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On several classes of additively non-regular C-semirings

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Abstract. In this paper, the authors study several classes of additively non-regular C-semirings whose additive idempotents are central, including the generalized C-rpp semirings, C-rpp semirings, generalized C-abundant semirings and C-abundant semirings. After introducing the concept of generalized C-rpp semirings, the authors obtain their equivalent characterizations, and show that a semiring is a generalized C-rpp semiring if and only if it is a strong b-lattice of additively left cancellative halfrings, and if and only if it is a subdirect product of a b-lattice and an additively left cancellative halfring. Also, the authors give the constructions of C-rpp semirings, generalized C-abundant semirings and C-abundant semirings. Consequently, the corresponding results of Clifford semirings and generalized Clifford semirings in [7] and [29] are extended and generalized.

1. Introduction

A semiring is an algebra $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that both $(R, +)$ and $(R, \cdot)$ are semigroups and such that the distributive laws

$$x(y + z) \approx xy + xz \quad \text{and} \quad (x + y)z \approx xz + yz$$

are satisfied.

The additive identity (if it exists) of a semiring R is called zero and denoted by 0. An additively commutative semiring R with a zero satisfying $0x = x0 = 0$

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for all $x \in R$, is called a hemiring. A halfring is a hemiring whose additive reduct $(R,+)$ is a cancellative monoid, i.e., for any $a,b,c \in R$, $a+b=a+c$ or $b+a=c+a$ implies $b=c$. A skew-ring $(R,+,\cdot)$ [29] is a semiring whose additive reduct $(R,+)$ is a group, not necessarily an abelian group. An additively cancellative skew-halfring (additively left cancellative skew-halfring, respectively) is a semiring whose additive reduct is an additively cancellative monoid (left cancellative monoid, respectively), not necessarily to be additively commutative. Also, a semiring $(R,+,\cdot)$ is said to be a b-lattice [29] if its additive reduct $(R,+)$ is a semilattice and its multiplicative reduct $(R,\cdot)$ is a band.

The algebraic theory of semirings have some important applications in automation theory, optimization theory and models of discrete event networks etc. There are a series of papers in the literature considering semirings (for example, see [2], [7]–[10], [16]–[17], [20]–[21], [23]–[32]).

Since semirings are generalizations of distributive lattices, b-lattices, rings, skew-rings, skew-halfrings and left skew-halfrings, it is interesting to use those semirings to establish the constructions of some semirings. In [2], Bandelt and Petrich introduced Bandelt–Petrich Construction in semirings and described the semirings with regular addition which is a subdirect products of a distributive lattice and a ring. In [7], Ghosh established the constructions of strong distributive lattice of semirings which include the Bandelt–Petrich Construction, and characterized all semirings which are subdirect products of a distributive lattice and a ring. In particular, the authors introduced Clifford semirings, and showed that a semiring is a Clifford semiring if and only if it is a strong distributive lattice of rings, and if and only if it is an inverse subdirect product of a distributive lattice and a ring. Later, Sen, Maity and Shum in [29] defined the Clifford semiring which is a completely regular and an additively inverse semiring such that the set of its additive idempotents is a distributive sublattice as well as a k-ideal (without assuming that its additive reduct is commutative) and verified that a semiring is a Clifford semiring if and only if it is a strong distributive lattice of skew-rings. Meanwhile, they introduced generalized Clifford semirings which are completely regular and inverse semirings such that its additive idempotent set is a k-ideal, and obtained that a semiring is a generalized Clifford semiring if and only if it is a strong b-lattice of skew-rings, and if and only if it is an additively inverse semiring and is a subdirect product of a b-lattice and a skew-ring. It is not hard to see that all the semirings studied in [2], [7] and [29] are additively regular.

On the other hand, as we know, in order to generalize regular semigroups, new Green's relations, namely, the Green's $*$-relations on a semigroup have been introduced as follows (see [3], [19], or [22]):

$$\mathcal{L}^* = \{(a,b) \in S \times S : (\forall x, y \in S^1) ax = ay \Leftrightarrow bx = by\},$$

$$\mathcal{R}^* = \{(a,b) \in S \times S : (\forall x, y \in S^1) xa = ya \Leftrightarrow xb = yb\},$$

$$\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*,$$

$$\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*.$$

It is clear that $\mathcal{L} \subseteq \mathcal{L}^*$, $\mathcal{R} \subseteq \mathcal{R}^*$, $\mathcal{H} \subseteq \mathcal{H}^*$, $\mathcal{D} \subseteq \mathcal{D}^*$. A semigroup S is abundant [3] if its each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent, a semigroup S is an rpp semigroup (a lpp semigroup, respectively) if its each $\mathcal{L}^*$-class ($\mathcal{R}^*$-class, respectively) contains an idempotent (see [5]). A semigroup S is a C-rpp semigroup ([5]) if its every $\mathcal{L}^*$-class contains an idempotent and $E(S)$ is central. Dually, we will get the definition of C-lpp semigroups. A semigroup S is said to be a C-abundant semigroup if it is abundant and $E(S)$ is central, i.e., it is both a C-lpp semigroup and a C-rpp semigroup. In general, abundant semigroups, C-rpp semigroups, C-lpp semigroups and C-abundant semigroups are not regular, so we will call them non-regular semigroups in the following. There are also a series of papers in the literature considering non-regular semigroups (for example, see [1], [3]–[5], [11]–[15], [18] etc.).

In this paper, we will study several classes of additively non-regular C-semirings whose additive idempotents are central, including the generalized C-rpp semirings, C-rpp semirings, generalized C-abundant semirings and C-abundant semirings. Our purpose is to extend the results of Clifford semirings and generalized Clifford semirings in [29] and the semirings which are subdirect products of a distributive lattice and a ring in [7] to the non-regular C-semirings. We will show that a semiring is a generalized C-rpp semiring (C-rpp semiring, generalized C-abundant semiring, C-abundant semiring, respectively) if and only if it is a strong b-lattice (strong distributive lattice, strong b-lattice, strong distributive lattice, respectively) of additively left cancellative (left cancellative, cancellative, cancellative, respectively) halfrings, and if and only if it is a subdirect product of a b-lattice (distributive lattice, b-lattice, distributive lattice, respectively) and an additively left cancellative (left cancellative, cancellative, cancellative, respectively) halfring.

For notations and terminologies not mentioned in this paper, the readers are referred to [3], [8] or [29].

2. Generalized C-rpp semirings and C-rpp semirings

In this section, we will study the classes of generalized C-rpp semirings and C-rpp semirings, and show that a semiring is a generalized C-rpp semiring (C-rpp semiring, respectively) if and only if it is a strong b-lattice (strong distributive lattice, respectively) of additively left cancellative halfrings, and if and only if it is a subdirect product of a b-lattice (distributive lattice, respectively) and an additively left cancellative halfring. Also, we will give some other characterizations of such semirings.

Let $(R, +, \cdot)$ be a semiring. We denote the Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ on additive reduct $(R, +)$ by $\overset{+}{\mathcal{L}}$, $\overset{+}{\mathcal{R}}$, $\overset{+}{\mathcal{H}}$, respectively. These are also equivalence relations on semiring $(R, +, \cdot)$. Now, we introduce Green's $*$-relations $\overset{+}{\mathcal{L}}{}^*$, $\overset{+}{\mathcal{R}}{}^*$, $\overset{+}{\mathcal{H}}{}^*$ on semiring R which are given by

$$\overset{+}{\mathcal{L}}{}^* = \{(a, b) \in R \times R : (\forall x, y \in R^1) a + x = a + y \Leftrightarrow b + x = b + y\},$$

$$\overset{+}{\mathcal{R}}{}^* = \{(a, b) \in R \times R : (\forall x, y \in R^1) x + a = y + a \Leftrightarrow x + b = y + b\},$$

$$\overset{+}{\mathcal{H}}{}^* = \overset{+}{\mathcal{L}}{}^* \cap \overset{+}{\mathcal{R}}{}^* .$$

It is clear that $\overset{+}{\mathcal{L}} \subseteq \overset{+}{\mathcal{L}}{}^*$, $\overset{+}{\mathcal{R}} \subseteq \overset{+}{\mathcal{R}}{}^*$, $\overset{+}{\mathcal{H}} \subseteq \overset{+}{\mathcal{H}}{}^*$ on $(R, +, \cdot)$. In particular, if R is an additively regular semiring, $\overset{+}{\mathcal{L}} = \overset{+}{\mathcal{L}}{}^*$, $\overset{+}{\mathcal{R}} = \overset{+}{\mathcal{R}}{}^*$, $\overset{+}{\mathcal{H}} = \overset{+}{\mathcal{H}}{}^*$ [4]. In general, Green's equivalence relations $\overset{+}{\mathcal{L}}{}^*$, $\overset{+}{\mathcal{R}}{}^*$ and $\overset{+}{\mathcal{H}}{}^*$ are not congruences on $(R, +, \cdot)$.

For a semiring R, we denote by $E^+(R)$ the set of all additive idempotents of R. For any $e, f \in E^+(R)$, we write $e \leq_+ f$ if $e + f = f = f + e$. Remark that $\leq_+$ is a partial order which is compatible with the multiplication.

In the following, we will introduce the concepts of strong b-lattice and strong distributive lattice of semirings.

Definition 1 (Definition 2.3 in [29]). Let T be a b-lattice and $\{R_\alpha : \alpha \in T\}$ be a family of pairwise disjoint semirings which are indexed by the elements of T. For each $\alpha \leq \beta$ in T, we now embed $R_\alpha$in R_β via a semiring monomorphism $\phi_{\alpha,\beta}$ satisfying the following conditions:

(1.1) $\phi_{\alpha,\alpha} = I_{R_\alpha}$, the identity mapping on R_α;

(1.2) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \leq \beta \leq \gamma$;

(1.3) $R_\alpha\phi_{\alpha,\beta}R_\beta\phi_{\beta,\gamma} \subseteq R_{\alpha\beta}\phi_{\alpha\beta,\gamma}$ if $\alpha + \beta \leq \gamma$, i.e., $\alpha + \beta + \alpha\beta \leq \gamma$.

On $R = \cup_{\alpha \in Y} R_\alpha$, we define addition $+$ and multiplication $\cdot$ for $a \in R_\alpha$, $b \in R_\beta$, as follows:

$$a + b = a\phi_{\alpha,\alpha+\beta} + b\phi_{\beta,\alpha+\beta} \tag{1.4}$$

and

$$a \cdot b = c \in R_{\alpha\beta}$$

such that

$$c\phi_{\alpha\beta,\alpha+\beta} = a\phi_{\alpha,\alpha+\beta} \cdot b\phi_{\beta,\alpha+\beta}.$$

Same as the notation of strong semilattice of semigroups, we denote the above system by$R = \langle T, R_\alpha, \phi_{\alpha,\beta} \rangle$ and call it the strong b-lattice T of the semirings $R_\alpha, \alpha \in T$.

In an obvious way, we may replace b-lattice T in the above definition by distributive lattice D, $R = \langle D, R_\alpha, \phi_{\alpha,\beta} \rangle$ and call it strong distributive lattice D of the semirings $R_\alpha, \alpha \in D$.

Lemma 1 (Theorem 2.4 in [29])**.** *The system $R = \langle T, R_\alpha, \phi_{\alpha,\beta} \rangle$ defined above is a semiring.*

Lemma 2 ([5])**.** *A semigroup S is a C-rpp semigroup if and only if it is a strong semilattice of left cancellative monoids.*

By Lemma 2, we can dually obtain that a semigroup S is a C-lpp semigroup if and only if it is a strong semilattice of right cancellative monoids.

From [5], it is also known that a semigroup $(S,+)$ is called a [right, left, respectively]adequate semigroup if its idempotents commute and every $\mathcal{L}^*$-class and $\mathcal{R}^*$-class [$\mathcal{L}^*$-class, $\mathcal{R}^*$-class, respectively] contain a unique idempotent. For an element a of such a semigroup, the unique idempotent in the $\mathcal{L}^*$-class [$\mathcal{R}^*$-class, respectively] containing a is denoted by $a^*[a^+]$. A [right, left, respectively] adequate semigroup S is called [right, left, respectively] type A if [$e + a = a + (e+a)^*, a + e = (a+e)^+ + a$] $e + a = a + (e+a)^*$ and $a + e = (a+e)^+ + a$ for $a \in S$ and $e \in E^+(S)$.

By the definition of C-rpp semigroups, it is not hard to see that a right type A semigroup is a C-rpp semigroup.

Lemma 3 (Corollary 2.8 in [4])**.** *Let $(S,+)$ be a right type A semigroup with semilattice of idempotents $E = E(S)$ and μ_L the largest congruence contained in $\mathcal{L}^*$. Then the following conditions are equivalent:*

(1) *$S/\mu_L \cong E$;*

(2) *$\mu_L = \mathcal{L}^*$;*

(3) *E is central in S;*

(4) *S is a strong semilattice of left cancellative monoids.*

Lemma 4 (Proposition 2.9 in [4])**.** *Let $(S, +)$ be a adequate semigroup with semilattice of idempotents $E = E(S)$ and μ the largest congruence contained in $\mathcal{H}^*$. Then the following conditions are equivalent:*

(1) *$S/\mu \cong E$;*

(2) *$\mu = \mathcal{H}^*$;*

(3) *E is central in S;*

(4) *S is a strong semilattice of cancellative monoids.*

From Lemma 2, it is known that a semigroup S is a C-rpp semigroup if and only if it is a strong semilattice of left cancellative monoids. Now, we will similarly give the definition of generalized C-rpp semirings and then investigate some of their equivalent characterizations and constructions.

Definition 2. A semiring R is said to be a generalized C-rpp semiring if it is a strong b-lattice of additively left cancellative shew-halfring.

In the following, for any $a \in R$, the unique idempotent in the $\overset{+}{\mathcal{H}}{}^*$-class containing a is denoted by a^0.

Theorem 1. *Assume that R is a generalized C-rpp semiring. Then the following conditions hold:*

(GCR1) *$(R, +)$ is a C-rpp semigroup;*

(GCR2) *$E^+(R)$ is a b-lattice;*

(GCR3) *for any $a, b \in S$, $(ab)^0 + a^0b^0 = a^0b^0$;*

(GCR4) *if $a^0 = b^0$ and $a + e = b + e$ for $a, b \in R$ and some $e \in E^+(R)$, then $a = b$.*

Proof. Assume that R is a generalized C-rpp semiring, then it is a strong b-lattice of additively left cancellative skew-halfrings, say $R = \langle T, R_\alpha, \phi_{\alpha,\beta} \rangle$, where R_α are the additively left cancellative skew-halfrings in which the zero of additive reduct is denoted by 0_α and T is a b-lattice.

i) Since R is a strong b-lattice of left additively cancellative skew-halfrings R_α, $(R, +)$ is a strong semilattice of left cancellative monoids $(R_\alpha,+)$, by Lemma 2, $(R,+)$ is a C-rpp semigroup, and condition (GCR1) holds.

ii) Notice that $E^+(R) = \{0_\alpha \mid \alpha \in T\} \cong T$, where T is a b-lattice, then $(E^+(R), +, \cdot)$ is also a b-lattice, and condition (GCR2)holds.

iii) We will show that $a^0 = 0_\alpha$ for any $a \in R_\alpha$ at first. In fact, notice that $(R, +)$ is a strong semilattice of left cancellative monoids $(R_\alpha, +)$, by Lemma 3, we obtain that $a \overset{+}{\mathcal{L}^*} = \overset{+}{R_\alpha}$ for any $a \in R_\alpha$. And then, $a^0 = 0_\alpha$. Thus, for any $a \in R_\alpha$, $b \in R_\beta$, we have $(ab)^0 = 0_{\alpha\beta} = 0_\alpha 0_\beta = a^0 b^0$, $(ab)^0 + a^0 b^0 = a^0 b^0$. The condition (GCR3)holds.

iv) Assume that $a^0 = b^0$ and $a + e = b + e$ for $a, b \in R$ and some $e \in E^+(R)$, then there exist $\alpha, \beta \in T$, s.t., $a, b \in R_\alpha$ and $e \in E(R_\beta)$. Now let $f = a^0 + e$. Then

$$a + f = b + f,$$

i.e.,

$$a\varphi_{\alpha,\alpha+\beta} + f\varphi_{\alpha+\beta,\alpha+\beta} = b\varphi_{\alpha,\alpha+\beta} + f\varphi_{\alpha+\beta,\alpha+\beta}.$$

Since $\varphi_{\alpha+\beta,\alpha+\beta}$ is a monomorphism, we have

$$f\varphi_{\alpha+\beta},_{\alpha+\beta} = f,$$

where f is the additive identity of $S_{\alpha+\beta}$. And then, we have

$$a\varphi_{\alpha,\alpha+\beta} = b\varphi_{\alpha,\alpha+\beta}.$$

Also, notice that $\varphi_{\alpha,\alpha+\beta}$ is a monomorphism, we immediately get $a = b$. The condition (GCR4) holds. □

Actually, the converse of the above theorem also holds. To show this, we need the following proposition.

Proposition 1. *If R satisfies the conditions* (GCR1)–(GCR3), *then the following conclusions hold:*

(1) *$\mathcal{L}^*$ is a semiring congruence;*

(2) *$R/\mathcal{L}^* \cong E^+(R)$.*

Proof. (1) Assume that R satisfies the conditions (GCR1)–(GCR3), we will show that $\mathcal{L}^*$ is a semiring congruence.

Firstly, since (GCR1) holds, by Lemma 3, $(R, +)$ is a strong semilattice Y of left cancellative monoids $(R_\alpha, +)$, where $Y \cong (E^+(R), +)$. By Lemma 3 again, $(R, +)/\mathcal{L}^* \cong (E^+(R), +)$, we obtain that $\mathcal{L}^*$ is a semilattice congruence on $(R, +)$. To show that $\mathcal{L}^*$ is a semiring congruence, we only need to prove that $\mathcal{L}^*$ is a multiplicative congruence on $(R, \cdot)$, i.e., for any $a, b \in R$,

$$(ab)^0 = a^0 b = ab^0 = a^0 b^0.$$

In fact, for any $a, b \in R$, since $a^0b, ab^0, a^0b^0 \in E^+(R)$, by condition (GCR2), we have

$$(ab)^0 = [(a + a^0)(b + b^0)]^0 = (ab + a^0b + ab^0 + ab^0)^0 = (ab)^0 + a^0b + ab^0 + a^0b^0.$$

And then

$$(ab)^0 + a^0b^0 = (ab)^0.$$

Together with (GCR3), we have

$$(ab)^0 = a^0b^0.$$

Notice that $(ab)^0 + a^0b = (ab)^0$ and $(ab)^0 + ab^0 = (ab)^0$ also hold, we immediately get

$$(ab)^0 = a^0b = ab^0 = a^0b^0.$$

(2) Define a mapping

$$\phi : S/\mathcal{L}^* \to E^+(S), \quad a\mathcal{L}^* \mapsto a^0.$$

It is a routine way to check that ϕ is bijective, and

$$(a\mathcal{L}^* + b\mathcal{L}^*)\phi = [(a + b)\mathcal{L}^*]\phi = (a + b)^0 = a^0 + b^0,$$
$$[(a\mathcal{L}^*)(b\mathcal{L}^*)]\phi = [(ab)\mathcal{L}^*]\phi = (ab)^0 = a^0b^0.$$

Thus, $S/\mathcal{L}^* \cong E^+(S)$. □

Now, we have the following theorem.

Theorem 2. *A semiring R is a generalized C-rpp semiring if and only if it satisfies the conditions* (GCR1)–(GCR4).

Proof. We only need to show the sufficiency. By Proposition 1, it is known that if S satisfies the conditions (GCR1)–(GCR4), then $\mathcal{L}^*$ is a semiring congruence on $(R, +, .)$, and $R/\mathcal{L}^* \cong E^+(R)$ is a b-lattice. Also, for any $a \in R$, notice that $(L_a^*, +)$ is an additively left cancellative monoid, we obtain that R is a b-lattice of additively left cancellative skew-halfrings.

For any $e, f \in E^+(R)$ with $e \leq_+ f$, define a mapping

$$\phi_{e,f} : L_e^* \to L_f^*, \quad a \mapsto a + f.$$

In the following, we will show that $R = \langle E^+(R), L_e^*, \phi_{e,f} \rangle$ is a strong b-lattice of the additively left cancellative skew-halfrings $L_e^*, e \in E^+(R)$. That is, we will show that $\phi_{e,f}$ satisfies the conditions of strong b-lattice.

For any $a, b \in L^*_e$,

$$(a+b)\phi_{e,f} = a+b+f = a+(f+b+f) = (a+f)+(b+f) = a\phi_{e,f} + b\phi_{e,f}.$$

Also, since $af \in L^*_{ef} \cap E^+(R)$, we have $af = ef$. And then,

$$\begin{aligned}(ab)\phi_{e,f} &= ab+f = ab+e+f = ab+(e+f)^2 = ab+e+ef+fe+f\\ &= ab+af+fb+f = (a+f)(b+f) = a\phi_{e,f}b\phi_{e,f}.\end{aligned}$$

Hence, $\phi_{e,f}$ is a semiring morphism.

For any $a, b \in L^*_e$, if $a\phi_{e,f} = b\phi_{e,f}$, we have $a+f = b+f$. Notice that $(L^*_e, +)$ is an additively left cancellative monoid, we will get $a^0 = b^0 = e$. It follows from (GCR4) that $a = b$. Thus, $\phi_{e,f}$ is a semiring monomorphism.

Moreover, we can check that the monomorphism $\phi_{e,f}$ satisfies the conditions (1.1)–(1.4) of Definition 1.

(i) $\phi_{e,e}$ is clearly an identity morphism.

(ii) For any $e, f, g \in E^+(R)$ with $e \leq_+ f \leq_+ g$, we have

$$a\phi_{e,f}\phi_{f,g} = a+f+g = a+g = a\phi_{e,g}.$$

Hence, $\phi_{e,f}\phi_{f,g} = \phi_{e,g}$.

(iii)For any $e, f, g \in E^+(R)$, if $e+f \leq_+ g$, then for any $a \in L^*_e$, $b \in L^*_f$,

$$\begin{aligned}a\phi_{e,g}b\phi_{f,g} &= (a+g)(b+g) = ab+ag+gb+g = ab+eg+gf+g\\ &= ab+g = (ab)\phi_{ef,g}.\end{aligned}$$

(iv) For any $e, f \in E^+(R)$, $a \in L^*_e$, $b \in L^*_f$, we have

$$\begin{aligned}a\phi_{e,e+f} + b\phi_{f,e+f} &= (a+e+f)+(b+e+f) = a+f+b+e\\ &= a+b+e+f = a+b;\end{aligned}$$

$$\begin{aligned}a\phi_{e,e+f}b\phi_{f,e+f} &= (a+e+f)(b+e+f) = ab+a(e+f)+(e+f)b+(e+f)\\ &= ab+(e+f) = (ab)\phi_{ef,e+f}\end{aligned}$$

Thus, we have shown that R is a strong b-lattice of additively left cancellative skew-halfrings. And then it is a generalized C-rpp semiring. □

Example 1. Let $(A, +)$ and $(B, +)$ be the infinite cyclic monoids generated by a and b respectively. Let $M = A \cup B \cup \{0\}$ with additive identity 0 and addition $+$ defined by

$$ma + nb = (m+n)b, nb + ma = (n+m)a$$

for any $m, n \in N^+$. Also, we define the multiplication $\cdot$ of M as follows: $s_1 \cdot s_2 = 0$ for any $s_1, s_2 \in M$, it is a routine way to check that $(M, +, \cdot)$ is an additively left cancellative skew-halfring.

On the other hand, let $D = \{e, f\}$ such that $e+e = e \cdot e = e, f+f = e+f = f+e = f \cdot f = e \cdot f = f \cdot e = f$. Then $(D, +, \cdot)$ is a b-lattice.

Now, construct the direct product of D and M, and denote it by R, i.e., $R = D \times M$. Then, we can check that $E^+(R) = D \times \{e_M\}$, where e_M is the identity element of $(M, +)$. It is also not hard to check that $(R, +, .)$ is a semiring which satisfies the following conditions:

(i) $(R, +)$ is a C-rpp semigroup;

(ii) $(E^+(R), +, \cdot)$ is a b-lattice;

(iii) for any $a, b \in R$, $(ab)^0 + a^0b^0 = a^0b^0$;

(iv) if $a^0 = b^0$ and $a + e = b + e$ for $a, b \in R$ and some $e \in E^+(R)$, then $a = b$.

By Theorem 2, $(R, +, .)$ is just a generalized C-rpp semiring.

Next, we will give another construction of generalized C-rpp semirings. Recall that a subdirect product algebra T is a subalgebra of a direct product of algebras such that the projection mapping from the algebra T to each of its components is surjective.

Theorem 3. *A semiring R is a generalized C-rpp semiring if and only if it is a subdirect product of a b-lattice and an additively left cancellative shew-halfring.*

Proof. ($\Leftarrow$) Suppose that R is a subdirect product of a b-lattice T and an additively left cancellative shew-halfring M. Consider $R \subseteq T \times M$. For each $\alpha \in T$, let $R_\alpha = (\{\alpha\} \times M) \cap R$. Then R_α is an additively left cancellative shew-halfring for each $\alpha \in T$ and $R = \cup_{\alpha \in T} R_\alpha$. Now for each pair $\alpha, \beta \in T$ with $\alpha \leq_+ \beta$, define a mapping

$$\phi_{\alpha,\beta} : R_\alpha \to R_\beta, (\alpha, r)\phi_{\alpha,\beta} = (\beta, r).$$

Then $\phi_{\alpha,\beta}$ is clearly a monomorphism satisfying the conditions $\phi_{\alpha,\alpha} = I_{R_\alpha}$ and $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \leq_+ \beta \leq_+ \gamma$ for $\alpha, \beta, \gamma \in T$.

Let $\alpha, \beta, \gamma \in T$ be such that $\alpha + \beta \leq_+ \gamma$. Denote $a = (\alpha, r) \in R_\alpha$, $b = (\beta, r') \in R_\beta$. And then

$$a + b = (\alpha, r) + (\beta, r') = (\alpha + \beta, r + r') \in R_{\alpha+\beta}$$

and

$$ab = (\alpha, r)(\beta, r') = (\alpha\beta, rr') \in R_{\alpha\beta}.$$

Now, we have

$$(a\phi_{\alpha,\gamma})(b\phi_{\beta,\gamma}) = (\alpha\beta, rr')\phi_{\alpha\beta,\gamma} = (ab)\phi_{\alpha\beta,\gamma}.$$

Also, since

$$a+b = (\alpha,r)+(\beta,r') = (\alpha+\beta,r+r') = (\alpha+\beta,r)+(\alpha+\beta,r') = a\phi_{\alpha,\alpha+\beta}+b\phi_{\beta,\alpha+\beta}$$

and

$$\begin{aligned}(a\phi_{\alpha,\alpha+\beta})(b\phi_{\beta,\alpha+\beta}) &= (\alpha+\beta,r)(\alpha+\beta,r') = (\alpha+\beta,rr') \\ &= (\alpha\beta,rr')\phi_{\alpha\beta,\alpha+\beta} = (ab)\phi_{\alpha\beta,\alpha+\beta},\end{aligned}$$

R is a strong b-lattice of additively left cancellative shew-halfrings. Hence, R is a generalized C-rpp semiring.

($\Rightarrow$) Assume that R is a generalized C-rpp semiring. We will show that it is a subdirect product of a b-lattice and an additively left cancellative shew-halfring by the following steps.

Firstly, from Proposition 1, $\mathcal{L}^*$ is a semilattice congruence on $(R,+)$ and a semiring congruence on R. Also, since R is a generalized C-rpp semiring, we have $aa^0 = a$ for any $a \in R$, and then $a = aa^0\mathcal{L}^*a^2$. Hence, $R/\mathcal{L}^*$ is an idempotent semiring with the semilattice additive reduct and band multiplicative reduct, i.e., $R/\mathcal{L}^*$ is a b-lattice.

Secondly, define a binary relation

$$\theta = \{(a,b) \mid (\exists e \in E^+(R))a+e = b+e\}.$$

It can be easily seen that θ is an equivalence relation on R. Moreover, θ is the minimum additively left cancellative shew-halfring congruence on $(R,+,.)$. In fact, by the Proposition 1.7 in [18], θ is a minimum left cancellative monoid congruence on the additive reduct $(R,+)$. Also, if $a\theta b$ for some $a,b \in R$, there exists $e \in E^+(R)$ such that $a+e = b+e$. Now, for any $c \in R$, we have

$$ac+ec = bc+ec, ca+ce = cb+ce.$$

Notice that $ce, ec \in E^+(R)$. We immediately obtain that

$$ac\theta bc, ca\theta cb.$$

Thus, we have shown that θ is the minimum additively left cancellative shew-halfring congruence on $(R,+,.)$. This also shows that R/θ is an additively left cancellative shew-halfring.

Finally, define a mapping

$$\Phi : R \to R/\theta \times R/\mathcal{L}^*, \quad a \mapsto (a\theta, a\mathcal{L}^*).$$

It is a routine way to check that R can be embedded into $R/\theta \times R/\mathcal{L}^*$, and the projection mapping from R into each of its components is surjective. Consequently, R is a subdirect product of a b-lattice and an additively left cancellative shew-halfring. □

So we have obtained some constructions and characterizations of generalized C-rpp semirings. In the following, we will investigate another class of additive non-regular C-semirings, called C-rpp semirings.

Definition 3. A semiring R is said to be a C-rpp semiring if it is a strong distributive lattice of additively left cancellative shew-halfrings.

Theorem 4. *Assume that R is a C-rpp semiring. Then the following conditions hold:*

(CR1) *$(R, +)$ is a C-rpp semigroup;*

(CR2) *$(E^+(R), +, \cdot)$ is a distributive lattice;*

(CR3) *for any $a, b \in R$, $(ab)^0 + a^0b^0 = a^0b^0$;*

(CR4) *if $a^0 = b^0$ and $a + e = b + e$ for $a, b \in R$ and some $e \in E^+(R)$,then $a = b$.*

Conversely, if a semiring R satisfies the conditions (CR1)–(CR4)*, then it is a C-rpp semiring.*

Proof. (⇒) From Definition 2 and Definition 3, it is known that a C-rpp semiring is a generalized C-rpp semiring. Thus, by Theorem 2, (CR1), (CR3),(CR4) hold. We only need to prove that (CR2) holds.

Assume that S is a C-rpp semiring. Then it is a strong distributive lattice of additively left cancellative skew-halfrings, say $R = \langle D, R_\alpha, \phi_{\alpha,\beta}\rangle$, where each R_α is an additively left cancellative skew-halfring in which the additive identity is denoted by 0_α and T is a distributive lattice. Notice that $E^+(R) = \{0_\alpha \mid \alpha \in T\} \cong T$, we immediately obtain that $(E^+(R), +, \cdot)$ is also a distributive lattice. Hence, (GC2) holds.

(⇐) Assume that the semiring R satisfies the conditions (CR1)–(CR4), then by Theorem 2, it is clearly a generalized C-rpp semiring. Also, note that (CR2) holds. By analogy with the discussions of Theorem 1, R is a C-rpp semiring. □

Example 2. Let $(M = A \cup B \cup \{0\}, +, \cdot)$ be an additively left cancellative skew-halfring as defined in Example 1. Let $D = \{e, f\}$ be such that $e + e = e \cdot e =$

$e+f=f+e=e, f+f=f\cdot f=e\cdot f=f\cdot e=f$. Then $(D,+,\cdot)$ is a distributive lattice.

Now, Construct the direct product of D and M and denote it by R, i.e., $R=D\times M$. Clearly, $E^+(R)=D\times\{e_M\}$, where e_M is the identity element of $(M,+)$. It is also not hard to check that $(R,+,.)$ is a semiring which satisfies the following conditions:

(i) $(R,+)$ is a C-rpp semigroup;

(ii) $(E^+(R),+,\cdot)$ is a distributive lattice;

(iii) for any $a,b\in R$, $(ab)^0+a^0b^0=a^0b^0$;

(iv) if $a^0=b^0$ and $a+e=b+e$ for $a,b\in R$ and some $e\in E^+(R)$, then $a=b$.

Thus, by Theorem 4, $(R,+,.)$ is a C-rpp semiring.

Further, by analogy with the discussions of the subdirect decompositions of generalized C-rpp semirings, we have the following theorem.

Theorem 5. *A semiring R is a C-rpp semiring if and only if it is a subdirect product of a distributive lattice and an additively left cancellative skew-halfring.*

3. Generalized C-abundant semirings and C-abundant semirings

In this section, we will study generalized C-abundant semirings and C-abundant semirings, and will show that a semiring is a generalized C-abundant semiring (C-abundant semiring, respectively) if and only if it is a strong b-lattice (strong distributive lattice, respectively) of additively cancellative skew-halfrings, and if and only if it is a subdirect product of a b-lattice (distributive lattice, respectively) and an additively cancellative skew-halfring. Also, we will give some characterizations of such semirings.

Firstly, by Lemma 2 and its dual, we immediately have

Lemma 5. *A semigroup S is a C-a(or C-abundant) semigroup if and only if it is a strong semilattice of cancellative monoids.*

Definition 4. A semiring R is said to be a generalized C-abundant semiring if it is a strong b-lattice of additively cancellative skew-halfrings.

Theorem 6. *Assume that R is a generalized C-abundant semiring, then the following conditions hold:*

(GCA1) *$(R,+)$ is a C-abundant semigroup;*

(GCA2) *$(E^+(R),+,\cdot)$ is a b-lattice;*

(GCA3) *for any* $a, b \in R$, $(ab)^0 + a^0b^0 = a^0b^0$*;*

(GCA4) *if* $a^0 = b^0$ *and* $a + e = b + e$ *for* $a, b \in S$ *and some* $e \in E^+(R)$, *then* $a = b$.

PROOF. From Definition 2 and Definition 4, it is known that, a generalized C-abundant semiring is a generalized C-rpp semiring, then the conditions (GCA2)–(GCA4) hold. We only need to show that condition (GCA1) holds.

Actually, if R is a generalized C-abundant semiring, then it is a strong b-lattice of additively cancellative skew-halfrings, say $R = \langle T, R_\alpha, \phi_{\alpha,\beta}\rangle$, where each R_α is an additively cancellative skew-halfring and T is a b-lattice. It follows that $(R, +)$ is strong semilattice of cancellative monoids $(R_\alpha, +)$, i.e., $(R, +)$ is a C-abundant semigroup. Thus, the condition (GCA1) holds. □

Proposition 2. *Assume that a semiring* R *satisfies the conditions* (GCA1)–(GCA3). *Then the following conclusions hold:*

(1) $\mathcal{H}^*$ *is a semiring congruence;*

(2) $R/\mathcal{H}^* \cong E^+(R)$.

PROOF. (1) Assume that semiring R satisfies the conditions (GCA1)–(GCA3). We will show that $\mathcal{H}^*$ is a semiring congruence.

Firstly, since condition (GCA1) holds, by Lemma 4 or Lemma 5, $(R, +)$ is a strong semilattice Y of cancellative monoids R_α, where $Y \cong (E^+(R), +)$. By Lemma 4 again, we obtain that $\mathcal{H}^* = \mathcal{L}^* = \mathcal{R}^*$ is a semilattice congruence on $(R, +)$. And then, by analogy with with the discussions of Proposition 1, we can get $\mathcal{H}^* = \mathcal{L}^* = \mathcal{R}^*$ is a semiring congruence.

(2) Define a mapping

$$\phi : R/\mathcal{H}^* \to E^+(R), a\mathcal{H}^* \mapsto a^0.$$

It is not hard to check that ϕ is bijective, and

$$(a\mathcal{H}^* + b\mathcal{H}^*)\phi = [(a + b)\mathcal{H}^*]\phi = (a + b)^0 = a^0 + b^0,$$

$$[(a\mathcal{H}^*)(b\mathcal{H}^*)]\phi = [(ab)\mathcal{H}^*]\phi = (ab)^0 = a^0b^0.$$

Thus, $R/\mathcal{H}^* \cong E^+(R)$. □

Theorem 7. *A semiring* R *is a generalized* C*-abundant semiring if and only if it satisfies the conditions* (GCA1)–(GCA4).

Proof. We only need to show the sufficiency. By Proposition 2, it is known that if S satisfies the conditions (GCA1)–(GCA4), then $\mathcal{H}^*$ is a semiring congruence on $(R,+,.)$, and $R/\mathcal{H}^* \cong E^+(R)$ is a b-lattice. Also, notice that $(H_a^*,+)$ is an additively cancellative monoid, we obtain that R is a b-lattice of additively cancellative skew-halfrings.

For any $e,f \in E^+(R)$ with $e \leq_+ f$, define mapping

$$\phi_{e,f} : H_e^* \to H_f^*, a \mapsto a+f.$$

In the following, we begin to show that $R = \langle E^+(R), R_e, \phi_{e,f}\rangle$ is a strong b-lattice of the semirings $R_e, e \in E^+(R)$.

For any $a,b \in H_e^*$,

$$(a+b)\phi_{e,f} = a+b+f = a+(f+b+f) = (a+f)+(b+f) = a\phi_{e,f} + b\phi_{e,f}.$$

Also, since $af \in H_{ef}^* \cap E^+(R)$, we have $af = ef$. And then,

$$\begin{aligned}(ab)\phi_{e,f} &= ab+f = ab+e+f = ab+(e+f)^2 = ab+e+ef+fe+f \\ &= ab+af+fb+f = (a+f)(b+f) = a\phi_{e,f}b\phi_{e,f}.\end{aligned}$$

Hence, $\phi_{e,f}$ is a semiring morphism.

For any $a,b \in H_e^*$,if $a\phi_{e,f} = b\phi_{e,f}$, we have $a+f = b+f$. Notice that $(H_e^*,+)$ is an additively cancellative monoid, we have $a^0 = b^0 = e$. By condition (GCA4), we have $a = b$. Thus, $\phi_{e,f}$ is a semiring monomorphism.

Moreover,we can check that the monomorphism $\phi_{e,f}$ satisfies the conditions (1.1)–(1.4) of Definition 1.

(i) $\phi_{e,e}$ is clearly an identity morphism.

(ii) For any $e,f,g \in E^+(R)$ with $e \leq_+ f \leq_+ g$, we have

$$a\phi_{e,f}\phi_{f,g} = a+f+g = a+g = a\phi_{e,g}.$$

Hence, $\phi_{e,f}\phi_{f,g} = \phi_{e,g}$.

(iii) For any $e,f,g \in E^+(S)$, if $e+f \leq_+ g$, then for any $a \in H_e^*$, $b \in H_f^*$,

$$a\phi_{e,g}b\phi_{f,g} = (a+g)(b+g) = ab+ag+gb+g = ab+eg+gf+g = (ab)\phi_{ef,g},$$

i.e.,

$$\phi_{e,g}b\phi_{f,g} = \phi_{ef,g}.$$

(iv) For any $e, f \in E^+(R)$, $a \in H^*_e$, $b \in H^*_f$, we have

$$\begin{aligned} a\phi_{e,e+f} + b\phi_{f,e+f} &= (a+e+f) + (b+e+f) = a+f+b+e \\ &= a+b+(e+f) = a+b; \end{aligned}$$

$$\begin{aligned} a\phi_{e,e+f} b\phi_{f,e+f} &= (a+e+f)(b+e+f) = ab + a(e+f) + (e+f)b + (e+f) \\ &= ab + (e+f) = (ab)\phi_{ef,e+f}. \end{aligned}$$

Thus, we have shown that R is a strong b-lattice of additively cancellative skew-halfrings, and then it is a generalized C-abundant semiring. □

Example 3. Let T be a b-lattice and M an additively cancellative skew-halfring. Construct the direct product of T and R, and denote it by M, i.e., $R = T \times M$. Then, we can check that $E^+(R) = T \times \{e_M\}$, where e_M is the identity element of $(M, +)$. We can also check that $(R, +, .)$ is a semiring which satisfies the conditions (GCA1)–(GCA4). Thus, by Theorem 7, $(R, +, .)$ is really a generalized C-abundant semiring.

Theorem 8. *A semiring R is a generalized C-abundant semiring if and only if it is a subdirect product of a b-lattice and an additively cancellative skew-halfring.*

Proof. ($\Leftarrow$) By Theorem 3 and its dual, the sufficiency is clear.

($\Rightarrow$) Assume that R is a generalized C-rpp semiring, we will show that it is a subdirect product of a b-lattice and an additively cancellative skew-halfring by the following steps.

Firstly, from Proposition 2, $\mathcal{H}^*$ is a semilattice congruence on $(R, +)$ and a semiring congruence on R. Also, since R is a generalized C-a semiring, we have $aa^0 = a = a^0a$ for any $a \in R$, and then $a = aa^0\mathcal{H}^*a^2$. Hence, $R/\mathcal{H}^*$ is an idempotent semiring with the semilattice additive reduct and band multiplicative reduct, i.e., $R/\mathcal{H}^*$ is a b-lattice.

Secondly, define a binary relation

$$\theta = \{(a, b) \mid (\exists e \in E^+(R))a + e = b + e\}.$$

It can be easily seen that θ is an equivalence relation on $(R, +, .)$. Moreover, θ is the minimum additively cancellative skew-halfring congruence on $(R, +, .)$. In fact, by the Proposition 1.7 in [18] and its dual, θ is a minimum cancellative monoid congruence on the additive reduct $(R, +)$. Also, if $a\theta b$ for some $a, b \in R$, there exists $e \in E^+(R)$ such that $a + e = b + e$. Now, for any $c \in R$, we have

$$ac + ec = bc + ec, ca + ce = cb + ce.$$

Notice that $ce, ec \in E^+(R)$, we immediately obtain that

$$ac\theta bc, ca\theta cb.$$

Thus, we have shown that θ is the minimum additively cancellative skew-halfring congruence on $(R, +, .)$. This also shows that R/θ is an additively cancellative skew-halfring.

Finally, define a mapping

$$\Phi : R \to R/\theta \times R/\mathcal{H}^*, \quad a \mapsto (a\theta, a\mathcal{H}^*).$$

It is a routine way to check that R can be embed into $R/\theta \times R/\mathcal{H}^*$,and the projection mapping from R into each of its components is surjective. Consequently, R is a subdirect product of a b-lattice and an additively cancellative skew-halfring. □

Remark 1. From Theorem 8, we can see that the class of generalized C-abundant semirings is actually a general extension of the class of generalized Clifford semirings studied in [29].

At the end of this section, we will study C-abundant semirings.

Definition 5. A semiring R is said to be a C-abundant semiring if it is a strong distributive lattice of additively cancellative skew-halfrings.

Some characterizations of such semirings are also given below.

Theorem 9. *If R is a C-abundant semiring, then the following conditions hold:*

(CA1) $(R, +)$ *is a C-abundant semigroup;*

(CA2) $(E^+(R), +, \cdot)$ *is a distributive lattice;*

(CA3) *for any $a, b \in R$,$(ab)^0 + a^0b^0 = a^0b^0$;*

(CA4) *if $a^0 = b^0$ and $a + e = b + e$ for $a, b \in S$ and some $e \in E^+(R)$,then $a = b$.*

Conversely, if a semiring R satisfies the conditions (CA1)–(CA4), *then it is a C-abundant semiring.*

Proof. (⇒) By Definition 4 and Definition 5, a C-a semiring is clearly a generalized C-abundant semiring. Thus, by Theorem 6, condition (CA1), (CA3), (CA4) hold. We only need to prove that condition (CA2) holds.

Assume that R is a C-abundant semiring. Then it is a strong distributive lattice of additively cancellative skew-halfrings, say $R = \langle D, R_\alpha, \phi_{\alpha,\beta}\rangle$, where each R_α is an additively cancellative skew-halfring in which the additive identity is denoted by 0_α and D is a distributive lattice. Notice that $E^+(R) = \{0_\alpha \mid \alpha \in$

$D\} \cong D$, we immediately obtain that $(E^+(R), +, \cdot)$ is also a distributive lattice. (CA2) holds.

($\Leftarrow$) Assume that the semiring R satisfies the conditions (CA1)–(CA4), then by Theorem 7, it is clearly a generalized C-abundant semiring. Also, note that (CA2) holds, by analogy with the discussions of Theorem 1, together with Definition 5, R is a C-abundant semiring. □

Example 4. Let D be a distributive lattice and M an additively cancellative skew-halfring. Construct the direct product of D and M, and denote it by R, i.e., $R = D \times M$. Then, $E^+(R) = D \times \{e_M\}$, where e_M is the identity element of $(M, +)$. We can also check that $(R, +, .)$ is a semiring which satisfies the conditions (CA1)–(CA2). Thus, by Theorem 9, $(R, +, .)$ is a C-abundant semiring.

By analogy with the discussions of the subdirect decompositions of the generalized C-a semirings, we will have the following theorem.

Theorem 10. *A semiring R is a C-abundant semiring if and only if it is a subdirect product of a distributive lattice and an additively cancellative skew-halfring.*

Remark 2. From Theorem 10, we can see that the class of C-abundant semirings is actually a general extension of the one of Clifford semirings studied in [7] and [29].

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