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Reduction of singularities of holomorphic maps of $\mathbf{C}^2$ tangent to the identity

By FENG RONG (Shanghai)

Abstract. We study the singularities of holomorphic maps of $\mathbf{C}^2$ tangent to the identity. We will adapt the method of Hironaka for the resolution of singularities of algebraic varieties to our study. In particular, we introduce a new numerical invariant associated to such maps at the singularities, called "adapted order", which behaves well under blow-ups. Though other methods for desingularizing such maps exist, our approach has the advantage that it has no essential obstruction for generalizing to higher dimensions.

1. Introduction

In [2], Abate generalized the well-known Leau–Fatou Flower Theorem in one-dimentional holomorphic dynamics to dimension two. The proof of this generalization suggests an interesting connection between discrete and continuous local holomorphic dynamics. Following this connection, some new results in the study of holomorphic maps, which have similar counterparts in the study of vector fields, have been obtained (see e.g. [3], [8], [9]).

We studied holomorphic maps of $\mathbf{C}^n$ tangent to the identity with absolutely isolated singularities in [9]. However, the method used in [9] does not apply to the case when the singularities are not absolutely isolated. For the general case, we

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propose to use the idea of Hironaka for the resolution of singularities of algebraic varieties ([6], [7]). In this paper, we deal with dimension two. A similar approach in the study of plane vector fields has been carried out by Cano ([5]).

Our main result is the following reduction theorem (see Section 2 for precise definitions).

Theorem 1. *Let f be a holomorphic map of $\mathbf{C}^2$, tangent to the identity at an isolated fixed point. Then there exists a finite sequence of blow-ups, which reduces f to a map whose adapted order at each of its singularities is less than or equal to one.*

2. Reduction of singularities

Let us first recall some basic definitions (cf. [3]). For simplicity, we give all definitions in dimension two.

Let M be a two-dimensional complex manifold and f a holomorphic self-map of M with $p \in M$ as a fixed point. Assume that f is *tangent to the identity* at p, that is $df_p = \mathrm{id}$. Write $f = (f_1, f_2)$, with $f_j(z) = z_j + g_j(z)$. The *order* of f at p is $\nu(f,p) = \min\{\nu(g_1), \nu(g_2)\}$, where $\nu(g_j)$ is the least $i \geq 0$ such that $P_{j,i}$ is not identically zero in the homogeneous expansion of g_j, $g_j = P_{j,0} + P_{j,1} + \dots$, with $\deg P_{j,i} = i$ or $P_{j,i} \equiv 0$. We always assume that $\nu(f,p) < \infty$. Set $l = \mathrm{g.c.d.}(g_1, g_2)$ and $g_j = l g_j^o$, with both l and g_j^o defined up to units in $\mathcal{O}_{M,p}$. The *pure order* of f at p is $\nu_o(f,p) = \min\{\nu(g_1^o), \nu(g_2^o)\}$. We say that p is a *singular point* or a *singularity* of f if $\nu_o(f,p) \geq 1$.

Let $P = (P_1, P_2)$ be a 2-tuple of homogeneous polynomials of degree ν in $\mathbf{C}^2$. A *characteristic direction* for P is a vector $v \in \mathbf{P}^1$ such that $P(v) = \lambda v$ for some $\lambda \in \mathbf{C}$. It is a *nondegenerate* characteristic direction if $\lambda \neq 0$, and *degenerate* otherwise. A *characteristic direction* for f at p is a characteristic direction for $P_{\nu(f,p)} = (P_{1,\nu(f,p)}, P_{2,\nu(f,p)})$. A *singular direction* for f at p is a characteristic direction for $P^o_{\nu_o(f,p)} = (P^o_{1,\nu_o(f,p)}, P^o_{2,\nu_o(f,p)})$. Here $P_{j,\nu(f,p)} = P^o_{j,\nu_o(f,p)} R_\sigma$, with R_σ being the first nonzero term in the homogeneous expansion of l and $\sigma = \nu(f,p) - \nu_o(f,p)$. The set of singular directions is clearly either a discrete set of points of $\mathbf{P}^1$ or the whole of $\mathbf{P}^1$. If the set of singular directions is a discrete set, we say that f is *non-dicritical* at p, otherwise we say that f is *dicritical* at p.

Let $\pi : \tilde{M} \to M$ be the blow-up of M at p. Then, there exists a unique map $\tilde{f}$, the *blow-up* of f at p, such that $\pi \circ \tilde{f} = f \circ \pi$ (see [1]). By [3, Lemma 2.2], if p is non-dicritical then a direction $v \in \mathbf{P}^1$ is singular for f if and only if it is a singularity of $\tilde{f}$. And by [9, Lemma 2.5], if p is dicritical then the pure order of $\tilde{f}$

at any of its singularities is less than the pure order of f at p. Thus the crucial part of the proof for Theorem 1 is to deal with non-dicritical points. Therefore, without loss of generality, we will assume that no dicritical points occur during the blow-ups.

We now start our investigation of holomorphic maps of $\mathbf{C}^2$ tangent to the identity at an isolated fixed point, which we assume to be the origin O.

First of all, note that the *pure order* is not a "good" numerical invariant, since it might increase after blow-ups at non-dicritical points. Therefore we are going to use the *adapted order* instead. Let M be a two-dimensional manifold and f a holomorphic self-map of M piontwise fixing $S \subset M$, where S is an analytic variety of M with only normal crossings as singularities. (For our purpose, we have either f being the map we start with and $S = O$ or f being the blow-up map and S being the exceptional divisor.) Let $p \in S$ be a singularity of f and let e_p be the number of irreducible components of S through p. Then $e_p = 0$, 1 or 2. If $e_p = 0$, choose local coordinates (x, y) and write f at p as

$$\begin{cases} x_1 = x + a(x, y) \\ y_1 = y + b(x, y). \end{cases} \tag{1}$$

If $e_p = 1$, choose local coordinates (x, y) such that the irreducible component is given by $\{x = 0\}$ (or $\{y = 0\}$). Write f at p as

$$\begin{cases} x_1 = x + x^\kappa x a(x, y) \\ y_1 = y + x^\kappa b(x, y), \end{cases} \tag{2}$$

or

$$\begin{cases} x_1 = x + y^\tau a(x, y) \\ y_1 = y + y^\tau y b(x, y). \end{cases} \tag{3}$$

If $e_p = 2$, choose local coordinates (x, y) such that the two irreducible components are given by $\{x = 0\}$ and $\{y = 0\}$ respectively. Write f at p as

$$\begin{cases} x_1 = x + x^\kappa y^\tau x a(x, y) \\ y_1 = y + x^\kappa y^\tau y b(x, y). \end{cases} \tag{4}$$

Here $\kappa, \tau \geq 1$ are the biggest possible powers and $a(x, y)$ and $b(x, y)$ are relatively prime in $\mathcal{O}_{M,p}$. The *adapted order* $\nu_1(f, p)$ of f at p is then equal to $\min\{\nu(a), \nu(b)\}$.

Remark 2. Under our assumptions, the blow-up map will always be "tangential" along a fixed component, and thus can be written in the form of either

(1), (2), (3) or (4). In the "non-tangential" case, one can define "adapted order" to be just "pure order". (For the definition of "tangential" and the dynamics in case the fixed point of the original map is not isolated, see e.g. [4])

One readily checks that the adapted order is well-defined (cf. [2, Lemma 2.2]). And it is easy to verify that if p is a singularity of f and q is a singularity of $\tilde{f}$, the blow-up of f at p, then $\nu_1(\tilde{f}, q) \leq \nu_1(f, p)$. (To appreciate the difference between ν_0 and ν_1, one can look at the blow-up $x = \tilde{x}\tilde{y}$, $y = \tilde{y}$ at O for $x_1 = x + x^3$, $y_1 = y + x(x + y^2)$.)

If f is of the form (2) (resp. (3)) at p and $\nu_1(f, p) = \nu(b)$ (resp. $\nu_1(f, p) = \nu(a)$), then p is said to be of *type zero*. Otherwise p is said to be of *type one*.

For an element $g \in \mathcal{O}_{M,p}$ with order $\nu(g) \geq r$, $r \in \mathbf{N}$, denote by $\mathfrak{T}^r(g)$ the affine plane $\mathbf{C}^2$ if $r < \nu(g)$ and the *strict tangent space* (i.e. the maximum linear subvariety of the tangent cone of $g = 0$ leaving it invariant by translations) if $r = \nu(g)$.

Set $\mu = \nu_1(f, p)$. We define the *directrix* $\mathfrak{T}(f, p)$ of f at p as follows. If $e_p = 0$ or 2, then

$$\mathfrak{T}(f, p) = \mathfrak{T}^\mu(a) \cap \mathfrak{T}^\mu(b).$$

If p is of type zero (resp. one) and f is of the form (3) (resp. (2)), then

$$\mathfrak{T}(f, p) = \mathfrak{T}^\mu(a).$$

If p is of type zero (resp. one) and f is of the form (2) (resp. (3)), then

$$\mathfrak{T}(f, p) = \mathfrak{T}^\mu(b).$$

By definition, we always have that $\dim \mathfrak{T}(f, p) \leq 1$. If $\dim \mathfrak{T}(f, p) = 1$, the directrix defines a closed point $\mathbf{P}(\mathfrak{T}(f, p))$ in the exceptional divisor of the blow-up with center p. We have the following

Lemma 3. *Let $\tilde{f}$ be the blow-up of f at p. If q is a closed point of the exceptional divisor such that $\nu_1(\tilde{f}, q) = \nu_1(f, p)$, then $\dim \mathfrak{T}(f, p) = 1$ and $q = \mathbf{P}(\mathfrak{T}(f, p))$.*

Proof. Assume that f is of the form (2) and p is of type zero. Set $\mu = \nu_1(f, p)$ and write $b_\mu(x, y)$ for the leading homogeneous polynomial in the homogeneous expansion of $b(x, y)$.

In the chart $(x = \tilde{x}, y = \tilde{x}(\tilde{y} + \zeta))$ centered in q, $\tilde{f}$ is of the form

$$\begin{cases} \tilde{x}_1 = \tilde{x} + \tilde{x}^{\kappa+\mu-1} \cdot O(\tilde{x}), \\ \tilde{y}_1 = \tilde{y} + \tilde{x}^{\kappa+\mu-1} \cdot (b_\mu(1, \tilde{y} + \zeta) + O(\tilde{x})) \end{cases}$$

Since $\nu_1(\tilde{f}, q) = \mu$, we have $\nu(b_\mu(1, \tilde{y}+\zeta)) = \mu$. This implies that $b_\mu(x,y) = c(y-\zeta x)^\mu$, where c is a constant. Therefore $\mathfrak{T}(f,p) = \{y-\zeta x = 0\}$ and $q = \mathbf{P}(\mathfrak{T}(f,p))$.

In the chart $(x = \tilde{x}\tilde{y}, y = \tilde{y})$ centered in q, $\tilde{f}$ is of the form

$$\begin{cases} \tilde{x}_1 = \tilde{x} + \tilde{x}^{\kappa+1}\tilde{y}^{\kappa+\mu-1} \cdot (-b_\mu(x,1) + O(\tilde{y}), \\ \tilde{y}_1 = \tilde{y} + \tilde{x}^{\kappa+1}\tilde{y}^{\kappa+\mu-1} \cdot O(\tilde{y}) \end{cases}$$

Since $\nu_1(\tilde{f}, q) = \mu$, we have $\nu(b_\mu(x,1)) = \mu$. This implies that $b_\mu(x,y) = cx^\mu$, where c is a constant. Therefore $\mathfrak{T}(f,p) = \{x = 0\}$ and $q = \mathbf{P}(\mathfrak{T}(f,p))$.

The argument for other cases is similar and we leave it to the interested reader. □

Due to the above lemma, we will then focus on points p with $\dim \mathfrak{T}(f,p) = 1$ and $e_p \geq 1$. In suitable coordinates, the map f takes exactly one of the following forms:

I.	(2)	type zero	$\mathfrak{T}(f,p) = \mathfrak{T}^\mu(b) = (y=0)$.
II.	(3)	type zero	$\mathfrak{T}(f,p) = \mathfrak{T}^\mu(a) = (y=0)$.
III.	(2)	type one	$\mathfrak{T}(f,p) = \mathfrak{T}^\mu(a) = (y=0)$.
IV.	(3)	type one	$\mathfrak{T}(f,p) = \mathfrak{T}^\mu(b) = (y=0)$.
V.	(4)	$\zeta \in \mathbf{C}$	$\mathfrak{T}(f,p) = (y - \zeta x = 0)$.

As in [7], for a power series $g = \sum_{i,j} g_{i,j} x^i y^j \in \mathbf{C}[[x,y]]$ and $r \in \mathbf{N}$, set

$$\gamma^r(g; x, y) = \min \left\{ \frac{i}{r-j};\ j < r,\ g_{i,j} \neq 0 \right\}.$$

The following facts are easily verified:

$$\gamma^r(g; x, y) < 1 \Longleftrightarrow \nu(g) < r, \tag{5}$$

and if $\nu(g) = r$, then

$$\gamma^r(g; x, y) > 1 \Longleftrightarrow \mathfrak{T}^r(g) = (y = 0)$$

and

$$\gamma^r(\tilde{g}; \tilde{x}, \tilde{y}) = \gamma^r(g; x, y) - 1, \quad \tilde{g} = g \circ \pi \cdot \tilde{x}^{-r}\ (\pi : (\tilde{x}, \tilde{y}) \mapsto (x, y) = (\tilde{x}, \tilde{x}\tilde{y})). \tag{6}$$

We define $\gamma(f, p; x, y)$ to be equal to

$$\begin{array}{ll} \min\{\gamma^\mu(ya; x, y), \gamma^\mu(b; x, y)\}, & \text{if I,} \\ \min\{\gamma^\mu(a; x, y), \gamma^\mu(yb; x, y)\}, & \text{if II,} \\ \min\{\gamma^{\mu+1}(ya; x, y), \gamma^{\mu+1}(b; x, y)\}, & \text{if III,} \\ \min\{\gamma^{\mu+1}(a; x, y), \gamma^{\mu+1}(yb; x, y)\}, & \text{if IV,} \\ \min\{\gamma^\mu(a; x, y), \gamma^\mu(b; x, y)\}, & \text{if V.} \end{array}$$

Since a and b are relatively prime, one has that $\gamma(f,p;x,y)<\infty$ if $\mu\geq 2$.

Set $\gamma=\gamma(f,p;x,y)$. If $\gamma\in\mathbf{N}$, a *γ-preparation* is a change of coordinates of the form $(x'=x, y'=y+\lambda x^\gamma)$, $\lambda\in\mathbf{C}$. One can make successive γ-preparations to increase $\gamma(f,p;x,y)$. If $\mu\geq 2$, this increase is finite since a and b are relatively prime (a and b are relatively prime in $\mathcal{O}_{M,p}$, if and only if they are relatively prime in the completion $\hat{\mathcal{O}}_{M,p}$). We say that f is *γ-prepared with respect to* (x,y) if one of the following holds:

(i) $\gamma\notin\mathbf{N}$;

(ii) $\gamma\in\mathbf{N}$ and f takes the form II, IV or V;

(iii) $\gamma\in\mathbf{N}$ and γ does not increase after any γ-preparation.

We then define $\gamma(f,p)$ to be the minimum $\gamma(f,p;x,y)$, where (x,y) runs over all γ-prepared situations. (If $\mu=1$, one may have that $\gamma(f,p)=\infty$.)

Consider a sequence of blow-ups

$$p=p_0\xleftarrow{\pi_0}p_1\xleftarrow{\pi_1}\cdots, \tag{7}$$

with p_i being the center of π_i and lying in the exceptional divisor of π_{i-1}. Let f_i be the blow-up map at p_i. We say that the sequence (7) is *stationary of order* μ if $\mu=\nu_1(f_i,p_i)$ remains constant. By Lemma 3, there is at most one possible stationary sequence for each p_0.

One readily checks that, in a stationary sequence, the forms of blow-up maps change as follows:

$$\begin{array}{rcl} \mathrm{I} & \longrightarrow & \mathrm{I},\\ \mathrm{II} & \longrightarrow & \mathrm{V},\\ \mathrm{III} & \longrightarrow & \mathrm{I,II,III},\\ \mathrm{IV} & \longrightarrow & \mathrm{V},\\ \mathrm{V} & \longrightarrow & \mathrm{I,II,III,V}. \end{array}$$

Lemma 4. *Let $\{p_i\}$ be a stationary sequence of order μ. If f_i takes the form* II *at p_i for some $i>0$, then*
(i) *the stationary sequence terminates at p_i, if $\mu\geq 2$;*
(ii) *the stationary sequence terminates at p_{i+1}, if $\mu=1$.*

Proof. If $e_{p_{i-1}}\geq 1$ then, since f_i takes the form II at p_i, we know that f_{i-1} takes the form III or V at p_{i-1}.

First, suppose that f_{i-1} takes the form III at p_{i-1}. We can then write f_{i-1} as

$$\begin{cases} x_1=x+x^\kappa xa(x,y)\\ y_1=y+x^\kappa b(x,y),\end{cases}$$

where $a(x,y) = cy^\mu + O(\mu+1)$ with $c \neq 0$ and $b(x,y) = O(\mu+1)$. After the blow-up ($\tilde{x} = x, \tilde{y} = y/x$) and then interchanging $\tilde{x}$ and $\tilde{y}$, we see that f_i is of the form

$$\begin{cases} \tilde{x}_1 = \tilde{x} + \tilde{y}^{\kappa+\mu}\tilde{a}(\tilde{x},\tilde{y}) \\ \tilde{y}_1 = \tilde{y} + \tilde{y}^{\kappa+\mu}\tilde{y}\tilde{b}(\tilde{x},\tilde{y}), \end{cases}$$

where $\tilde{a}(\tilde{x},\tilde{y}) = d\tilde{y}^\mu + e\tilde{x}^{\mu+1} + O(\mu+1)$ with $d \neq 0$ and $\tilde{b}(\tilde{x},\tilde{y}) = c\tilde{x}^\mu + O(\tilde{y})$. After the blow-up ($\bar{x} = \tilde{x}, \bar{y} = \tilde{y}/\tilde{x}$), we can then write f_{i+1} as

$$\begin{cases} \bar{x}_1 = \bar{x} + \bar{x}^{\kappa+2\mu-1}\bar{y}^{\kappa+\mu}\bar{x}\bar{a}(\bar{x},\bar{y}) \\ \bar{y}_1 = \bar{y} + \bar{x}^{\kappa+2\mu-1}\bar{y}^{\kappa+\mu}\bar{y}\bar{b}(\bar{x},\bar{y}), \end{cases}$$

where $\bar{a}(\bar{x},\bar{y}) = d\bar{y}^\mu + e\bar{x} + O(\bar{x}^2)$ and $\bar{b}(\bar{x},\bar{y}) = -d\bar{y}^\mu + (c-e)\bar{x} + O(\bar{x}^2) + O(\bar{x}\bar{y})$.

We see that $\nu_1(f_{i+1}, p_{i+1}) = 1$. Therefore, if $\mu \geq 2$ then the stationary sequence terminates at p_i. If $\mu = 1$, we have $\dim \mathfrak{T}(f_{i+1}, p_{i+1}) = 0$ and thus the stationary sequence terminates at p_{i+1} by Lemma 3.

Second, suppose that f_{i-1} takes the form V at p_{i-1}. We can then write f_{i-1} as

$$\begin{cases} x_1 = x + x^\kappa y^\tau x a(x,y) \\ y_1 = y + x^\kappa y^\tau b(x,y), \end{cases}$$

where $a(x,y) = c(y-\zeta x)^\mu + O(\mu+1)$ and $b(x,y) = d(y-\zeta x)^\mu + O(\mu+1)$, with $|c| + |d| \neq 0$ and $\zeta \neq 0$. After the blow-up ($\tilde{x} = x, \tilde{y} = y/x - \zeta$) and then interchanging $\tilde{x}$ and $\tilde{y}$, we see that f_i is of the form

$$\begin{cases} \tilde{x}_1 = \tilde{x} + \tilde{y}^{\kappa+\tau+\mu}(\tilde{x}+\zeta)^{\tau+1}\tilde{a}(\tilde{x},\tilde{y}) \\ \tilde{y}_1 = \tilde{y} + \tilde{y}^{\kappa+\tau+\mu}(\tilde{x}+\zeta)^{\tau+1}\tilde{y}\tilde{b}(\tilde{x},\tilde{y}), \end{cases}$$

where $\tilde{b}(\tilde{x},\tilde{y}) = e\tilde{y}^\mu + O(\mu+1)$ with $e \neq 0$ and $\tilde{b}(\tilde{x},\tilde{y}) = c\tilde{x}^\mu + O(\tilde{y})$ with $c \neq 0$ (since we necessarily have $c = d$). We then see that we can argue exactly as above.

Now suppose that $e_{p_{i-1}} = 0$. We then necessarily have $i-1 = 0$. By Lemma 3, we can write f_0 in suitable local coordinates (x,y) as

$$\begin{cases} x_1 = x + a(x,y) \\ y_1 = y + b(x,y), \end{cases}$$

where $a(x,y) = cy^\mu + O(\mu+1)$ and $b(x,y) = dy^\mu + O(\mu+1)$ with $|c| + |d| \neq 0$. Since f_1 takes the form II at p_1, we necessarily have $d = 0$ (and thus $c \neq 0$) and after the blow-up ($\tilde{x} = x, \tilde{y} = y/x$) and then interchanging $\tilde{x}$ and $\tilde{y}$, we can write f_1 as

$$\begin{cases} \tilde{x}_1 = \tilde{x} + \tilde{y}^{\mu-1}\tilde{a}(\tilde{x}, \tilde{y}) \\ \tilde{y}_1 = \tilde{y} + \tilde{y}^{\mu-1}\tilde{y}\tilde{b}(\tilde{x}, \tilde{y}), \end{cases}$$

where $\tilde{a}(\tilde{x}, \tilde{y}) = e\tilde{y}^{\mu} + O(\mu + 1)$ with $e \neq 0$ and $\tilde{b}(\tilde{x}, \tilde{y}) = c\tilde{x}^{\mu} + O(\tilde{y})$. We can then again argue as above.

This completes the proof. □

Lemma 5. *Let $\{p_i\}$ be a stationary sequence of order μ. If f_i takes the form* I *(resp.* III*, resp.* IV*, resp.* V*) at p_i and f_{i+1} takes the form* I *(resp.* III*, resp.* V*, resp.* V*) at p_{i+1}, then*

$$\gamma(f_{i+1}, p_{i+1}) \leq \gamma(f_i, p_i) - 1.$$

Proof. Assume, without loss of generality, that $\gamma(f_i, p_i) < \infty$.

Let (x, y) be local coordinates around p_i such that f_i is γ-prepared with respect to (x, y), where $\gamma = \gamma(f_i, p_i; x, y)$. In each of the cases we are concerned with, the coordinates centered in p_{i+1} are ($\tilde{x} = x, \tilde{y} = y/x$). By (6), we have that

$$\gamma(f_{i+1}, p_{i+1}; \tilde{x}, \tilde{y}) = \gamma(f_i, p_i; x, y) - 1.$$

It is not difficult to see that f_{i+1} is $(\gamma - 1)$-prepared with respect to $(\tilde{x}, \tilde{y})$ if f_i is γ-prepared with respect to (x, y). Therefore the lemma follows from the above equality and the definition of $\gamma(f, p)$. □

For stationary sequences of order greater or equal to two, we have the following

Theorem 6. *All the possible stationary sequences of order greater or equal to two have finite length.*

Proof. If f_i takes the form III or V at p_i and f_{i+1} takes the form II at p_{i+1}, then, by Lemma 4 (i), the stationary sequence terminates at p_{i+1}.

If f_i takes the form I (resp. III, resp. IV, resp. V) at p_i and f_{i+1} takes the form I (resp. III, resp. V, resp. V) at p_{i+1}, then, by Lemma 5, we have

$$\gamma(f_{i+1}, p_{i+1}) \leq \gamma(f_i, p_i) - 1.$$

We can now conclude using the above inequality and (5). □

It is obvious that Theorem 1 is a corollary of the above theorem. To finish the paper, we give a brief discussion on stationary sequences of order one.

Lemma 7. *Assume that f takes the form* I *at p. Let (x, y) be local coordinates around p such that f is γ-prepared with respect to (x, y) for $\gamma = \gamma(f, p; x, y)$. Set $\gamma' = \gamma(f, p)$ and assume that $\gamma > \gamma'$. Then $\gamma' \in \mathbf{N}$ and there exists a change of coordinates $(x' = x, y' = y + \lambda x^{\gamma'})$ such that $\gamma(f, p; x', y') = \gamma'$ and that f is γ'-prepared with respect to (x', y').*

Proof. By the assumption, we can write f as

$$\begin{cases} x_1 = x + x^{\kappa} x a(x, y) \\ y_1 = y + x^{\kappa} b(x, y), \end{cases}$$

where $a(x, y) = O(\mu)$ and $b(x, y) = cy^{\mu} + O(\mu + 1)$, with $\mu = \nu_1(f, p)$ and $c \neq 0$.

Choose local coordinates $(\bar{x}, \bar{y})$ such that $\gamma(f, p; \bar{x}, \bar{y}) = \gamma'$ and f is γ'-prepared with respect to $(\bar{x}, \bar{y})$. Since f takes the form I at p, we have that $\bar{x} = u \cdot x$, $\bar{y} = v \cdot y + w \cdot x^n$, where $u, v, w \in \mathcal{O}_{M,p}$ with $u(0, 0) \neq 0$ and $v(0, 0) \neq 0$. Since $\gamma > \gamma'$, we also have that $w(0, 0) \neq 0$.

Since $b(x, y) = cy^{\mu} + O(\mu + 1)$, the term $\bar{x}^n \bar{y}^{\mu-1}$ appears in the expression of $\bar{y}_1$. Therefore, we have $n \geq \gamma'$. Since terms $x^i y^j$ create terms $\bar{x}^{i+kn} \bar{y}^l$ with $k + l = j$ and $i > \gamma'(\mu - j)$, we have

$$\frac{i + kn}{\mu - l} > \frac{\gamma'(\mu - k - l) + kn}{\mu - l} = \gamma' + \frac{k(n - \gamma')}{\mu - l} \geq \gamma'.$$

Therefore, we actually have $\gamma' = n \in \mathbf{N}$.

Now consider the change of coordinates $(x' = x, y' = y + \lambda x^{\gamma'})$, with $\lambda \neq 0$. By the above discussion, we have that $\gamma(f, p; x', y') = \gamma'$. Due to the presence of the term $x'^n y'^{\mu-1}$, we see that γ' does not increase after any γ'-preparation. Therefore f is γ'-prepared with respect to (x', y'). □

For stationary sequences of order equal to one, we have the following

Theorem 8. *The length of a stationary sequence of order one is infinite if and only if there exists a step p_l such that f_l takes the form* I *at p_l and $\gamma(f_l, p_l) = \infty$. Moreover, if $i \geq l$ then f_i takes the form* I *at p_i and $\gamma(f_i, p_i) = \infty$.*

Proof. Assume that $\{p_i\}$ is a stationary sequence of order one with infinite length. If $\gamma(f_i, p_i) < \infty$ for all i, then we can show as in Theorem 6 that the stationary sequence has finite length, a contradiction. Therefore, we have that $\gamma(f_l, p_l) = \infty$ for some l. Since $\nu_1(f_l, p_l) = 1$, by the definition of $\gamma(f, p)$, one readily checks that f_l takes the form I at p_l. If $\gamma(f_i, p_i) < \infty$ for some $i > l$, then by Lemma 5 and (5) we have that the stationary sequence has finite length, a contradiction. Therefore, the last statement holds.

For the converse, assume that f_l takes the form I at p_l and $\gamma(f_l, p_l) = \infty$ for some l. Suppose that $\gamma(f_{l+1}, p_{l+1}) = \gamma < \infty$. We can choose local coordinates (x, y) such that $\gamma(f_l, p_l; x, y) > \gamma + 1$. Then $\gamma(f_{l+1}, p_{l+1}; \tilde{x}, \tilde{y}) > \gamma$ for $(\tilde{x} = x, \tilde{y} = y/x)$. By Lemma 7, there exists a change of coordinates $(\tilde{x}' = \tilde{x}, \tilde{y}' = \tilde{y} + \lambda\tilde{x}^\gamma)$ such that $\gamma(f_{l+1}, p_{l+1}; \tilde{x}', \tilde{y}') = \gamma$ and that f_{l+1} is γ-prepared with respect to $(\tilde{x}', \tilde{y}')$. Now consider the change of coordinates $(x' = x, y' = y + \lambda x^{\gamma+1})$. We have that $\gamma(f_l, p_l; x', y') = \gamma$ and that f_l is $(\gamma+1)$-prepared with respect to (x', y'), a contradiction. Therefore, we must have $\gamma(f_{l+1}, p_{l+1}) = \infty$. □

Remark 9. In [9], we gave "final forms" of singularities which are persistent under blow-ups. In the two-dimensional case, it is easy to see that a *simple point* of type A) takes the form I with $\gamma(f, p) = \infty$ and a *simple point* of type B) and a *non-dicritical simple corner* has adapted order 0 (see [9] for precise definitions).

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FENG RONG
DEPARTMENT OF MATHEMATICS
SHANGHAI JIAO TONG UNIVERSITY
800 DONG CHUAN ROAD
SHANGHAI, 200240
P.R. CHINA

E-mail: frong@sjtu.edu.cn
URL: www.math.sjtu.edu.cn/faculty/frong

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