Bezout duo and distributive partial skew power series rings

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Abstract. In this paper we consider rings R with a partial action α of $\mathbb{Z}$ on R. We study necessary and sufficient conditions for the partial skew power series rings $R[[x;\alpha]]$ to be right duo and right Bezout.

Introduction

Partial actions of groups have been introduced in the theory of operator algebras giving powerful tools of their study (see [3] and the literature quoted therein). In [3], the authors introduced partial actions on rings in a pure algebraic context and studied partial skew group rings. In [3], the authors defined a partial action as follows: let R be a ring with an identity 1_R and let $\mathbb{Z}$ the additive group of integers. A partial action α of $\mathbb{Z}$ on R is a collection of ideals S_i, $i \in \mathbb{Z}$ and isomorphisms of rings $\alpha_i : S_{-i} \to S_i$ such that the following conditions hold:

(i) $S_0 = R$ and α_0 is the identity map of R;

(ii) $S_{-(i+j)} \supseteq \alpha_i^{-1}(S_i \cap S_{-j})$;

(iii) $\alpha_j \circ \alpha_i(a) = \alpha_{j+i}(a)$, for any $a \in \alpha_i^{-1}(S_i \cap S_{-j})$.

The above properties easily imply that $\alpha_j(S_{-j} \cap S_i) = S_j \cap S_{i+j}$ and $\alpha_{-i} = \alpha_i^{-1}$, for all $i, j \in \mathbb{Z}$.

Following [2], the *partial skew Laurent polynomial ring* $R\langle x;\alpha\rangle$, in an indeterminate x, is the set of all finite formal sums $\sum_{i=-n}^{m} a_i x^i$, $a_i \in S_i$, where the addition is defined in the usual way and the multiplication is defined by

$(a_i x^i)(a_j x^j) = \alpha_i(\alpha_{-i}(a_i)a_j)x^{i+j}$, for any $i, j \in \mathbb{Z}$. The *partial skew polynomial ring* $R[x; \alpha]$ is the subring of $R\langle x; \alpha \rangle$ whose elements are the polynomials $\sum_{i=0}^{n} a_i x^i$, $a_i \in S_i$.

Given a partial action α of $\mathbb{Z}$ on R, an enveloping action is a ring T containing R together with a global action $\beta = \{\sigma^i : i \in \mathbb{Z}\}$ on T, where σ is an automorphism of T such that the partial action α_i is given by the restriction of σ^i ([3], Definition 4.2). Note that T does not necessarily have an identity, since the group acting on R is infinite. It is shown in ([3], Theorem 4.5) that a partial action α has an enveloping action if and only if all the ideals S_i are generated by central idempotents of R.

When α has an enveloping action (T, σ), where σ is an automorphism of T, we may consider that R is an ideal of T and the following properties hold:

(i) $T = \sum_{i \in \mathbb{Z}} \sigma^i(R)$;

(ii) $S_i = R \cap \sigma^i(R)$, for every $i \in \mathbb{Z}$;

(iii) $\alpha_i(a) = \sigma^i(a)$, for all $i \in \mathbb{Z}$ and $a \in S_{-i}$.

In order to have associative rings and apply the results which are known for skew polynomial rings and for skew power series rings we assume throughout the paper that all ideals S_i are generated by central idempotents of R. The idempotent corresponding to S_i will be denoted by 1_i and the enveloping action of α by (T, σ), where σ is an automorphism of T. By condition (ii) above we have that $1_i = 1_R \sigma^i(1_R)$. This fact and conditions (i) and (iii) above will be used freely in the paper. Also the following remark will be used without further mention: if I is an ideal of R, then I is also an ideal of T. In fact, if $a \in I$ and $t \in T$ we have $ta = t1_R a \in Ra \subseteq I$ and, similarly, $at \in I$.

The *skew Laurent polynomial ring* $T\langle x; \sigma \rangle$ is the set of formal finite sums $\sum_{i=p}^{q} a_i x^i$, $a_i \in T$, with usual sum and the multiplication given by $xa = \sigma(a)x$, for all $a \in T$. The partial skew Laurent polynomial ring $R\langle x; \alpha \rangle$ is a subring of $T\langle x; \sigma \rangle$. Moreover, $R[x; \alpha]$ is a subring of the skew polynomial ring $T[x; \sigma]$.

The *partial skew power series ring* $R[[x; \alpha]]$, in an indeterminate x, is the set of all series $\sum_{i=0}^{\infty} a_i x^i$, $a_i \in S_i$, where the addition is defined in the usual way and the multiplication is defined by

$$\left(\sum_{i=0}^{\infty} a_i x^i \right)\left(\sum_{i=0}^{\infty} b_i x^i \right) = \sum_{i=0}^{\infty} c_i x^i$$

for any $a_i, b_i \in S_i$, where $c_i = a_0 b_i + a_1 \alpha_1(b_{i-1} 1_{-1}) + \cdots + a_i \alpha_i(b_0 1_{-i})$, for all $i \geq 0$. The partial skew polynomial ring $R[x; \alpha]$ is a subring of $R[[x; \alpha]]$.

The *skew power series ring* $T[[x;\sigma]]$ is the set of all series $\sum_{i=0}^{\infty} a_i x^i$, $a_i \in T$, with usual sum and the multiplication given by

$$\left(\sum_{i=0}^{\infty} a_i x^i \right) \left(\sum_{i=0}^{\infty} b_i x^i \right) = \sum_{i=0}^{\infty} c_i x^i$$

where $c_i = a_0 b_i + a_1 \sigma(b_{i-1}) + \cdots + a_i \sigma^i(b_0)$, for any $i \geq 0$. The skew polynomial ring $T[x;\sigma]$ is a subring of $T[[x;\sigma]]$, consisting of the series with a finite number of non zero coefficients. Moreover, $R[[x;\alpha]]$ is a subring of the skew power series ring $T[[x;\sigma]]$.

We recall some terminology from [2]. We say that an ideal I of R is an α-ideal (α-invariant ideal) if $\alpha_i(I \cap S_{-i}) \subseteq I \cap S_i$, for all $i \geq 0$, ($\alpha_i(I \cap S_{-i}) = I \cap S_i$, for all $i \in \mathbb{Z}$). Note that I is an α-ideal of R if and only if the set of all polynomials $\sum_{i \geq 0} a_i x^i$, where $a_i \in I \cap S_i$, is an ideal of $R[x;\alpha]$. A similar result holds in $R\langle x;\alpha \rangle$ if I is an α-invariant ideal of R.

A ring R is called *right (left) quasi-duo* if every maximal right (left) ideal of R is two-sided or, equivalently, every right (left) primitive homomorphic image of R is a division ring. We refer [8] for further information on quasi-duo rings.

Let $J(R)$ be the Jacobson radical of R. We have that R is right (left) quasi-duo if and only if $R/J(R)$ is right (left) quasi-duo and in this case $R/J(R)$ is a reduced ring. We will use this property in the paper without further mention.

The main purpose of this paper is to study necessary and sufficient conditions for the partial skew power series rings $R[[x;\alpha]]$ to be right duo and right Bezout. In Section 1, we present the results that are necessary to prove the principal result of this article, which generalize the results that appear in [10]. In Section 2, we prove our principal result, generalizing ([9], Theorem 1.6 and Corollary 3.1).

1. Preliminaries results

A ring S is reduced if S do not have non-zero nilpotent elements. For $\emptyset \neq X \subseteq S$ we denote by $r_S(X) = \{a \in S : Xa = 0\}$ the *right annihilator* of X. Moreover, if S is a reduced ring then, by ([10], 1.35(2)), we have that $r_S(a) = l_S(a)$ is a two-sided ideal of S, for all $a \in S$.

We begin with the following definition that appears in ([1], Section 3 Definition 1).

Definition 1.1. A partial action α of $\mathbb{Z}$ on a ring R is called partially α-rigid if for each $a \in S_j$ such that $a\alpha_j(a.1_{-j}) = 0$, $j \in \mathbb{Z}$, we have that $a = 0$.

Equivalently, α is a partial α-rigid action if, for each $0 \neq a \in S_j$, $j \in \mathbb{Z}$, we have that $a\alpha_j(a1_{-j}) \neq 0$.

The next result generalizes ([10], 6.52).

Lemma 1.1. *Let* $A = R[[x; \alpha]]$ *and* $B = R[x; \alpha]$. *The following conditions are equivalent:*

(1) *A is a reduced ring.*

(2) *B is a reduced ring.*

(3) *α is partially α-rigid.*

PROOF. $(1) \Rightarrow (2)$ Straightforward.

$(2) \Rightarrow (3)$ Let $a \in S_n$ such that $a\alpha_n(a.1_{-n}) = 0$, for some $n \geq 0$. Then $(ax^n)^2 = 0$ and, by assumption, we have that $ax^n = 0$. Hence, $a = 0$.

$(3) \Rightarrow (1)$ By assumption, we clearly have that R is a reduced ring.

Let $f = \sum_{n \geq 0} f_n x^n \in A$, with $f_n \in S_n$, for all $n \geq 0$, such that

$$f^2 = \sum_{n \geq 0} \left(\sum_{i+j=n} f_i \alpha_i(f_j 1_{-i}) \right) x^n = 0.$$

Since R is reduced, then $f_0 = 0$. Suppose that $f_j = 0$, for all $j < m$. By the fact that the coefficient of the term of degree $2m$ of f^2 satisfies

$$\sum_{i+j=2m} f_i \alpha_i(f_j 1_{-i}) = f_0 f_{2m} + \cdots + f_m \alpha_m(f_m 1_{-m}) + \cdots + f_{2m} \alpha_{2m}(f_0 1_{-2m})$$

$$= f_m \alpha_m(f_m 1_{-m}) = 0$$

and we have that $f_m = 0$. So, $f = 0$. $\qquad\square$

Definition 1.2. Let α be a partial action of $\mathbb{Z}$ on R. We say that R is an α-strongly regular ring if, for any $a \in R$ and $j \neq 0$, there exists $b_j \in R$ such that $\alpha_j(a1_{-j}) = \alpha_j(a1_{-j})^2 b_j$.

The following definition appears in [5].

Definition 1.3. (i) A ring S is said to be Von Newmman regular if, for each $a \in S$, there exists $b \in S$ such that $a = aba$.

(ii) A ring S is said to be strongly regular if, for any $a \in S$, there exists $b \in S$ such that $a = a^2 b$.

(iii) A ring S is said to be abelian if all idempotents of S are central.

From [5] we easily have that $(ii) \Rightarrow (i)$ and $(ii) \Rightarrow (iii)$.

Lemma 1.2. *R is an α-strongly regular ring if and only if S_k is a strongly regular ring, for all $k \neq 0$.*

PROOF. Suppose that R is an α-strongly regular ring and let $a \in S_k$, with $k \neq 0$. Then, for any $j \neq 0$, there exists $b_j \in R$ such that $\alpha_j(a1_{-j}) = \alpha_j(a1_{-j})^2 b_j$. In particular, for $j = -k$, we have that $\alpha_{-k}(a) = \alpha_{-k}(a)^2 b_{-k}$. Thus,

$$a = \alpha_k(\alpha_{-k}(a)1_{-k}) = a^2 \alpha_k(b_{-k}1_{-k}).$$

So S_k is strongly regular, for all $k \neq 0$.

Conversely, suppose that S_k is strongly regular, for all $k \neq 0$. Note that, for any $a \in R$, we have $a1_{-k} \in S_{-k}$. Thus, there exists $b_k \in S_{-k}$ such that $a1_{-k} = (a1_{-k})^2 b_k$. So

$$\alpha_k(a1_{-k}) = \alpha_k(a1_{-k})^2 \alpha_k(b)$$

and it follows that R is α-strongly regular.   □

A ring S is *semicommutative* if the right (left) annihilator of each element of S is a two-sided ideal. Equivalently, S is semicommutative if, for any $a, b \in S$ such that $ab = 0$, we have $aSb = 0$. Thus, every semicommutative ring is an abelian ring. Moreover, every reduced ring is a semicommutative ring.

Now we can prove the following result.

Lemma 1.3. *Suppose that R is a semicommutative ring such that, for any $a \in R$, there exists $b_j \in R$ satisfying $\alpha_j(a1_{-j}) = \alpha_j(a1_{-j})ab_j$, for any $j \geq 1$. Then R is an α-strongly regular ring.*

PROOF. By assumption, for each $a \in S_{-j}$, there exists $b_j \in R$ such that $\alpha_j(a)[1_R - ab_j] = 0$ and $r_R(\alpha_j(a))$ is a two-sided ideal of R. Thus $\alpha_j(m1_{-j})am \in r_R(\alpha_j(a))$ and

$$0 = \alpha_j(a)\alpha_j(m1_{-j})am = \alpha_j(am)am,$$

where $m = 1_R - ab_j \in r_R(\alpha_j(a))$. Hence, $0 = am = a(1_R - ab_j) = a - a^2 b_j$ and it follows that $a = a.1_{-j} = a^2(b_j1_{-j})$. So, S_j is a strongly regular ring, for all $j \neq 0$ and by Lemma 1.2, R is α-strongly regular.   □

The following definition appears in [10].

Definition 1.4. (i) A ring S is said to be right distributive if, for any right ideals J, K, L of R, we have that $(J + K) \cap L = (J \cap L) + (K \cap L)$.

(ii) A ring S is said to be right Bezout if any two-finitely generated right ideal of S is a principal right ideal.

From now on, we denote $B = R[x; \alpha]$ and $A = R[[x; \alpha]]$. The next result generalizes ([10], 6.55((3), (5), (6))) and the proof will be omitted since it follows from the original ideas adapted to our case.

Proposition 1.4. *Let $S(n) = B/(\sum_{i \geq n} S_i x^i)$ and*

$$I(n) = \left(\sum_{i \geq 1} S_i x^i \right) \Big/ \left(\sum_{i \geq n} S_i x^i \right),$$

for all $n \geq 1$. The following conditions hold.

(1) *$I(n)$ is a nilpotent ideal of $S(n)$ and $R \simeq S(n)/I(n)$, for all $n \geq 1$.*

(2) *If R is a strongly regular ring, then $S(n)/J(S(n))$ is a strongly regular ring.*

(3) *If R is a strongly regular ring, then $S(n)$ is a right distributive ring if and only if $S(n)$ is a right Bezout ring.*

We recall that a ring S is said to be *J-semisimple* if $J(S) = 0$, where $J(S)$ is the Jacobson radical of S.

Following [4], a partial action α of $\mathbb{Z}$ is said to be of *finite type* if, for any $z \in \mathbb{Z}$, there exists a finite subset $\{z_1, \ldots, z_n\}$ of $\mathbb{Z}$ such that $\sum_{1 \leq i \leq n} S_{z+z_i} = R$. It is not difficult to see that $R = \bigoplus_{i=1}^{n} D_{z+z_i}$, where D_{z+z_i} is an ideal of S_{z+z_i} generated by a central idempotent e_{z+z_i} of R, $1 = \sum_{i=1}^{n} e_{z+z_i}$ and $e_{z+z_i} e_{z+z_j} = 0$, for $i \neq j$.

The next result is well known and we put it here adapted for our case.

Lemma 1.5. *Suppose that α is a partial action of finite type. The following conditions hold.*

(1) *R is a J-semisimple ring if, and only if, S_j is a J-semisimple ring, for all $j \neq 0$.*

(2) *R is a von Neumann regular ring if, and only if, S_j is a von Neumann regular ring, for all $j \neq 0$.*

(3) *R is a strongly regular ring if, and only if, S_j is a strongly regular ring, for all $j \neq 0$.*

The next result generalizes ([10], 6.61).

Proposition 1.6. *Suppose that α is a partial action of finite type such that, for any $j \geq 1$ and $a \in R$, there exists $b_j \in R$ satisfying $\alpha_j(a1_{-j}) = \alpha_j(a1_{-j})ab_j$. The following conditions hold.*

(1) *$J(R) = 0$.*

(2) *If R is a semicommutative ring, then R is a strongly regular ring and all idempotent elements of R are α-invariant and central in $R[x; \alpha]$.*

PROOF. (1) Let $a \in J(S_{-j}) = J(R) \cap S_{-j}$, for $j \geq 1$. Then there exists $b_j \in R$ such that $\alpha_j(a)(1_R - ab_j) = 0$. Since $a \in J(R)$, then $1_R - ab_j \in U(R)$ and we obtain that $\alpha_j(a) = 0$. Thus, $J(S_{-j}) = 0$, for all $j \neq 0$. So, by Lemma 1.5(1), we have that R is a J-semisimple ring.

(2) By Lemma 1.6, R is an α-strongly regular ring and, by Lemma 1.5, S_j is a strongly regular ring, for all $j \neq 0$. Thus, by Lemma 1.5(3), R is a strongly regular ring and, by [5], we have that all idempotent elements of R are central. Note that, for all $j \geq 1$, there exists $u_j, v_j \in R$ such that

$$\alpha_j(e1_{-j}) = \alpha_j(e1_{-j})eu_j$$

$$\alpha_j((1_R - e)1_{-j}) = \alpha_j((1_R - e)1_{-j})(1_R - e)v_j.$$

Hence, $\alpha_j(e1_{-j}) + \alpha_j((1_R - e)1_{-j}) = 1_j$ and we have that

$$e1_j = e1_j\alpha_j(e1_{-j})eu_j + e1_j\alpha_j((1_R - e)1_{-j})(1_R - e)v_j = \alpha_j(e1_{-j}).$$

So, all the idempotents of R are α-invariant. □

It is convenient to remark that the homomorphic image of a right Bezout ring (right distributive ring) is a right Bezout ring (right distributive ring), see [10].

Next, we study conditions for the partial skew polynomial ring $R[x; \alpha]$ to be a Bezout ring and this generalizes ([10], 6.62).

Proposition 1.7. *Suppose that R is a strongly regular ring such that all idempotent elements of R are α-invariant. The following conditions hold.*

(1) *A is a reduced ring and B is a Bezout reduced subring of A.*

(2) *If $n \geq 1$, then $B/(\sum_{i \geq n} S_i x^i)$ is a Bezout distributive ring.*

PROOF. (1) Let $a \in S_j$ such that $a\alpha_j(a1_{-j}) = 0$, for some $j \in \mathbb{Z}$. Since R is a strongly regular ring then, by ([5]), we have that $a = ue$, where $u \in U(R)$ and $e \in R$ is a central idempotent of R. Thus,

$$0 = a\alpha_j(a1_{-j}) = ue\alpha_j(ue1_{-j}) = ue\alpha_j(u1_{-j})\alpha_j(e1_{-j}) = ue\alpha_j(u1_{-j})$$

and we have that $e1_j = 0$. Hence $a = a1_j = ue1_j = 0$ and we obtain that α is a partially α-rigid. So, by Lemma 1.1, A is reduced.

We claim that S is a right Bezout ring. In fact, let $D = fS + gS$ be a right ideal of S generated by $f, g \in S$ and $n = \min(\delta(f), \delta(g))$, where $\delta(h)$ denotes

the degree of a polynomial $h \in B$. We proceed by induction on n. Suppose that $n = \delta(f) \le \delta(g)$.

Let $f = \sum_{i=0}^{n} f_i x^i \in B$, where $f_i \in S_i$, for all $0 \le i \le n$. Since R is strongly regular we can write $f_n = a_n e_n$, where $a_n \in U(R)$ and $e_n \in R$ is central idempotent.

If $n = 0$, then $f = f_0 = a_0 e_0$. We define $h = e_0 + g(1_R - e_0)$ and note that $hf = h e_0 a_0 = f$. By the fact that all idempotents of R are α-invariant we have that $h(1_R - e_0 + g e_0) = g$. So, $D = fS + gS = hS$ is a right principal ideal of B.

Now we suppose, by induction, that $uB + vB$ is a right principal ideal of B, for any $u, v \in B$ such that $n > \min(\delta(u), \delta(v))$.

By assumption, all idempotents of R are α-invariant and we have that

$$D(1_R - e_n) = (fB + gB)(1_R - e_n) = fB(1_R - e_n) + gB(1_R - e_n)$$
$$= f(1_R - e_n)B + g(1_R - e_n)B.$$

It is not difficult to see that

$$f_n \alpha_n((1_R - e_n)1_{-n}) = f_n(1_R - e_n)1_n = f_n(1_R - e_n) = a_n e_n(1_R - e_n) = 0.$$

Hence, $\delta(f(1_R - e_n)) < n$ and it follows that $D(1_R - e_n)$ is a right principal ideal of B.

Note that we can suppose that $\delta(g) = \delta(f) = n$ because, if $\delta(g) = p > n$, we take

$$l = f e_n \alpha_{-n}(a_n^{-1} g_p 1_n) x^{p-n} - g e_n$$

and we have that $D = Bf + Bg = Bf + Bl$. The coefficient of the term of degree p of $l \in B$ satisfies

$$f_n e_n a_n^{-1} g_p 1_n - g_p e_n = 0$$

and we obtain that $\delta(l) < p$.

We define $q = f e_n \alpha_{-n}(a_n^{-1} g_n) - g e_n$ and note that

$$q_n = f_n \alpha_n(e_n 1_{-n})a_n^{-1} g_n - g_n e_n = f_n e_n a_n^{-1} g_n - g_n e_n = a_n e_n a_n^{-1} g_n - g_n e_n = 0.$$

Hence, $\delta(q) < n$. Moreover,

$$De_n = (fB + gB)e_n = f e_n B + g e_n B = f e_n B + qB,$$

where $\min(\delta(f e_n), \delta(q)) < n$ and it follows that De_n is a right principal ideal of S. So, D is a principal right ideal of B.

(2) Let $\psi : B \to B/(\sum_{i \ge n} S_i x^i)$ be the canonical epimorphism, for all $n \ge 1$. By (1), B is a Bezout ring and we obtain that $B/(\sum_{i \ge n} S_i x^i)$ is a Bezout ring as homomorphic image of B. So, by Proposition 1.4(3), $B/(\sum_{i \ge n} S_i x^i)$ is a distributive ring. $\square$

The *order* of a series $f \in A$, denoted by $\mathcal{O}(f)$, is the minor power of x on f with non zero coefficient. We consider, for all $j \geq 1$, the set

$$M_j = \{f \in A : \mathcal{O}(f) \geq j + 1\}.$$

The next result generalizes ([10], 6.68((1), (3), (4), (5), (6), (7))) and the proof follows the principals ideas of them adapted to our case.

Proposition 1.8. *Let $A(n) = A/M_{n-1}$ and $I(n) = M_0/M_{n-1}$, for all $n \geq 1$. The following conditions hold.*

(1) *For all $f, g \in A$, $\sum_{i \geq 0}(f1_j x^j g)^i$ is the inverse element of $1_R - f1_j x^j g \in A$, for all $j \geq 1$.*

(2) *$M_0 \subseteq J(A)$.*

(3) *$A/M_{n-1} \simeq B/\sum_{i \geq n} S_i x^i$, for all $n \geq 1$.*

(4) *$I(n)$ is a nilpotent ideal of $A(n)$ and $R \simeq A(n)/I(n)$, for all $n \geq 1$.*

(5) *If R is a strongly regular ring then $A/J(A)$ and $A(n)/J(A(n))$ are strongly regular rings.*

(6) *If R is a strongly regular ring then A (respectively $A(n)$) is a right distributive ring if, and only if, A (respectively $A(n)$) is a right Bezout ring.*

Let M be a right R-module. A *subfactor* of M is a submodule of M/N, where N is a submodule of M. Moreover, if N is a subfactor of a distributive module M, then $End(N/J(N))$ is a reduced ring, see ([10], 2.54). Our next result generalizes ([10], 6.63).

Proposition 1.9. *Suppose that α is a partial action of finite type, $\pi_j : A \to A/M_j$ is the canonical epimorphism and $\pi_j(A)$ is a right distributive ring, for all $j \geq 1$. The following conditions hold.*

(1) *For any $a \in R$ and $j \geq 1$, there exists an element $b_j \in R$ such that $\alpha_j(a1_{-j}) = \alpha_j(a1_{-j})ab_j$.*

(2) *R is a strongly regular ring, $\pi_j(A)$ is a distributive Bezout ring, for all $j \geq 1$ and all idempotent elements of R are α-invariant and central in A.*

PROOF. (1) Let $a \in R$ and $j \geq 1$. By assumption, $\pi_j(A)$ is right distributive and there exists

$$\overline{f} = \pi_j(f) = f + M_j$$

$$\overline{g} = \pi_j(g) = g + M_j$$

$$\overline{h} = \pi_j(h) = h + M_j$$

$$\overline{s} = \pi_j(s) = s + M_j$$

in $\pi_j(A)$, where $f, g, h, s \in A$, such that

$$1_R + M_j = f + g + M_j$$

$$af + M_j = 1_j x^j h + M_j$$

$$1_j x^j g + M_j = as + M_j$$

Thus, there exists $l_1, l_2, l_3 \in M_j$ such that

$$1_R = f + g + l_1$$

$$af = 1_j x^j h + l_2$$

$$1_j x^j g = as + l_3$$

and we have that

$$\begin{aligned}
\alpha_j(a1_{-j})x^j = 1_j x^j a 1_R &= 1_j x^j a(f + g + l_1) = 1_j x^j (af) + 1_j x^j (ag) + 1_j x^j a l_1 \\
&= 1_j x^j (1_j x^j h + l_2) + \alpha_j(a1_{-j}) 1_j x^j g + 1_j x^j a l_1 \\
&= 1_j 1_{2j} x^{2j} h + 1_j x^j l_2 + \alpha_j(a1_{-j})[as + l_3] + 1_j x^j a l_1.
\end{aligned}$$

Note that the coefficients of degree j in this equality satisfies $\alpha_j(a1_{-j}) = \alpha_j(a1_{-j})as_j$, where s_j is the coefficient of the term of degree j in the series $s \in A$. So, the result follows.

(2) By Proposition 1.6, R is a J-semisimple ring and

$$R \simeq A/M_0 \simeq (A/M_j)/(M_0/M_j) = \pi_j(A)/\pi_j(M_0).$$

Thus, there exists a surjective ring homomorphism $\psi : \pi_j(A) \to R$ and it follows that R is a right distributive ring as homomorphic image of $\pi_j(A)$. By ([10], 2.54), we have that $R \simeq End(R) = End(R/J(R))$ is a reduced ring and, in particular, semicommutative.

By Proposition 1.6(2), R is a strongly regular ring such that all idempotent elements of R are α-invariant and central in A. By Proposition 1.7(2), $B/(\sum_{i \geq j} S_i x^i)$ is a distributive Bezout ring, for all $j \geq 1$. By Proposition 1.8(3), $\pi_j(A) = A/M_j \simeq B/\sum_{i \geq j+1} S_i x^i$ is a distributive Bezout ring, for all $j \geq 1$. $\square$

Definition 1.5. A ring S is said to be right quasi-duo if all right maximal ideals of S are two-sided ideals.

The next result generalizes ([10], 6.64).

Proposition 1.10. *Suppose that α is a partial action of finite type and*

$$B/\left(\sum_{i \geq j+1} S_i x^i\right)$$

is a right Bezout ring, for all $j \geq 1$. If R is either right quasi-duo or semicommutative then, for any $a \in R$ and $j \geq 1$, there exists an element $b_j \in R$ such that $\alpha_j(a1_{-j}) = \alpha_j(a1_{-j})ab_j$. Moreover, R is a strongly regular ring, all idempotent elements of R are α-invariant and central in B and $B/(\sum_{i \geq j+1} S_i x^i)$ is a distributive Bezout ring, for all $j \geq 1$.

PROOF. Suppose that R is a right quasi-duo ring and consider the canonical epimorphism $\pi_j : B \to B/(\sum_{i \geq j+1} S_i x^i)$, for all $j \geq 1$. Then

$$\pi_j\left(\sum_{i \geq 1} S_i x^i\right) = \left(\sum_{i \geq 1} S_i x^i\right) / \left(\sum_{i \geq j+1} S_i x^i\right)$$

is a left nilpotent ideal of $\pi_j(B)$ and, in particular, $\pi_j(\sum_{i \geq 1} S_i x^i) \subseteq J(\pi_j(S))$. Note that,

$$\pi_j(B)/\pi_j\left(\sum_{i \geq 1} S_i x^i\right) = \left(B/\left(\sum_{i \geq j+1} S_i x^i\right)\right) / \left(\left(\sum_{i \geq 1} S_i x^i\right) / \left(\sum_{i \geq j+1} S_i x^i\right)\right)$$

$$\simeq B/\left(\sum_{i \geq 1} S_i x^i\right) \simeq R.$$

Thus, we can consider the surjective homomorphism

$$\gamma : R \to \pi_j(B)/J(\pi_j(B))$$

and we have that $\pi_j(B)/J(\pi_j(B))$ is a right quasi-duo ring. Hence, $\pi_j(B)$ is a right quasi-duo ring. By assumption, $\pi_j(B)$ is a right Bezout ring and by ([10], 2.35), $\pi_j(B)$ is a right distributive ring. So, the result follows from Proposition 1.9.

Now, suppose that R is a semicommutative ring. Let $a \in R$ and $1_j x^j \in S$. By assumption, $\pi_j(B)$ is a right Bezout ring and we have that $\overline{aS + 1_j x^j S}$ is a right principal ideal of $\pi_j(B)$, for all $j \geq 1$. Thus, there exists $f, g, m, u \in B$ and $l_1, l_2 \in \sum_{i \geq j+1} S_i x^i$ such that

$$(ag + 1_j x^j m)f = a + l_1$$

$$(ag + 1_j x^j m)u = 1_j x^j + l_2.$$

Hence, we get $ag_0 f_0 = a$, $ag_0 u_0 = 0$ and

$$ag_j \alpha_j(u_0 1_{-j}) + ag_{j-1}\alpha_{j-1}(u_1 1_{1-j}) + \cdots + ag_0 u_j + \alpha_j(m_0 u_0 1_{-j}) = 1_j. \quad (1)$$

Since $r_R(ag_0)$ is a two-sided ideal of R then $f_0 m_0 u_0 \in r_R(ag_0)$. By the fact that $a = ag_0 f_0$, we obtain that $\alpha_j(am_0 u_0 1_{-j}) = 0$. Let $b = g_j \alpha_j(u_0 1_{-j}) + g_{j-1}\alpha_{j-1}(u_1 1_{1-j}) + \cdots + g_0 u_j$. By (1), we get $ab + \alpha_j(m_0 u_0 1_{-j}) = 1_j$ and

$$\begin{aligned}
\alpha_j(a1_{-j}) &= \alpha_j(a1_{-j})[ab + \alpha_j(m_0 u_0 1_{-j})] \\
&= \alpha_j(a1_{-j})ab + \alpha_j(am_0 u_0 1_{-j}) = \alpha_j(a1_{-j})ab.
\end{aligned}$$

Thus, by Proposition 1.6, R is a strongly regular ring. So, R is a right duo ring and, in particular, right quasi-duo. $\square$

We say that a right R-module M is $\aleph_0$-*algebrically compact* if any system $L(a_{ij}, m_i, M)$ with a countable number of linear equations

$$\left\{ \sum_{j=0}^{t(i)} x_j a_{ij} = m_i \right\}_{i=0}^{\infty}$$

with coefficients $a_{ij} \in R$, $m_i \in M$ and with a countable number of variables $\{x_j\}_{j=0}^{\infty}$ assuming values on M such that all finite subsystem of this system has a solution, then $L(a_{ij}, m_i, M)$ has a solution on M.

Following [10], a right R-module E is *flat* if, for each left R-module monomorphism $u : M_1 \to M_2$, the group homomorphism $f : E \otimes M_1 \to E \otimes M_2$ defined by

$$f(a \otimes b) = (id_E \otimes u)(a \otimes b) = a \otimes u(b)$$

is a monomorphism.

Our next result generalizes ([10], 6.69(2)).

Proposition 1.11. *Let R be an $\aleph_0$-injective strongly regular ring such that all idempotent elements of R are α-invariant. Then A is a reduced distributive Bezout ring and all submodules of an A-module flat are flat.*

PROOF. We show that A is a right distributive ring, i.e., following ([10], 2.4) it is equivalent to show that, for any $u = \sum_{i \geq 0} u_i x^i$ and $v = \sum_{i \geq 0} v_i x^i \in A$, there exists $f = \sum_{i \geq 0} f_i x^i$, $g = \sum_{i \geq 0} g_i x_i$, $h = \sum_{i \geq 0} h_i x^i$, $k = \sum_{i \geq 0} k_i x^i \in A$ such that $1_R = f + g$, $fu = hv$ and $gv = ku$. In fact, we have a system of three equations on A equivalent to a countable system of linear equations Q on R_R with a countable number of left variables $f_i, g_i, h_i, k_i \in R$, for all $i \geq 0$, with right

coefficients in R depending of fixed variables $\{u_i, v_i \in R\}$, where each equation has only a finite number of variables.

Let N be a finite subsystem of Q. Then there exists $n \geq 1$ such that N doesn't have variables $f_i, g_i, h_i, k_i \in R$, for all $i \geq n$. By Proposition 1.8(3), $A/M_{n-1} \simeq B/\sum_{i \geq n} S_i x^i$ and, by Proposition 1.7(2), A/M_{n-1} is a distributive Bezout ring, for all $n \geq 1$. Thus, by ([10], 2.4), the extension of N to the quotient A/M_{n-1} is a solvable system. Hence, all finite subsystem of Q is solvable in R.

So, by ([10], 4.88), R is right $\aleph_0$-algebrically compact and we have that Q is solvable in R. The result follows from Propositions 1.7(1), 1.8(6) and ([10], 4.21). $\qquad\square$

Our next result generalizes ([10], 6.70).

Proposition 1.12. *Suppose that α is a partial action of finite type. If each right ideal of A generated by two elemens is flat, then R is a von Neumann regular ring.*

PROOF. Since α is of finite type, then $R = \bigoplus_{j=1}^n D_j$, where $D_j = Re_j$, for all $1 \leq j \leq n$ and $\{e_j : 1 \leq j \leq n\}$ is a set of central orthogonal idempotents. According to Lemma 1.5(2), it is enough to show that D_j is a von Neumann regular ring, for all $1 \leq j \leq n$. In fact, let $a \in D_j$. Then $a1_j x^j = 1_j x^j \alpha_{-j}(a)$ and, by ([10], 4.24), there exists $f, g, t \in A$ such that

$$af = 1_j x^j g$$

$$(1_R - f)1_j x^j = t\alpha_{-j}(a).$$

Thus, we have that $af_0 = 0$ and $(1_R - f_0)1_j = t_j a$. Hence,

$$a = a - af_0 = a(1_R - f_0) = a(1_R - f_0)1_j = at_j a. \qquad\square$$

The proof of the next proposition is straightforward.

Proposition 1.13. *If $L = \bigoplus_{i \in \mathcal{I}} R_i$ is a direct sum of a family of $\aleph_0$-injective strongly regular rings, then L is an $\aleph_0$-injective strongly regular ring.*

Now, we are prepared to prove the principal results of this section. The next result generalizes ([10], 6.71).

Theorem 1.14. *Suppose that α is a partial action of finite type. The following conditions are equivalent:*

(1) *A is a right distributive ring.*

(2) A is a right Bezout ring and R is either right quasi-duo or semicommutative.

(3) A is either right Bezout or right distributive ring and R is a strongly regular ring.

(4) A is a reduced distributive Bezout ring and all submodules of a flat A-module is flat.

(5) R is an $\aleph_0$-injective strongly regular ring and all idempotent elements of R are α-invariant.

PROOF. During this proof we consider the canonical epimorphism $\pi_n : A \rightarrow A/M_n$ for any $n \geq 1$, where $M_n = \sum_{i \geq n} S_i x^i$.

(1) $\Rightarrow$ (3) Note that $\pi_n(A)$ is a right distributive ring as homomorphic image of A. Thus by Proposition 1.9(2), R is a strongly regular ring.

(2) $\Rightarrow$ (3) Since $\pi_n(A)$ is a right Bezout ring (as homomorphic image of A) then, by Proposition 1.8(3), we have that

$$\pi_n(A) = A/M_n \simeq B/ \sum_{i \geq n+1} S_i x^i.$$

Thus, $B/ \sum_{i \geq n+1} S_i x^i$ is a right Bezout ring, for all $n \geq 1$. So, by Proposition 1.10, R is a strongly regular ring.

(5) $\Rightarrow$ (4) Follows directly from Proposition 1.11.

(4) $\Rightarrow$ (3) Note that $\pi_n(A)$ is a right distributive ring as homomorphic image of the right distributive ring A. Thus, by Proposition 1.9(2), R is a strongly regular ring.

(3) $\Rightarrow$ (1) If R is a strongly regular ring and A is a right Bezout ring then, by Proposition 1.8(6), A is a right distributive ring.

(3) $\Rightarrow$ (2) Let R be a strongly regular ring. Then R is a right duo ring and, in particular, R is a semicommutative right quasi-duo ring. Moreover, if A is a right distributive ring then, by Proposition 1.8(6), A is a right Bezout ring.

(3) $\Rightarrow$ (5) Suppose that R is a strongly regular ring. Thus, by Proposition 1.8(6), A is a right distributive ring and so $\pi_n(A)$ is a right distributive ring as homomorphic image of A. By Proposition 1.9(2), all idempotent elements of R are α-invariant and, by Proposition 1.7(1), A is a reduced ring. From ([10], 4.21) we have that all (right or left) submodules of a flat A-module are flat. Thus, by ([10], 4.18), all right ideals of A generated by two elements are flat.

We show that S_n is $\aleph_0$-injective, for any $n \neq 0$. In fact, let $\{c_i\}_{i \geq 0}$ be a countable subset of S_n and $\{e_i\}_{i \geq 0}$ a countable set of central mutually orthogonal

idempotents of S_n. Then, there exists a countable set $\{a_i\}_{i \geq 0}$ of S_{-n} such that $a_i = \alpha_{-n}(c_i)$, for all $i \geq 0$. Consider

$$u = e_i x^n$$
$$v = c_i x^n e_i x^n = c_i e_i x^{2n}$$
$$w = e_i x^n c_i x^n = e_i \alpha_n(c_i 1_{-n}) x^{2n}.$$

Note that $wu = e_i \alpha_n(c_i 1_{-n}) x^{3n} = uv$. Moreover, there exists $f = \sum_{i \geq 0} f_i x^i$, $g = \sum_{i \geq 0} g_i x^i$, $h = \sum_{i \geq 0} h_i x^i \in A$, such that

$$(1_R - f)v = gu \tag{2}$$
$$uf = wh. \tag{3}$$

From (2), we obtain that

$$g e_i x^n = (1_R - f)c_i e_i x^{2n} = c_i e_i x^{2n} - f c_i e_i x^{2n}$$

where $g_n e_i = c_i e_i - e_i f_0 c_i$.

From (3), we obtain that $e_i x^n f = e_i \alpha_n(c_i 1_{-n}) x^{2n} h$, where $e_i \alpha_n(f_0 1_{-n}) = 0$. Since $e_i \in R$ is α-invariant, then $e_i f_0 1_{-n} = 0$ and we have that

$$(e_i f_0 x^n)^2 = e_i f_0 \alpha_n(e_i f_0 1_{-n}) x^{2n} = 0.$$

By the fact that A is a reduced ring we obtain $e_i f_0 = 0$. So

$$g_n e_i = c_i e_i - e_i f_0 c_i = c_i e_i,$$

for all $i \geq 0$ and, by ([10], 4.88), S_n is an $\aleph_0$-injective ring, for all $n \neq 0$. Therefore, by Proposition 1.13, R is $\aleph_0$-injective. $\qquad\square$

Now, we prove the second principal result of this section, which generalizes ([10], 6.72).

Theorem 1.15. *Suppose that α is a partial action of finite type and R is an abelian ring such that all idempotent elements of R are α-invariant. The following conditions are equivalent:*

(1) *All submodules of a flat A-module are flat.*

(2) *All right ideals of A generated by two elements are flat.*

(3) *R is a strongly $\aleph_0$-injective regular ring.*

PROOF. (1) $\Rightarrow$ (2) Follows directly from ([10], 4.18).

(2) $\Rightarrow$ (3) By Proposition 1.12, R is a von Neumann regular ring. By ([5], Theorem 3.5), R is strongly regular. Now, using an analogous process used in Theorem 1.14 ((3) $\Rightarrow$ (5)), we have that R is an $\aleph_0$-injective ring.

(3) $\Rightarrow$ (1) Follows from Theorem 1.14. $\qquad\square$

2. Our principal result

During this section, α is a partial action of $\mathbb{Z}$ on a ring with unit R such that (T, σ) is its enveloping action where $\sigma : T \to T$ is an automorphism of T. From now on we denote $A = R[[x; \alpha]]$ and $B = R[x; \alpha]$.

Our first result generalizes ([9], Proposition 2.2).

Proposition 2.1. *If A is a right duo ring, then R is a right duo ring such that all idempotent elements of R are α-invariant.*

PROOF. Let $a \in R$. Then $ba \in Aa \subseteq aA$, for all $b \in R$, because A is right duo and there exists $f = \sum_{n \geq 0} f_n x^n \in A$ such that $ba = af$. Thus $ba = af_0 \in aR$ and it follows that $Ra \subseteq aR$. So, R is a right duo ring.

Now, let $e \in R$ be an idempotent element. Note that all right duo rings are semicommutative and, in particular, abelian. Hence, we have that e is central in A and

$$e.1_j x^j = 1_j x^j.e = \alpha_j(e.1_{-j}) x^j$$

for all $j \geq 1$. So, e is α-invariant. $\qquad \square$

For all $l \in \mathbb{Z}$, $\sigma^l(R)[[x; \sigma]]$ is the set of all series in the form $\sum_{i=0}^{\infty} \sigma^i(a_i) x^i$ and it is not difficult to see that is a right ideal of $T[[x; \sigma]]$.

The proof of the next lemma follows the same ideas of the analogous result in ordinary power series rings.

Lemma 2.2. *Let $l \in \sigma^{-i}(R)[[x; \sigma]] \subseteq T[[x; \sigma]]$ be a serie with coefficients in $\sigma^{-i}(R)$ such that its independent term is invertible in $\sigma^{-i}(R)$. Then there exists $t \in \sigma^{-i}(R)[[x; \sigma]]$ such that $lt = \sigma^{-i}(1_R)$.*

The next result generalizes ([9], Proposition 2.4).

Proposition 2.3. *Suppose that R is a $\aleph_0$-injective strongly regular ring such that all idempotents of R are α-invariant. Then all right principal ideals of $A = R[[x; \alpha]]$ are generated by a series whose coefficients are central idempotents mutually orthogonal in R.*

PROOF. Let $f = \sum_{n \geq 0} f_n x^n \in A$, where $f_n \in S_n$, for all $n \geq 0$ and fA a right principal ideal generated by f.

By assumption, R is a strongly regular ring, and we can write $f_n = d_n u_n$, where $d_n \in R$ is a central idempotent and $u_n \in U(R)$, for all $n \geq 0$. Note that $d_n \in S_n$, for all $n \geq 0$, because $f_n = f_n 1_n = (1_n d_n) u_n$.

Let

$$e_n = d_n(1_R - d_{n-1}) \ldots (1_R - d_0) \in S_n$$

for all $n \geq 0$ and consider $g = \sum_{n\geq 0} e_n x^n \in A$. Then, for $i > j$,

$$e_i e_j = d_i(1_R - d_{i-1})\ldots(1_R - d_j)\ldots(1_R - d_0).d_j(1_R - d_{j-1})\ldots(1_R - d_0) = 0$$

and it folows that the coefficients of g are central idempotents and mutually orthogonal. We consider $X = \{\alpha_{-m}(f_{m+n}1_m)\}_{m,n\geq 0}$ be a countabel subset of R. Since R is $\aleph_0$-injective strongly regular then, by ([10], 4.88), there exists a countable subset $\{b_n\}_{n\geq 0}$ of R such that

$$b_n e_m = \alpha_{-m}(f_{m+n}1_m)e_m$$

for any $m, n \geq 0$. By assumption, all idempotents in R are α-invariant and, for any $m, n \geq 0$,
$$e_m \alpha_m(b_n 1_{-m}) = e_m f_{m+n}. \tag{4}$$

We claim that $fA \subseteq gA$. In fact, let $h = \sum_{n\geq 0} b_n 1_n x^n \in A$ such that $gh = \sum_{n\geq 0} c_n x^n$, where $c_n = \sum_{i+j=n} e_i \alpha_i(b_j 1_j 1_{-i})$. By the fact that $f_n = d_n u_n = d_n^2 u_n = d_n f_n$ we have that by (4),

$$c_n = \sum_{i+j=n} e_i \alpha_i(b_j 1_j 1_{-i}) = \sum_{i+j=n} e_i \alpha_i(b_j 1_{-i})1_n$$

$$= \sum_{i+j=n} e_i f_{i+j} 1_n = \left(\sum_{i=0}^{n} e_i\right) f_n = \left[\left(\sum_{i=0}^{n} e_i\right)d_n\right]f_n \in S_n.$$

and

$$\sum_{i=0}^{n-1} e_i + \prod_{i=0}^{n-1}(1_R - d_i) = e_0 + e_1 + \cdots + e_{n-1} + (1_R - d_0)\ldots(1_R - d_{n-1})$$

$$= e_0 + e_1 + \ldots + e_{n-1} + (1_R - d_0)\ldots(1_R - d_{n-2})1_R - (1_R - d_0)\ldots(1_R - d_{n-2})d_{n-1}$$

$$= e_0 + e_1 + \cdots + e_{n-2} + (1_R - d_0)\ldots(1_R - d_{n-2})$$

where, by induction, we have

$$\sum_{i=0}^{n-1} e_i + \prod_{i=0}^{n-1}(1_R - d_i) = 1_R.$$

Thus,

$$\left(\sum_{i=0}^{n} e_i\right)d_n = (e_0 + e_1 + \cdots + e_{n-1})d_n + e_n d_n = \sum_{i=0}^{n-1} e_i d_n + e_n$$

$$= \sum_{i=0}^{n-1} e_i d_n + \prod_{i=0}^{n-1}(1_R - d_i)d_n = \left[\sum_{i=0}^{n-1} e_i + \prod_{i=0}^{n-1}(1_R - d_i)\right]d_n = 1_R d_n = d_n$$

and

$$c_n = \left[\left(\sum_{i=0}^{n} e_i\right) d_n\right] f_n = d_n f_n = f_n$$

for all $n \geq 0$. Hence $gh = f$ and then, $fA \subseteq gA$.

Next, we show that $gA \subseteq fA$. In fact, since all idempotents of R are α-invariant, then $1_n = \alpha_n(1_{-n}) = 1_n 1_{-n}$ and we have that

$$1_{-n} = \alpha_{-n}(1_n) = \alpha_{-n}(1_n 1_{-n}) = 1_n 1_{-n}.$$

Hence, $1_n = 1_{-n}$, for all $n \geq 0$.

Note that $g = fp$ if and only if, $e_n = \sum_{i+j=n} f_i \alpha_i(p_j 1_{-i})$, for all $n \geq 0$, where $p = \sum_{n \geq 0} p_n x^n \in A$, with $p_n \in S_n$, for all $n \geq 0$. In this case, we obtain a system with a countable number of linear equations such that

$$e_n = \alpha_{-n}(e_n) = \alpha_{-n}\left(\sum_{i+j=n} f_i \alpha_i(p_j 1_{-i})\right)$$

$$= \sum_{i+j=n} \alpha_{-n}(f_i 1_n)\alpha_{i-n}(p_j 1_{n-i})1_{-n} = \sum_{i+j=n} \alpha_{-n}(f_i 1_n)\alpha_{-j}(p_j).$$

From ([10], 4.88) we have that R is $\aleph_0$-algebraically compact by (5) and the system

$$e_n = \sum_{i+j=n} \alpha_{-n}(f_i 1_n)x_j \tag{5}$$

is solvable in R if, and only if, all its finite subsystem is solvable in R. Now, we show that the finite subsystems are solvable in R.

We clearly have that

$$fe_i = \sum_{n \geq 0} f_n x^n e_i = \sum_{n \geq 0} f_n \alpha_n(e_i 1_{-n})x^n = \sum_{n \geq 0} f_n e_i 1_n x^n = \sum_{n \geq 0} f_n e_i x^n.$$

Thus, for all $n < i$,

$$f_n e_i = u_n d_n e_i = (u_n d_n) d_i(1_R - d_{i-1})\dots(1 - d_n)\dots(1_R - d_0) = 0$$

and we have that $fe_i = \sum_{n \geq i} f_n e_i x^n$.

Let $h_1 = \alpha_{-i}(u_i 1_i) + \sum_{j \geq 1} \alpha_{-i}(f_{i+j} 1_i)x^j \in A$. Then,

$$e_i x^i . h_1 = e_i u_i x^i + \sum_{j \geq 1} e_i f_{i+j} x^{i+j} = e_i f_i x^i + \sum_{j \geq 1} e_i f_{i+j} x^{i+j} = \sum_{n \geq i} f_n e_i x^n.$$

Each element $e_i \in S_i$ is α-invariant and

$$
\begin{aligned}
fe_i = \sum_{n \geq i} f_n e_i x^n = e_i x^i.h_1 &= x^i e_i \left[\alpha_{-i}(u_i 1_i) + \sum_{j \geq 1} \alpha_{-i}(f_{i+j} 1_i) x^j \right] \\
&= e_i x^i \left[\sigma^{-i}(u_i) + \sum_{j \geq 1} \sigma^{-i}(f_{i+j}) x^j \right] = e_i x^i h_2
\end{aligned}
$$

where $h_2 = \sigma^{-i}(u_i) + \sum_{j \geq 1} \sigma^{-i}(f_{i+j}) x^j \in \sigma^{-i}(R)[[x;\sigma]]$ has invertible independent term in $\sigma^{-i}(R)$ and we have that by Lemma 2.2, there exists $t \in \sigma^{-i}(R)[[x;\sigma]]$ such that $h_2 t = \sigma^{-i}(1_R)$. Thus

$$
e_i x^i = e_i x^i \sigma^{-i}(1_R) = e_i x^i (h_2 t) = (e_i x^i h_2) t = (fe_i) t.
$$

Define $q_m = \sum_{i=0}^{m} e_i t$, for all $m \geq 0$ and it follows that

$$
e_0 + e_1 x + \cdots + e_m x^m = \sum_{i=0}^{m} e_i x^i = \sum_{i=0}^{m} fe_i t = f \sum_{i=0}^{m} e_i t = fq_m. \tag{6}
$$

If $q_m = \sum_{n \geq 0} a_n x^n \in A$, with $y_j 1_{-j} = \alpha_{-j}(a_j)$, for all $j \geq 0$, then

$$
\begin{aligned}
fq_m = \sum_{n \geq 0} f_n x^n . \sum_{n \geq 0} a_n x^n &= \sum_{n \geq 0} \left(\sum_{i+j=n} f_i \alpha_i(a_j 1_{-i}) \right) x^n \\
&= \sum_{n \geq 0} \left(\sum_{i+j=n} f_i \alpha_i(\alpha_j(y_j 1_{-j}) 1_{-i}) \right) x^n = \sum_{n \geq 0} \left(\sum_{i+j=n} f_i \alpha_n(y_j 1_{-n}) \right) x^n.
\end{aligned}
$$

By (6),

$$
\sum_{i=0}^{m} e_i x^i = \sum_{n \geq 0} \left(\sum_{i+j=n} f_i \alpha_n(y_j 1_{-n}) \right) x^n
$$

and we obtain that $e_n = \sum_{i+j=n} f_i \alpha_n(y_j.1_{-n})$, for all $0 \leq n \leq m$. So,

$$
e_n = \alpha_{-n}(e_n) = \sum_{i+j=n} \alpha_{-n}(f_i 1_n) y_j
$$

for all $0 \leq n \leq m$. □

Lemma 2.4. *Suppose that R is an $\aleph_0$-injective strongly regular ring such that all idempotents are α-invariant. Then $A = R[[x;\alpha]]$ is a right duo ring.*

PROOF. Let $f = \sum_{n\geq 0} f_n x^n \in A$ and $g = \sum_{n\geq 0} g_n x^n \in A$. Then by Proposition 2.3, the coefficients of f are central idempotents that are mutually ortogonal in R. We consider the countable subset $\{\alpha_{-n}(g_i 1_n)\}_{i,n\geq 0}$ of R. By ([10], 4.88), there exists a countable subset $\{t_i\}_{i\geq 0}$ of R such that $f_n t_i = e_n \alpha_{-n}(g_i 1_n)$, for any $i, n \geq 0$. Since all idempotents of R are α-invariant, then

$$f_n \alpha_n(t_i 1_{-n}) = f_n g_i \tag{7}$$

for any $i, n \geq 0$.

Let $h = \sum_{n\geq 0} t_n.1_n x^n \in A$. Then

$$gf = \left(\sum_{n\geq 0} g_n x^n\right)\left(\sum_{n\geq 0} f_n x^n\right) = \sum_{n\geq 0}\left(\sum_{i+j=n} g_i f_j\right)x^n$$

$$= \left(\sum_{n\geq 0}\left(\sum_{i+j=n} g_i f_j\right)x^n\right)1_R = \sum_{n\geq 0}\left(\sum_{i+j=n} g_i f_j\right)1_n x^n$$

and, by (7),

$$fh = \left(\sum_{n\geq 0} f_n x^n\right)\left(\sum_{n\geq 0} t_n 1_n x^n\right) = \sum_{n\geq 0}\left(\sum_{i+j=n} f_i \alpha_i((t_j 1_j)1_{-i})\right)x^n$$

$$= \sum_{n\geq 0}\left(\sum_{i+j=n} f_i \alpha_i(t_j 1_{-i})\right)1_n x^n = \sum_{n\geq 0}\left(\sum_{i+j=n} f_i g_j\right)1_n x^n.$$

Hence $gf = fh$ and, so, $Af \subseteq fA$.   □

Now we are in condition to prove our principal result of this article and generalizes ([9], Theorem 1.6 and Corollary 3.1).

Theorem 2.5. *Suppose that α is a partial action of finite type. The following conditions are equivalent:*

(1) *A is a right Bezout and right duo ring.*

(2) *A is a right Bezout and reduced ring.*

(3) *A is a right Bezout and right quasi-duo ring.*

(4) *A is a right Bezout and semicommutative ring.*

(5) *A is a right distributive and right duo ring.*

(6) *A is a right distributive and reduced ring.*

(7) *A is a right distributive ring.*

(8) *All right ideals of A generated by two elements are flat, R is an abelian ring and all idempotents of R are α-invariant.*

(9) *A is a right duo ring and all submodules of a flat A-module are flat.*

(10) *All submodules of a flat A-module are flat, R is an abelian ring and all idempotent of R are α-invariant.*

(11) *R is an $\aleph_0$-injective strongly regular ring and all idempotents of R are α-invariant.*

(12) *A is a right Bezout ring and R is a right duo ring.*

(13) *A is a right Bezout ring and R is a reduced ring.*

(14) *A is a right Bezout ring and R is a right quasi-duo ring.*

(15) *A is a right Bezout ring and R is a semicommutative ring.*

(16) *A is a right Bezout ring and R is a strongly regular ring.*

(17) *R is a right duo $\aleph_0$-injective von Neumann regular and all idempotents of R are α-invariant.*

PROOF. By Theorem 1.14, (7) and (11) are equivalent. By Theorem 1.15, (8), (10) and (11) are equivalent. Clearly (5) $\Rightarrow$ (7). Since (7) $\Leftrightarrow$ (11) and by Lemma 2.4, then A is a right duo ring. Thus, (7) $\Rightarrow$ (5).

By the fact that (7) $\Rightarrow$ (11) we have that R is strongly regular and all idempotents of R are α-invariant. Let $a \in S_j$ such that $a\alpha_j(a1_{-j}) = 0$, for some $j \in \mathbb{Z}$. Since S_j is strongly regular, for all $j \in \mathbb{Z}$, then $a = ue$, where $u \in U(S_j)$ and $e = e^2 \in S_j$. Hence,

$$0 = a\alpha_j(a1_{-j}) = ue\alpha_j(ue1_{-j}) = ue\alpha_j(u1_{-j})$$

and, then $e = e1_j = 0$. So, α is partially α-rigid and by Lemma 1.1, A is a reduced ring.

Clearly (6) $\Rightarrow$ (7).

By (7) $\Rightarrow$ (11) and Lemma 2.4, we obtain that A is a right duo ring. Moreover, by Theorem 1.14, A is a right Bezout ring. Hence (7) $\Rightarrow$ (1). Since all right duo rings are right quasi-duo, then (1) $\Rightarrow$ (3). By ([10], 2.35), all right Bezout rings and right quasi-duo rings are right distributive and it follows that (3) $\Rightarrow$ (7).

From (7) $\Rightarrow$ (1) and (7) $\Rightarrow$ (6) it follows that (7) $\Rightarrow$ (2) and since all reduced rings are semicommutative then (2) $\Rightarrow$ (4).

If A is semicommutative then $R \subseteq A$ is semicommutative. Hence, by Theorem 1.14, (4) $\Rightarrow$ (7).

All right duo ring are semicommutative and, in particular, are abelian. By Proposition 2.1, we have that (9) $\Rightarrow$ (10).

Conversely, assume (10). Since (10) $\Leftrightarrow$ (11) then, by Lemma 2.4, A is right duo and, we obtain (9).

Clearly (12) $\Rightarrow$ (14). By Theorem 1.14, (14) $\Rightarrow$ (16). Since all strongly regular rings are duo, then (16) $\Rightarrow$ (12).

Again, by Theorem 1.14, (7), (15) and (16) are equivalent. By the fact that all reduced rings are semicommutative, then $(13) \Rightarrow (15)$.

Note that all strongly regular rings are reduced then $(16) \Rightarrow (13)$ and we obtain that (13) and (15) are equivalent.

Suppose (17). Then R is an $\aleph_0$-injective von Neumann regular right duo ring. Thus, R is abelian and, by ([5], Theorem 3.5), R is strongly regular. Hence, we get (11).

Suppose (11). Then R is $\aleph_0$-injective strongly regular. In particular, R is duo and, by ([5], Theorem 3.5), R is abelian von Neumann regular. Then we get (17). $\qquad\qquad\square$

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