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Delay-dependent stability analysis of fuzzy Cohen–Grossberg neural networks with impulse

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Abstract. In this work, a class of fuzzy Cohen–Grossberg neural networks (FCGNNs) with time-varying delays and impulse are considered. Applying differential inequality techniques, some sufficient conditions for the existence, uniqueness and global exponential stability of equilibrium point for the addressed neural network are obtained. Moreover an example illustrates the effectiveness of obtained results.

1. Introduction

In recent years, Cohen–Grossberg neural networks [1] have been extensively studied due to their extensive applications in many fields such as pattern recognition, computing associative memory, signal and image processing and so on. In reality, During hardware implementation, time delays occur due to finite switching speed of the amplifiers and communication time. Time delay may lead to an oscillation and furthermore, to instability of networks. Therefore, the study of stability of neural networks with delay is practically required. Up to now, many results are obtained for the neural networks with delays (see, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16].)

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However, in mathematical modeling of real world problems, we encounter some inconveniences besides delays, namely, the complexity and the uncertainty or vagueness. For the sake of taking vagueness into consideration, fuzzy theory is viewed as a more suitable setting. Based on traditional CNNs, YANG and YANG [17], [18] first introduced fuzzy cellular neural networks (FCNNs), which integrates fuzzy logic into the structure of traditional CNNs and maintains local connectedness among cells. Unlike previous CNNs, FCNN is a very useful paradigm for image processing problems, which has fuzzy logic between its template input and/or output besides the sum of product operation. Recently, some results on stability and other behaviors have been derived for fuzzy neural networks with or without delay (see,[17], [18], [19], [20], [21], [22], [23].)

In generally speaking, dynamical systems are often classified into two categories of either continuous-time or discrete-time systems. These two dynamic systems are widely studied in population models and neural networks, yet there is somewhat new category of dynamical systems, which is neither continuous-time nor purely discrete-time; these are called dynamical systems with impulses. A fundamental theory of impulsive differential equations has been developed in [29]. For instance, in the implementation of neural networks, the state of the networks is subject to instantaneous perturbations and experiences abrupt changes at certain instants, which may be caused by switching phenomenon, frequency change or other sudden noise that it exhibits impulsive effects [24], [25], [26], [27], [28]. Neural networks are often subject to impulsive perturbations that in turn affect dynamical behaviors of the systems.

Motivated by the aforementioned discussion, in this paper we aim to investigate the global exponential stability for fuzzy Cohen–Grossberg neural networks (FCGNNs) with both time-varying delays and impulsive effects. To the best of our knowledge, the dynamical behavior of FCGNNs with delays and impulses are seldom considered. In [30], LI, LI and YE investigated the stability of FCGNNs with constant delays and impulsive. However, The time delays of neural networks vary usually with time. Therefore, it is necessary to consider fuzzy Cohen–Grossberg neural networks with time-varying delays and impulsive effects which described by the following system.

$$\begin{cases} \dfrac{dx_i(t)}{dt} = \alpha_i(x_i(t))\Big[-\beta_i(x_i(t)) + \sum\limits_{j=1}^{n} a_{ij} f_j(t-\tau_{ij}(t)) \\ \qquad + \bigwedge\limits_{j=1}^{n} T_{ij}u_j + \bigvee\limits_{j=1}^{n} H_{ij}u_j + I_i + \bigwedge\limits_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij}(t))) \\ \qquad + \bigvee\limits_{j=1}^{n} c_{ij} f_j(x_j(t-\tau_{ij}(t)))\Big], \quad t \neq t_k,\ t \geq t_0, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = \Delta_{ik}(x_i(t_k)), \quad t = t_k. \end{cases} \tag{1.1}$$

for $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots$, where $x_i(t)$ is the ith neuron state, $\alpha_i(x_i(t))$ represents an amplification function, $\beta_i(x_i(t))$ is an appropriately behaved function, f_j denote activation functions. a_{ij} is the element of feed-back template, b_{ij}, c_{ij} are elements of fuzzy feed-back MIN template and fuzzy feed-back MAX template, respectively. T_{ij} and H_{ij} are elements of fuzzy feed-forward MIN template and fuzzy feed-forward MAX template, respectively. $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operations. u_i and I_i denote input and bias of the ith neuron, respectively. t_k is called impulsive moment and satisfies $0 < t_1 < t_2 < \ldots$, $\lim_{k\to\infty} t_k = +\infty$. $x_i(t_k^-)$ and $x_i(t_k^+)$ denote the left limit and the right limit at t_k, respectively; $\Delta_k(x(t_k)) = (\Delta_{1k}(x_1(t_k)), \Delta_{2k}(x_2(t_k)), \ldots, \Delta_{nk}(x_n(t_k)))^T$, $\Delta_{ik}(x_i(t_k))$ shows impulsive perturbation of the ith neuron at t_k. $\tau_{ij}(t)$ correspond to transmission delays and satisfy $0 < \tau_{ij}(t) < \tau, \tau'_{ij}(t) < 0$, where τ is constant.

Remark 1.1. If $\Delta x_i(t_k) = 0$ $(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots.)$ Then system (1.1) becomes continuous FCGNNs

$$\frac{dx_i(t)}{dt} = \alpha_i(x_i(t)) \Bigg[-\beta_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(t-\tau_{ij}(t)) + \bigwedge_{j=1}^{n} T_{ij}u_j + \bigvee_{j=1}^{n} H_{ij}u_j \\ + I_i + \bigwedge_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij}(t))) + \bigvee_{j=1}^{n} c_{ij} f_j(x_j(t-\tau_{ij}(t)))\Bigg] \tag{1.2}$$

Throughout this paper, we assume that

(A1) $\alpha_i(u)$ is a continuous function and $0 < \underline{\alpha_i} \leq \alpha_i(u) \leq \overline{\alpha_i}$ ($\underline{\alpha_i}$ and $\overline{\alpha_i}$ are constants) for all $u \in R$, $i = 1, 2, \ldots, n$.

(A2) There exists a positive diagonal matrix $\beta = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)$ such that

$$\frac{\beta_i(x) - \beta_i(y)}{x - y} \geq \beta_i > 0, \quad x \neq y,\ i = 1, 2, \ldots, n.$$

(A3) There exist positive diagonal matrices $L = \mathrm{diag}(L_1, L_2, \ldots, L_n)$ such that

$$|f_j(x) - f_j(y)| \leq L_j|x - y|, \quad x \neq y,\ j = 1, 2, \ldots, n.$$

(A4) There exist constants $\delta_i > 0$ $(i = 1, 2, \ldots, n)$ such that

$$\beta_i > \frac{1}{2}\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}| + |c_{ij}|)L_j + \frac{1}{2}\sum_{j=1}^{n}\frac{\delta_j}{\delta_i}(|a_{ji}| + |b_{ji}| + |c_{ji}|)L_i.$$

(A5) The time delay $\tau_{ij}(t)$ is assumed to be time-varying and satisfy $0 < \tau_{ij}(t) < \tau$, $0 < \tau'_{ij}(t) < 1$ $(i, j = 1, 2, \ldots, n)$ where τ is a constant.

The organization of the rest of this paper is as follows. In Section 2, we introduce some notations and definitions and state some preliminary results needed in later sections. In Section 3, we establish our main results by constructing a proper Lyapunov functional. In Section 4, we give an example to illustrate our results.

2. Preliminaries and some notations

Let $PC(I, R^n) = \{x : I \to R^n \mid x$ is continuous everywhere except for some $t_k \in I$ at which $x(t_k^-)$ and $x(t_k^+)$ exist with $x(t^-) = x(t_k),\ k \in z\}$, where $I \subseteq R$ is an interval. For a matrix $A = (a_{ij})_{n\times n} \in R^{n\times n}$, $|A|$ denote the absolute value matrix given $|A| = (|a_{ij}|)_{n\times n}$, for $x = (x_1, x_2, \ldots, x_n)^T$, $|x|$ denotes the absolute-value vector given by $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$. For any $x \in R^n$, let vector norms $\|x\|_2$ and $\|x\|_\infty$ be defined as

$$\|x\|_2 = \left(\sum_{i=1}^{n}|x_i|^2\right)^{1/2}, \|x\|_\infty = \max_{1\leq i\leq n}|x_i|.$$

In order to obtain our results, we give the following definition.

Definition 2.1. Let $x(t) = (x_1(t), \ldots, x_n(t))^T$. $x(t) : R \to R^n$ is called a solution of system (1.1) with the initial conditions $x(s) = \phi(s) \in PC([t_0 - \tau, t_0], R^n)$, if $x(t)$ is continuous at $t \neq t_k$ and $t \geq t_0, x(t_k) = Z(t_k^-)$ and $Z(t_k^+)$ exists, $x(t)$ satisfies system (1.1) for $t \geq t_0$ under the initial condition. Especially, a point $x^* \in R^n$ is called an equilibrium point of system (1.1), if $x(t) = x^*$ is a solution of system (1.1).

Definition 2.2. An equilibrium point of system (1.1) $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ is said to be globally exponential stability, if for any solution $x(t)$ with the initial condition $\phi \in PC((-\infty, t_0], R^n)$, there exist positive constants $M > 1, \lambda > 0$ such that

$$\|x(t) - x^*\|_2 \le M\|\phi - x^*\|e^{-\lambda(t-t_0)},$$

for all $t \ge t_0$, where $\|\phi - x^*\| = \sup_{-\tau\le\theta\le 0} \|\phi(\theta) - x^*\|$.

Definition 2.3. If $f(t) : R \to R$ is a continuous function, then the upper right derivative of $f(t)$ is defined as

$$D^+ f(t) = \limsup_{h\to 0^+} \frac{1}{h}(f(t+h) - f(t)).$$

Lemma 2.1 ([17])**.** *Suppose x and y are two states of system* (1.1)*, then we have*

$$\Big| \bigwedge_{j=1}^{n} b_{ij} f_j(x) - \bigwedge_{j=1}^{n} b_{ij} f_j(y)\Big| \le \sum_{j=1}^{n} |b_{ij}|\, |f_j(x) - f_j(y)|,$$

and

$$\Big| \bigvee_{j=1}^{n} c_{ij} f_j(x) - \bigvee_{j=1}^{n} c_{ij} f_j(y)\Big| \le \sum_{j=1}^{n} |c_{ij}|\, |f_j(x) - f_j(y)|.$$

Lemma 2.2 ([28])**.** *If $H(x)$ is locally invertible continuous mapping satisfying the following conditions:*

(i) *$H(x)$ is injective on R;*

(ii) *$\|H(x)\| \to \infty$ as $x \to \infty$, then $H(x)$ is homeomorphism of R^n onto itself.*

3. Main results

In this section, we will prove main results of this paper.

Theorem 3.1. *Under assumptions* (A1)–(A4)*, furthermore, suppose that the impulsive operator $\Delta x_{ik}(x_i(t_k))$ satisfies*

$$\begin{aligned} &\Delta x_{ik}(x_i(t_k)) = -\lambda_{ik}(x_i(t_k) - x^*),\\ &0 < \lambda_{ik} < 2,\ i = 1, 2, \dots, n; \quad k = 1, 2, \dots . \end{aligned} \tag{3.1}$$

Then system (1.1) *has a unique equilibrium point, which is globally exponentially stable.*

PROOF. Let $\widetilde{I}_i = \bigwedge_{j=1}^n T_{ij}u_j + \bigvee_{j=1}^n H_{ij}u_j + I_i,\ i = 1, 2, \dots, n$. We supposed that $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ is an equilibrium point of system (1.2), then x^* satisfies the following equation:

$$0 = -\beta_i(x_i^*) + \sum_{j=1}^n a_{ij} f_j(x_j^*) + \bigwedge_{j=1}^n b_{ij} f_j(x_j^*) + \bigvee_{j=1}^n c_{ij} f_j(x_j^*) + \widetilde{I}_i, \quad i = 1, 2, \dots, n.$$

Let $H(x) = (h_1(x), h_2(x), \dots, h_n(x))^T$, where

$$h_i(x) = -\beta_i(x_i^*) + \sum_{j=1}^n a_{ij} f_j(x_j^*) + \bigwedge_{j=1}^n b_{ij} f_j(x_j^*) + \bigvee_{j=1}^n c_{ij} f_j(x_j^*) + \widetilde{I}_i, \quad i = 1, 2, \dots, n.$$

In the following, we shall prove that $H(x)$ is a homeomorphism of R^n onto itself.

Firstly we show that $H(x)$ is an injective map on R^n. In fact there exist $x = (x_1, x_2, \dots, x_n)^T \in R^n$, $y = (y_1, y_2, \dots, y_n)^T \in R^n$, and $x \neq y$ such that $H(x) = H(y)$, then, for $i = 1, 2, \dots, n$,

$$\begin{aligned}\beta_i(x_i) - \beta_i(y_i) = &\left[\sum_{j=1}^n a_{ij} f_j(x_j^*) - \sum_{j=1}^n a_{ij} f_j(y_j^*)\right] + \left[\bigwedge_{j=1}^n b_{ij} f_j(x_j^*) - \bigwedge_{j=1}^n b_{ij} f_j(y_j^*)\right] \\ &+ \left[\bigvee_{j=1}^n c_{ij} f_j(x_j^*) - \bigvee_{j=1}^n c_{ij} f_j(y_j^*)\right]\end{aligned}$$

By use of (A2), (A3) and Lemma 2.1, we have, for $i = 1, 2, \dots, n$,

$$\beta_i |x_i - y_i| \leq \sum_{j=1}^n (|a_{ij}| + |b_{ij}| + |c_{ij}|) L_j |x_j - y_j|$$

Then there exist $\delta_i > 0$, applying the element inequality $2ab \leq a^2 + b^2$, we have, for $i = 1, 2, \dots, n$.

$$2\delta_i \beta_i |x_i - y_i|^2 \leq \delta_i \sum_{j=1}^n (|a_{ij}| + |b_{ij}| + |c_{ij}|) L_j (|x_i - y_i|^2 + |x_j - y_j|^2)$$

Take sum of both sides of the above inequality, we have

$$\sum_{i=1}^n 2\delta_i \beta_i |x_i - y_i| \leq \sum_{i=1}^n \delta_i \left\{\sum_{j=1}^n (|a_{ij}| + |b_{ij}| + |c_{ij}|) L_j (|x_i - y_i|^2 + |x_j - y_j|^2)\right\}$$

$$= \sum_{i=1}^{n} \delta_i \left\{ \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}| + |c_{ij}|) L_j + \sum_{i=1}^{n} (|a_{ji}| + |b_{ji}| + |c_{ji}|) L_i \right\} |x_i - y_i|^2$$

Therefore

$$\sum_{i=1}^{n} 2\delta_i \Big\{ \beta_i - \frac{1}{2} \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}| + |c_{ij}|) L_j - \frac{1}{2} \sum_{i=1}^{n} \frac{\delta_j}{\delta_i} (|a_{ji}| + |b_{ji}| + |c_{ji}|) L_i \Big\} |x_i - y_i|^2 \le 0 \tag{3.2}$$

From (A4), we can see that (3.2) holds if and only if $x_i = y_i (i = 1, 2, \dots, n)$. Therefore $x = y$.

Secondly, we prove that $\|H(x_m)\| \to +\infty$ as $\|x_m\| \to +\infty$, for any $\{x_m\}$ of R^n, Namely, we need check that $\|H(x_m) - H(0)\| \to \infty$ as $m \to \infty$.

If it is not true, there exists a sequence $\{x_m\} \subset R^n$ with the property that there is a subsequence of $\{x_m\}$ (for simplicity, denoted again by $\{x_m\}$), such that $\{\|H(x_m) - H(0)\|\}$ is bounded. i.e. for some constant $N_0 > 0$.

$$|h_i(x_m) - h_i(0)| \le N_0, \quad i = 1, 2, \dots, n, \ m = 1, 2, \dots, \tag{3.3}$$

where $h_i(x_m)$ is the ith component of $H(x_m)$. Denote $x_m^{(i)}$ as the ith component of x_m. We get, for $i = 1, 2, \dots, n$

$$\beta_i(x_m^{(i)}) + h_i(x_m) - h_i(0) \le \sum_{j=1}^{n} a_{ij} [f_j(x_m^{(j)}) - f_j(0)] + \bigwedge_{j=1}^{n} c_{ij} [f_j(x_m^{(j)}) - f_j(0)] + \bigvee_{j=1}^{n} c_{ij} [f_j(x_m^{(j)}) - f_j(0)] \tag{3.4}$$

Furthermore,

$$\beta_i |x_m^{(i)}| - |h_i(x_m) - h_i(0)| \le \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}| + |c_{ij}|) L_j |x_m^{(j)}|$$

Multiply both sides of above inequality by $2\delta_i$, applying the element inequality $2ab \le a^2 + b^2$, we get, for $i = 1, 2, \dots, n$,

$$2\delta_i \beta_i |x_m^{(i)}| - 2\delta_i |h_i(x_m) - h_i(0)| \le \delta_i \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}| + |c_{ij}|) L_j (|x_m^{(i)}| + |x_m^{(j)}|).$$

Taking sum of both sides of the above inequality, we have

$$\sum_{i=1}^{n}\delta_i\left\{\beta_i-\frac{1}{2}\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|+|c_{ij}|)L_j-\frac{1}{2}\sum_{j=1}^{n}\frac{\delta_j}{\delta_i}(|a_{ji}|+|b_{ji}|+|c_{ji}|)L_i\right\}|x_m^{(i)}|^2$$
$$\leq\sum_{j=1}^{n}2\delta_i|h_i(x_m)-h_i(0)||x_m^{(i)}|.$$

By (A4) and (3.3), it follows that

$$p_0\sum_{i=1}^{n}|x_m^{(i)}|^2\leq\sum_{i=1}^{n}2\delta_iN_0|x_m^{(i)}|\leq N\sum_{i=1}^{n}|x_m^{(i)}|, \tag{3.5}$$

where $N=\max_{1\leq i\leq n}2\delta_iN_0$ and

$$p_0=\min_{1\leq i\leq n}\delta_i\left\{\beta_i-\frac{1}{2}\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|+|c_{ij}|)L_j\right.$$
$$\left.-\frac{1}{2}\sum_{j=1}^{n}\frac{\delta_j}{\delta_i}(|a_{ji}|+|b_{ji}|+|c_{ji}|)L_i\right\}>0$$

Since all the kinds of norms defined on R^n are equivalent, there exists a constant $D>0$, such that $\|x\|_1\leq D\|x\|_2$. From (3.5), we have

$$p_0\|x_m\|_2^2\leq nN\|x_m\|_1\leq nND\|x_m\|_2$$

Namely,

$$\|x_m\|_2\leq\frac{nN}{p_0}D<\infty$$

which is a contradiction to $\|x_m\|\to\infty$ as $m\to\infty$. Hence $\|H(x_m)\|\to\infty$ $(m\to\infty)$. By Lemma 2.2, we conclude that $H:R^n\to R^n$ is a homomorphism, which implies system (1.2) has a unique equilibrium $x^*=(x_1^*,x_2^*,\dots,x_n^*)^T$. From (3.1), we know that x^* is also unique equilibrium of system (1.1).

In the following, we will prove the unique equilibrium of system (1.1) is globally exponential stability.

Considering the first equation of system (1.1), we have

$$D^+|x_i(t)-x_i^*|$$
$$\leq\alpha_i(x_i(t))\left[\beta_i|x_i(t)-x_i^*|+\sum_{j=1}^{n}L_j(|a_{ij}|+|b_{ij}|+|c_{ij}|)|x_j(t-\tau_{ij}(t))-x_j^*|\right],$$

for all $t > t_0$, $t \neq t_k, k = 1, 2, \ldots$, also, for $i = 1, 2, \ldots, n$,

$$x_i(t_k^+) - x_i^* = -\lambda_{ik}(x_i(t_k) - x_i^*) + x_i(t_k) - x_i^* = (1 - \lambda)(x_i(t_k) - x_i^*).$$

Hence, for $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots$,

$$|x_i(t_k^+) - x_i^*| \leq |1 - \lambda||x_i(t_k) - x_i^*| \leq |x_i(t_k) - x_i^*|. \tag{3.6}$$

Consider the function

$$\Gamma_i(x) = \frac{x}{\underline{\alpha_i}} - \beta_i + \frac{1}{2}\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}| + |c_{ij}|)L_j e^{\tau x} + \frac{1}{2}\sum_{j=1}^{n}\frac{\delta_j}{\delta_i}(|a_{ji}| + |b_{ji}| + |c_{ji}|)L_i e^{\tau x}$$

From (3.1), we have $\Gamma_i(0) < 0$, and $\Gamma_i(x)$ is continuous. Since $\frac{d\Gamma_i(x)}{dx} > 0$, which implies that $\Gamma_i(x)$ is strictly increasing, there exist $\lambda_i > 0$, such that $\Gamma_i(\lambda_i) = 0$ for $i = 1, 2, \ldots, n$. Choose $0 < \lambda \leq \min_{1\leq i\leq n}\{\lambda_i\}$, then we have, for $i = 1, 2, \ldots, n$,

$$\begin{aligned}\frac{\lambda}{\underline{\alpha_i}} - \beta_i &+ \frac{1}{2}\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}| + |c_{ij}|)L_j e^{\lambda\tau} \\ &+ \frac{1}{2}\sum_{j=1}^{n}\frac{\delta_j}{\delta_i}(|a_{ji}| + |b_{ji}| + |c_{ji}|)L_i e^{\lambda\tau} < 0.\end{aligned} \tag{3.7}$$

Let $W_i(t) = e^{\lambda(t-t_0)}|x_i(t) - x_i^*|$, $i = 1, 2, \ldots, n$, then we have

$$\begin{aligned}&D^+W_i(t) \\ &\leq \lambda W_i(t) + \alpha_i(x_i(t))\Big[-\beta_i W_i(t) + \sum_{j=1}^{n} L_j e^{\lambda\tau}(|a_{ij}| + |b_{ij}| + |c_{ij}|)W_j(t - \tau_{ij}(t))\Big] \\ &\leq \alpha_i(x_i(t))\Big[\Big(\frac{\lambda}{\underline{\alpha_i}} - \beta_i\Big)W_i(t) + \sum_{j=1}^{n} L_j e^{\lambda\tau}(|a_{ij}| + |b_{ij}| + |c_{ij}|)W_j(t - \tau_{ij}(t))\Big]\end{aligned}$$

for $t > t_0$, $t \neq t_k$, $k = 1, 2, \ldots$. Also, for $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots$.

$$W_i(t_k^+) = e^{\lambda(t_k^+ - t_0)}|x_i(t_k^+) - x_i^*| \leq e^{\lambda(t_k - t_0)}|x_i(t_k) - x_i^*| = W_i(t_k),$$

Now we consider the following Lyapunov functional

$$V(t) = \sum_{i=1}^{n}\delta_i\Big[\frac{1}{\underline{\alpha_i}}W_i^2(t) + \sum_{j=1}^{n}L_j(|a_{ij}| + |b_{ij}| + |c_{ij}|)e^{\lambda\tau}\int_{t-\tau_{ij}(t)}^{t}W_j^2(s)ds\Big]$$

By $2ab \leq a^2 + b^2$, from (3.7), we obtain

$$
\begin{aligned}
D^+V(t) &= \sum_{i=1}^n \delta_i \Big[\frac{1}{\overline{\alpha_i}} 2W_i(t) D^+W_i(t) + \sum_{j=1}^n L_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) \\
&\quad \times e^{\lambda\tau} \left(W_j^2(t) - W_j^2(t-\tau_{ij}(t))(1-\tau'_{ij}(t)) \right) \Big] \\
&\leq \sum_{i=1}^n \delta_i \Big\{ 2W_i(t) \frac{\alpha_i(x_i(t))}{\overline{\alpha_i}} \Big[\Big(\frac{\lambda}{\underline{\alpha_i}} - \beta_i \Big) W_i(t) \\
&\quad + \sum_{j=1}^n L_j e^{\lambda\tau} (|a_{ij}| + |b_{ij}| + |c_{ij}|) W_j(t-\tau_{ij}(t)) \Big] \\
&\quad + \sum_{j=1}^n L_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) e^{\lambda\tau} W_j^2(t) - W_j^2(t-\tau_{ij}(t))(1-\tau'_{ij}(t)) \Big\} \\
&\leq \sum_{i=1}^n \delta_i \Big\{ 2 \frac{\alpha_i(x_i(t))}{\overline{\alpha_i}} \Big[\Big(\frac{\lambda}{\underline{\alpha_i}} - \beta_i \Big) + \frac{1}{2} \sum_{j=1}^n L_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) e^{\lambda\tau} \\
&\quad + \frac{1}{2} \sum_{j=1}^n \frac{\delta_j}{\delta_i} L_i(|a_{ji}| + |b_{ji}| + |c_{ji}|) e^{\lambda\tau} \Big] \Big\} W_i^2(t) \\
&\quad + \sum_{i=1}^n \delta_i \Big[\frac{\alpha_i(x_i(t))}{\overline{\alpha_i}} \sum_{j=1}^n L_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) e^{\lambda\tau} W_j^2(t-\tau_{ij}(t)) \\
&\quad - \sum_{j=1}^n L_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) e^{\lambda\tau} W_j^2(t-\tau_{ij}(t))(1-\tau'_{ij}(t)) \Big] \\
&\leq \sum_{i=1}^n \delta_i \Big\{ 2 \frac{\alpha_i(x_i(t))}{\overline{\alpha_i}} \Big[\Big(\frac{\lambda}{\underline{\alpha_i}} - \beta_i \Big) + \frac{1}{2} \sum_{j=1}^n L_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) e^{\lambda\tau} \\
&\quad + \frac{1}{2} \sum_{j=1}^n \frac{\delta_j}{\delta_i} L_i(|a_{ji}| + |b_{ji}| + |c_{ji}|) e^{\lambda\tau} \Big] \Big\} W_i^2(t) \\
&\quad + \sum_{i=1}^n \delta_i \Big[\Big(\frac{\alpha_i(x_i(t))}{\overline{\alpha_i}} - 1 \Big) \sum_{j=1}^n L_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) e^{\lambda\tau} W_j^2(t-\tau_{ij}(t)) \Big] \\
&\leq \sum_{i=1}^n \delta_i \Big\{ 2 \frac{\alpha_i(x_i(t))}{\overline{\alpha_i}} \Big[\Big(\frac{\lambda}{\underline{\alpha_i}} - \beta_i \Big) + \frac{1}{2} \sum_{j=1}^n L_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) e^{\lambda\tau} \\
&\quad + \frac{1}{2} \sum_{j=1}^n \frac{\delta_j}{\delta_i} L_i(|a_{ji}| + |b_{ji}| + |c_{ji}|) e^{\lambda\tau} \Big] \Big\} W_i^2(t) \leq 0
\end{aligned}
$$

for $t \geq t_0$, $t \neq t_k$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots$. Meanwhile

$$V(t_k^+) = \sum_{i=1}^{n} \delta_i \left[\frac{1}{\alpha_i} W_i^2(t_k^+) + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) e^{\lambda \tau} \int_{t_k^+ - \tau_{ij}(t)}^{t_k^+} W_j^2(s) ds \right]$$

$$\leq \sum_{i=1}^{n} \delta_i \left[\frac{1}{\alpha_i} W_i^2(t_k) + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) e^{\lambda \tau} \int_{t_k - \tau_{ij}(t)}^{t_k} W_j^2(s) ds \right] = V(t_k)$$

So we have

$$V(t) \leq V(0) \leq \eta \|\phi - x^*\|^2.$$

where η is a positive constant. ϕ is the initial condition of $x(t)$.

On the other hand,

$$\sum_{i=1}^{n} \delta_i W_i^2(t) \leq V(t) \leq \eta \|\phi - x^*\|^2.$$

i.e.

$$\sum_{i=1}^{n} \delta_i e^{\lambda(t-t_0)} |x_i(t) - x_i^*|^2 \leq \eta \|\phi - x^*\|^2.$$

So

$$\|x(t) - x_i^*\| \leq M \|\phi - x^*\| e^{-\lambda(t-t_0)}.$$

where $M = \eta / (\min_{1 \leq i \leq n}\{\delta_i\})^{\frac{1}{2}}$. It follows that the unique equilibrium x^* of system (1.1) is global exponentially stable. □

Corollary 3.2. *Assume that (A1)–(A3) hold and* $\beta - L(|A| + |B| + |C|)$ *is an M-matrix, then the unique equilibrium* $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ *is global exponentially stable.*

4. An illustrative example

In this section, we give an example to illustrate the results obtained.

Example 1. Considering the following fuzzy Cohen–Grossberg neural networks with time-varying delays.

$$\begin{cases} x_i'(t) = \alpha_i(x_i(t)) \Big[-\beta_i(x_i(t)) + \sum_{j=1}^{2} a_{ij} f_j(x_j(t - \tau_{ij}(t))) + I_i + \bigwedge_{j=1}^{2} T_{ij} u_j \\ \quad + \bigvee_{j=1}^{2} H_{ij} u_j + \bigwedge_{j=1}^{2} b_{ij} f_j(x_j(t - \tau_{ij}(t))) + \bigvee_{j=1}^{2} c_{ij} f_j(x_j(t - \tau_{ij}(t))) \Big], \\ \qquad\qquad t > 0,\ t \neq t_k,\ k = 1, 2, \ldots, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = \Delta_{ik}(x_i(t_k)), \quad t = t_k,\ i = 1, 2 \end{cases} \tag{4.1}$$

with the initial condition $x(s) = \phi(s) \in PC[t_0 - \tau, t_0]$, where

$$f_i(x_i) = \frac{1}{2}(|x_i + 1| + |x_i - 1|), \beta_i(x_i(t)) = \beta_i x_i(t), \tau_{ij}(t) = \frac{1}{1+e^t}.$$

It is clear that $\tau_{ij}(t) < 1$ and $\tau'_{ij}(t) < 0$.

Take

$$\begin{aligned}
&\alpha_1(x_1(t)) = 2 + \sin(x_1(t)),\ \alpha_1(x_2(t)) = 2 - \cos(x_2(t)),\ \alpha_2(x_1(t)) = 3 + \sin(x_1(t)),\\
&\alpha_2(x_2(t)) = 3 - \cos(x_2(t)),\ \beta_1 = \beta_2 = 6,\ a_{11} = 0.3, a_{12} = 0.4,\ a_{21} = -1,\\
&a_{22} = 0.2,\ b_{11} = 0.2,\ b_{12} = 0.3,\ b_{21} = 0.3,\ b_{22} = 0.2,\ c_{11} = 0.4,\\
&c_{12} = 0.3,\ c_{21} = 0.4,\ c_{22} = 0.5,\ T_{ij} = H_{ij} = u_j = 2(i,j = 1,2),\ I_1 = 2,\ I_2 = 1.
\end{aligned}$$

Furthermore, for $k = 1, 2, \dots,$ when

$$\begin{aligned}
\Delta x_1(t_k) &= -\left(1 + \frac{1}{3}\sin(1+k)\right)(x_1(t_k)), \quad t_k = 0.2 + 2(k-1)\pi,\\
\Delta x_2(t_k) &= -\left(1 + \frac{2}{3}\cos(2k)\right)(x_2(t_k)), \quad t_k = 0.2 + 2(k-1)\pi.
\end{aligned}$$

One can get that (3.1) hold. By directly calculating, it is easy to verify that (A1)–(A4) hold. Therefore all conditions of Theorem 3.1 are satisfied. So system (4.1) has a unique equilibrium, which is globally exponentially stable.

5. Conclusion

In this paper, the global exponential stability of impulsive fuzzy BAM neural networks with distributed delays and time-varying delays have been studied. Some sufficient conditions have been obtained to ensure the existence, uniqueness and global exponential stability of equilibrium point for impulsive fuzzy BAM neural networks with distributed delays and time-varying delays. The criteria of stability is simple and independent of time delay. Especially, the estimate of the exponential converging rate index was provided. Moreover an example is given to illustrate the effectiveness of our results obtained.

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