (f(u,v) - L) p(u) q(v)\, du dv \right| \\ &= \frac{1}{st} \Bigg| \left\{ \int_{P^{-1}(s/2)}^{P^{-1}(s)} + \int_{P^{-1}(s/4)}^{P^{-1}(s/2)} + \cdots + \int_{P^{-1}(s/2^m)}^{P^{-1}(s/2^{m-1})} + \int_0^{P^{-1}(s/2^m)} \right\} \\ &\quad \times \left\{ \int_{Q^{-1}(t/2)}^{Q^{-1}(t)} + \int_{Q^{-1}(t/4)}^{Q^{-1}(t/2)} + \cdots + \int_{Q^{-1}(t/2^n)}^{Q^{-1}(t/2^{n-1})} + \int_0^{Q^{-1}(t/2^n)} \right\} \\ &\quad \times (f(u,v) - L)\, p(u)\, q(v)\, du dv \Bigg| \le \varepsilon \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^m} \right) \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \right) \\ &\quad + \frac{1}{st} \left\{ \int_0^{P^{-1}(s)} \int_0^{Q^{-1}(2\rho_0)} + \int_0^{P^{-1}(2\rho_0)} \int_0^{Q^{-1}(t)} \right\} \\ &\quad \times |(f(u,v) - L|\, p(u)\, q(v)\, du dv < 2\varepsilon, \end{aligned} \tag{6.5}$$

provided that $\min\{s,t\}$ is so large that

$$\frac{1}{st}(\|f\|_\infty + |L|)\left\{P(P^{-1}(s))Q(Q^{-1}(2\rho_0)) + P(P^{-1}(2\rho_0))Q(Q^{-1}(t))\right\} = (\|f\|_\infty + |L|)\left\{\frac{2\rho_0}{t} + \frac{2\rho_0}{s}\right\} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary in (6.5), we have proved the desired limit in (6.2) which is equivalent to the existence of the limit in (4.3) which makes the proof complete. □

References

[1] I. Berkes, E. Csáki and L. Horváth, An almost sure central limit theorem under minimal conditions, *Statist. Probab. Lett.* **37** (1998), 67–76.

[2] G. H. Hardy, Divergent Series, *Clarendon Press, Oxford*, 1949.

[3] F. Móricz and B. E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Cambridge Philos. Soc.* **104** (1988), 283–294.

[4] F. Móricz, U. Stadtmüller and M. Thalmaier, Strong laws for blockwise $\mathcal{M}$-dependent random fields, *J. Theoret. Probab.* **21** (2008), 660–671.

[5] F. Móricz and U. Stadtmüller, Characterization of the convergence of weighted averages of sequences and functions, *Period. Math. Hungar.* **65** (2012), 135–145.

[6] A. Pringsheim, Elementare Theorie der unendlichen Doppelreihen, *München Berichte* **27** (1897), 101–152.

[7] P. Révész, The Laws of Large Numbers, *Akadémiai Kiadó, Budapest*, 1967.

[8] G. M. Robison, Divergent double sequences and series, *Trans. Amer. Math. Soc.* **28** (1926), 50–73.

[9] U. Stadtmüller, One-sided Tauberian theorems and double sequences, *Period. Math. Hungar.* **45** (2002), 135–136.

[10] A. Zygmund, Trigonometric Series, Vol. II, *Cambridge University Press*, 1959.

FERENC MÓRICZ
UNIVERSITY OF SZEGED
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
H-6720 SZEGED
HUNGARY

E-mail: moricz@math.u-szeged.hu

ULRICH STADTMÜLLER
UNIVERSITÄT ULM
ABT. ZAHLENTHEORIE UND
WAHRSCHEINLICHKEITSTHEORIE
D-89069 ULM
GERMANY

E-mail: ulrich.stadtmueller@uni-ulm.de

(Received July 11, 2012; revised April 5, 2013)