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A condition for a Finsler space to be a Riemannian space

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0. Introduction

In a Finsler space, there are known canonical Finsler connections, that is, the Cartan connection $C\Gamma$, the Berwald connection $B\Gamma$, the Rund connection $R\Gamma$ and the Hashiguchi connection $H\Gamma$.

The concept of the generalized Berwald P^1-connections was introduced by AIKOU and HASHIGUCHI [1] and MATSUMOTO [2]. In the previous paper [4], we considered a generalized Berwald P^1-connection Γ. Using this Finsler connection, we obtained a condition for a Finsler space to be a Riemannian space in each of the above-mentioned Finsler connections $C\Gamma$, $B\Gamma$, $R\Gamma$, $H\Gamma$ and Γ, respectively [4].

The purpose of the present paper is to improve these conditions for a Finsler space to be a Riemannian space and to make up these conditions into a condition valid for any of the Finsler connections mentioned above.

Throughout the present paper, the terminology and notations are those of MATSUMOTO's monograph [3].

1. Preliminaries

Let $F^n = (M, L)$ be a Finsler space with a Finsler metric $L(x, y)$, where x denotes a point of the underlying manifold M and y denotes a supporting element. The fundamental tensor g_{ij} is given by $g_{ij} = \frac{1}{2}\partial^2 L^2/\partial y^i \partial y^j$. We shall express a Finsler connection $F\Gamma(F_h{}^i{}_j,\, N^i{}_j,\, C_h{}^i{}_j)$ in terms of its coefficients. The $(v)hv$-torsion tensor $P^i{}_{jk}$ and the hv-curvature tensor $P_h{}^i{}_{jk}$ of a Finsler connection $F\Gamma$ are given by

$$P^i{}_{jk} = \dot\partial_k N^i{}_j - F_k{}^i{}_j, \tag{1.1}$$

and

$$P_h{}^i{}_{jk} = \dot\partial_k F_h{}^i{}_j - C_h{}^i{}_{k|j} + C_h{}^i{}_r P^r{}_{jk}, \tag{1.2}$$

respectively, where "$|$" denotes the h-covariant differentiation with respect to $F\Gamma$ and $\dot\partial_k = \partial/\partial y^k$.

As well-known, the Berwald connection $B\Gamma$ is given by $B\Gamma = (G_h{}^i{}_j, G^i{}_j, 0)$, where

$$G_h{}^i{}_j = \frac{1}{2}\dot\partial_h\dot\partial_j(\gamma_m{}^i{}_r y^m y^r),$$
$$\gamma_m{}^i{}_r = \frac{1}{2} g^{ij}(\partial g_{jm}/\partial x^r + \partial g_{jr}/\partial x^m - \partial g_{mr}/\partial x^j),$$
$$G^i{}_j = G_h{}^i{}_j y^h = \frac{1}{2}\dot\partial_j(\gamma_m{}^i{}_r y^m y^r).$$

Then the known Finsler connections $B\Gamma$, $H\Gamma$, $R\Gamma$, $C\Gamma$ and our Finsler connections Γ and Γ' are denoted by the following table:

Finsler connection	$F\Gamma$	$F_h{}^i{}_j$	$N^i{}_j$	$C_h{}^i{}_j$
Berwald	$B\Gamma$	$G_h{}^i{}_j$	$G^i{}_j$	0
Hashiguchi	$H\Gamma$	$G_h{}^i{}_j$	$G^i{}_j$	$g_h{}^i{}_j$
Rund	$R\Gamma$	$G_h{}^i{}_j - g_h{}^i{}_{j\|0}$	$G^i{}_j$	0
Cartan	$C\Gamma$	$G_h{}^i{}_j - g_h{}^i{}_{j\|0}$	$G^i{}_j$	$G_h{}^i{}_j$
	Γ	$G_h{}^i{}_j + Lg_h{}^i{}_j$	$G^i{}_j$	0
	Γ'	$G_h{}^i{}_j + Lg_h{}^i{}_j$	$G^i{}_j$	$g_h{}^i{}_j$

,

where $g_h{}^i{}_j = \frac{1}{2} g^{ir}\dot\partial_r g_{hj}$ and $g_h{}^i{}_{j|0} = g_h{}^i{}_{j|k} y^k$.

For the above-mentioned Finsler connections, the following equations are satisfied:

$$F_h{}^i{}_j = G_h{}^i{}_j - P^i{}_{jh}, \tag{1.3}$$

$$P_0{}^i{}_{jk} := P_h{}^i{}_{jk} y^h = P^i{}_{jk}. \tag{1.4}$$

Using (1.3), (1.2) is rewritten as follows:

$$P_h{}^i{}_{jk} = G_h{}^i{}_{j\cdot k} - P^i{}_{hj\cdot k} - C_h{}^i{}_{k|j} + C_h{}^i{}_r P^r{}_{jk}, \tag{1.5}$$

where $\cdot k = \dot{\partial}_k$.

2. The main result

In this section, we shall assume that the Finsler connection $F\Gamma$ is one of the Finsler connections $B\Gamma$, $H\Gamma$, $R\Gamma$, $C\Gamma$, Γ and Γ'.

Theorem. *Let $A_h{}^i{}_j = Lg_h{}^i{}_j$. Then a Finsler space F^n is Riemannian, if and only if*

$$P_h{}^i{}_{jk} = -A_h{}^i{}_{j\cdot k} \ . \tag{2.1}$$

To prove the theorem, we shall prove the following

Lemma. *Let F^n be a Finsler space with the hv-curvature tensor $P_h{}^i{}_{jk}$ satisfying* (2.1). *Then F^n is a Landsberg space satisfying*

$$P^i{}_{jk} = A_j{}^i{}_k. \tag{2.2}$$

Proof of the Lemma. Contracting (2.1) by y^h and paying attention to (1.4), we have (2.2). Thus, from (1.5), (2.1) and (2.2), we get

$$-A_h{}^i{}_{j\cdot k} = G_h{}^i{}_{j\cdot k} - A_h{}^i{}_{j\cdot k} - C_h{}^i{}_{k|j} + C_h{}^i{}_r\, A_j{}^r{}_k \ .$$

Hence we have

$$G_h{}^i{}_{j\cdot k} - C_h{}^i{}_{k|j} + C_h{}^i{}_r\, A_j{}^r{}_k = 0. \tag{2.3}$$

Case 1. If $F\Gamma$ is $H\Gamma$ or $C\Gamma$ or Γ', then we have $C_h{}^i{}_j = g_h{}^i{}_j$. In this case, contracting (2.3) by y^j, we have $g_h{}^i{}_{k|0} = 0$. This means that F^n is a Landsberg space.

Case 2. If $F\Gamma$ is $B\Gamma$ or $R\Gamma$ or Γ, then we have $C_h{}^i{}_j = 0$. Hence (2.3) reduces to $G_h{}^i{}_{j\cdot k} = 0$. This means that F^n is a Berwald space and hence a Landsberg space. Thus the lemma is proved.

Proof of the Theorem. The necessity of (2.1) is evident. Assume (2.1), then by the Lemma, we have that F^n is a Landsberg space satisfying (2.2). First, we shall consider cases of $H\Gamma$ and $B\Gamma$. For $H\Gamma$ and $B\Gamma$, the $(v)hv$-torsion tensor $P^i{}_{jk}$ vanishes. So from (2.2) we have $A_j{}^i{}_k = P^i{}_{jk} = 0$. Accordingly, F^n is a Riemannian space. Next, we shall consider cases of Γ and Γ'. For these Finsler connections we have $P^i{}_{jk} = -A_j{}^i{}_k$. So we have $A_j{}^i{}_k = P^i{}_{jk} = -A_j{}^i{}_k$. Hence $A_j{}^i{}_k = 0$. This means that F^n is Riemannian. Third, we shall consider cases of $C\Gamma$ and $R\Gamma$. For these Finsler connections, we have $P^i{}_{jk} = g_j{}^i{}_{k|0}$. So, noting that F^n is a Landesberg space, we obtain $A_j{}^i{}_k = P^i{}_{jk} = g_j{}^i{}_{k|0} = 0$. This means that F^n is Riemannian. Thus the proof of the Theorem is complete.

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