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On various concepts of nilpotence for expansions of groups

By ERHARD AICHINGER (Linz) and NEBOJŠA MUDRINSKI (Novi Sad)

Abstract. The group theoretic concept of *nilpotence* has been generalized in various ways to arbitrary universal algebras. We establish a relation between two such generalizations for expansions of groups.

1. Main results

The aim of this paper is to establish some relations between two properties of an algebraic structure: *nilpotence* and *supernilpotence*. Both of these properties generalize the group theoretic concept of nilpotence from groups to arbitrary universal algebras. For arbitrary algebras, the first property, nilpotence, has been studied in commutator theory [23], [10]. The second property, supernilpotence, is a (usually) stronger concept that appears, implicitly, in [10, Chapter XIV]. The name "supernilpotence" first appears in [2], [5], and since then, the concept of supernilpotence has been used in duality theory [6], clone theory [16], and for describing the structure of certain universal algebras [19]. In this paper, we prove that for a certain class of expanded groups nilpotence implies supernilpotence. In the remainder of this section, we give a precise formulation of the main results in the present paper, and defer a more detailed discussion on the various concepts of nilpotence to Section 2.

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We call an algebraic structure $\mathbf{V} = (V, \cdot, ^{-1}, 1, g_1, g_2, \ldots)$ an *expanded group* if $(V, \cdot, ^{-1}, 1)$ is a group. In expanded groups, every congruence is uniquely determined by the congruence class of 1; if a subset A of $\mathbf{V}$ is the congruence class of 1 for some congruence relation of $\mathbf{V}$, then A is called an *ideal* of $\mathbf{V}$ (cf. [15]). An n-ary operation f on V is called a *polynomial of* $\mathbf{V}$ if there are $l \in \mathbb{N}$, $v_1, \ldots, v_l \in V$, and a term t in the language of $\mathbf{V}$ such that $f(x_1, \ldots, x_n) = \mathsf{t}^{\mathbf{V}}(v_1, \ldots, v_l, x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in V$. The set of all n-ary polynomials of $\mathbf{V}$ will be abbreviated by $\mathsf{Pol}_n(\mathbf{V})$. The concept of nilpotence can be described by using the binary commutator operation, which has been defined for arbitrary algebras in [23], [10]. For expanded groups, commutators can be defined as follows.

Definition 1.1. Let $\mathbf{V}$ be an expanded group, and let A, B be ideals of $\mathbf{V}$. Then their *commutator* $[\![A, B]\!]$ is the ideal of $\mathbf{V}$ that is generated by the set $\{p(a, b) \mid a \in A,\ b \in B,\ p \in \mathsf{Pol}_2(\mathbf{V}),\ p(x, 1) = p(1, x) = 1 \text{ for all } x \in V\}$.

This definition is consistent with the definitions given in [10] and [21].

Definition 1.2. For $k \in \mathbb{N}_0$, we define an expanded group $\mathbf{V}$ to be *k-nilpotent* if and only if the lower central series of $\mathbf{V}$ defined by $\gamma_1(\mathbf{V}) := V$, $\gamma_n(\mathbf{V}) := [\![V, \gamma_{n-1}(\mathbf{V})]\!]$ for $n \geq 2$, satisfies $\gamma_{k+1}(\mathbf{V}) = \{1\}$. The expanded group $\mathbf{V}$ is *nilpotent* if there exists a $k \in \mathbb{N}_0$ such that $\mathbf{V}$ is k-nilpotent, and $\mathbf{V}$ is *nilpotent of class k* if k is minimal such that $\mathbf{V}$ is k-nilpotent.

The concept with which we compare nilpotence is *supernilpotence*, which we introduce in the next two definitions.

Definition 1.3. Let $\mathbf{V}$ be an expanded group, let $n \in \mathbb{N}$, and let p be an n-ary operation on $\mathbf{V}$. Then p is *absorbing* if for all $x_1, \ldots, x_n \in V$ with $1 \in \{x_1, \ldots, x_n\}$, we have $p(x_1, \ldots, x_n) = 1$.

Definition 1.4. Let $k \in \mathbb{N}_0$, and let $\mathbf{V}$ be an expanded group. Then $\mathbf{V}$ is *k-supernilpotent* if every $(k+1)$-ary absorbing polynomial of $\mathbf{V}$ is constant. We say that $\mathbf{V}$ is *supernilpotent* if there exists a $k \in \mathbb{N}_0$ such that $\mathbf{V}$ is k-supernilpotent, and $\mathbf{V}$ is *supernilpotent of class k* if k is minimal such that $\mathbf{V}$ is k-supernilpotent.

From these definitions, it is easy to see that $\mathbf{V}$ is 1-nilpotent if and only if $[\![V, V]\!] = \{1\}$, which is equivalent to $\mathbf{V}$ being 1-supernilpotent; in this case, $\mathbf{V}$ is called *abelian*.

There are several connections between nilpotence and supernilpotence, one being that every k-supernilpotent expanded group is k-nilpotent, hence supernilpotence implies nilpotence. We will review these results in Section 2. In the present note, we present a class of expanded groups for which nilpotence implies supernilpotence; moreover, we will obtain an upper bound for the class of supernilpotence of these expanded groups. This class is the class of multilinear expanded groups, which we will define now. We call an n-ary operation g on a group $(V, \cdot, ^{-1}, 1)$ *multilinear* if $n \geq 1$ and for all $i \in \{1, \dots, n\}$ and for all $(v_1, \dots, v_n) \in V^n$, the mapping $x \mapsto g(v_1, \dots, v_{i-1}, x, v_{i+1}, \dots, v_n)$ is a group endomorphism. Such operations are always absorbing. For $m \in \mathbb{N}$, an expanded group $\mathbf{V} = (V, \cdot, ^{-1}, 1, g_1, g_2, \dots)$ is *m-multilinear* if $m \geq 2$, all g_i are multilinear operations, and the arity of each g_i is at most m. $\mathbf{V}$ is *multilinear* if there is an $m \in \mathbb{N}$ such that $\mathbf{V}$ is m-multilinear; the smallest m for which $\mathbf{V}$ is m-multilinear is called the *degree of multilinearity* of $\mathbf{V}$. We note that by definition, this degree is always at least 2; for example, every group (with no further operations) is multilinear of degree 2. There are several examples of multilinear expanded groups: every group and every ring is a multilinear expanded group of degree 2; generalizing the second example, every distributive near-ring (cf. [8]) is 2-multilinear.

Now our first result is the following theorem.

Theorem 1.5. *Let $k, m \in \mathbb{N}$, $m \geq 2$, and let $\mathbf{V}$ be a multilinear expanded group with degree m of nilpotence class k. Then $\mathbf{V}$ is m^{k-1}-supernilpotent.*

The proof of this theorem will be given in Section 5. Although this bound is not tight for groups (for groups, m^{k-1} can be replaced by k in the statement of the theorem, see Section 2), it is tight in general:

Proposition 1.6. *Let $k, m \in \mathbb{N}$, $m \geq 2$, and let p be a prime. Then there exists a finite multilinear expanded group $\mathbf{V}$ with $|V| = p^k$, multilinearity degree m, nilpotence class k, and supernilpotence class m^{k-1}.*

The proof will be given by providing an example of such an expanded group in Section 6. Another example will show that neither the functions on an expanded group that preserve congruences and the nilpotence class, nor the functions that perserve congruences and the supernilpotence class, will form a clone in general.

On the way to prove Theorem 1.5, we will introduce higher commutators, and we establish a result (Theorem 4.10) that allows to compute these higher commutators in multilinear expanded groups.

2. An introduction to nilpotence for universal algebras

We start our comparison of two concepts of nilpotence from the following well-known Proposition:

Proposition 2.1. *Let* $\mathbf{G}$ *be a finite group. Then the following properties are equivalent.*

(1) $\mathbf{G}$ *is nilpotent;*

(2) $\mathbf{G}$ *is a direct product of groups of prime power order;*

(3) *there is a polynomial* $p \in \mathbb{Q}[x]$ *such that the free algebra over* n *generators in the variety generated by* $\mathbf{G}$ *has at most* $2^{p(n)}$ *elements.*

The equivalence of (1) and (2) is well known in finite group theory [22, 6.4.14]. The equivalence with condition (3) is given in [11] (cf. [20, Corollary 24.52]) and discussed, e.g., in [12, p. 163].

It is a natural question how this equivalence can be generalized to arbitrary universal algebras. For such algebras, commutator theory [10] provides a definition of nilpotence that generalizes the group theoretic concept. If one specializes the binary commutator operation defined in [10] or [18, Definition 4.150] to expanded groups, one obtains the commutator of two ideals defined via absorbing binary polynomials as it was introduced in Section 1; this is proved, e.g., in [3, Lemma 2.9]. Thus, in Section 1, nilpotence of expanded groups was defined in such a way that it is a special case of the concept of nilpotence defined for arbitrary algebras as defined in [12, p. 68], [10]. However, with this definition we obtain nilpotent finite expanded groups that fail to decompose into a direct product of algebras of prime power order. An example is the algebra $\mathbf{N}_6$ defined by $\mathbf{N}_6 := (\mathbb{Z}_6, +, f)$, where f is the unary function with $f(0) = f(3) = 3$, $f(1) = f(2) = f(4) = f(5) = 0$. This algebra and its clone of polynomial functions were studied in [3]. From this paper, we obtain that $\mathbf{N}_6$ is nilpotent, directly indecomposable, and the free algebra in the variety generated by $\mathbf{N}_6$ has at least 2^{2^n} elements.

If we take $\mathbf{G}$ to be a finite algebra with finitely many fundamental operations in a congruence modular variety, then, as a consequence of [7], [13, Theorem 3.14] and [12, Lemma 12.4], we obtain that the condition (3) of Proposition 2.1 holds if and only if $\mathbf{G}$ satisfies both of the conditions (1) and (2). Additionally, for a finite expanded group $\mathbf{G}$, condition (3) is equivalent to $\mathbf{G}$ being supernilpotent. A discussion of this last equivalence for expanded groups can be found in [1]. The definition of supernilpotence has been extended to all universal algebras [5, Definition 7.1], and the equivalence of (1) $\wedge$ (2) with (3) then carries over to all finite algebras with a Mal'cev term.

It is now natural to ask for the logical connections between nilpotence and supernilpotence. The main results are:

Theorem 2.2 ([5, Lemma 7.5])**.** *Let $k \in \mathbb{N}_0$, and let $\mathbf{V}$ be a k-supernilpotent expanded group. Then $\mathbf{V}$ is k-nilpotent.*

In [5], this is proved for all algebras with a Mal'cev term. A simpler proof for expanded groups is given in [1].

Theorem 2.3 ([2, Theorem 6.8] and [5, Corollary 6.15])**.** *Let $k \in \mathbb{N}_0$, and let $\mathbf{G}$ be a group. Then $\mathbf{G}$ is k-supernilpotent if and only if it is k-nilpotent.*

Nilpotence does in general not imply supernilpotence. Examples are the algebras $\mathbf{N}_6$ and the following algebra of infinite type:
$\mathbf{B} := (\mathbb{Z}_4, 2x_1, 2x_1x_2, 2x_1x_2x_3, \ldots)$. This algebra $\mathbf{B}$ is nilpotent of class 2, but not supernilpotent. However, [7, Theorem 2] by J. Berman and W. Blok yields, as a special case, the following result.

Theorem 2.4 ([7])**.** *Let $\mathbf{V}$ be a finite expanded group with finitely many fundamental operations. We assume that $\mathbf{V}$ is nilpotent and a direct product of expanded groups of prime power order. Then $\mathbf{V}$ is supernilpotent.*

Even for expanded groups, there is currently no proof of this result avoiding the methods of [10, Chapter VII] developed there for arbitrary congruence modular algebras. As a consequence, we cannot easily determine an upper bound on the class of supernilpotence of a given nilpotent algebra; in fact, from Proposition 1.6 one sees that for $k \geq 2$, there are finite k-nilpotent expanded groups of prime power order with arbitrary high degree of supernilpotence. While this shows that we cannot bound the supernilpotence class from above by a function of the nilpotence class alone, we can give such a bound if we restrict ourselves to multilinear expanded groups of given multilinearity degree. In fact, Theorem 1.5 provides such a bound for multilinear expanded groups. This class has been investigated in another context as well: In 2007, R. Willard remarked that every finite multilinear expanded group has a polynomial time algorithm for the subpower membership problem [24] (cf. [17]). We also remark that Theorem 1.5 is not restricted to finite structures.

On the way to establish Theorem 1.5, we will calculate the higher commutators for multilinear expanded groups. For an arbitrary universal algebra, the definition of higher commutators given in [9] is quite technical, but since in the present note, we work with expanded groups, a much easier introduction of higher commutators is possible. In fact, in [5, Corollary 6.12] it is shown that the higher commutators introduced by Bulatov in [9] specialize to the lattice of ideals of an expanded group $\mathbf{V}$ as follows:

Definition 2.5. Let $\mathbf{V}$ be an expanded group, and let $A_1, \dots, A_n$ be ideals of $\mathbf{V}$. We define *the n-ary commutator ideal of* $A_1, \dots, A_n$, and abbreviate it by $[\![A_1, \dots, A_n]\!]$, as the ideal of $\mathbf{V}$ that is generated by all $p(a_1, \dots, a_n)$, where p is an n-ary absorbing polynomial of $\mathbf{V}$, and $a_1 \in A_1, \dots, a_n \in A_n$.

For $n = 1$, we get $[\![A]\!] = A$ for every ideal A of $\mathbf{V}$, and for $n = 2$, we obtain the classical binary commutator operation for expanded groups (for a more detailed discussion, we refer to Section 2 of [3]; the definition of binary commutators for expanded groups was also given, independent of commutator theory, in [21]). Hence, an expanded group $\mathbf{V}$ is k-supernilpotent if and only if $[\![V, \dots, V]\!] = 0$ ($k+1$ repetitions of V). We will explain how to compute higher commutators in a multilinear expanded group of degree m from those higher commutators involving at most m arguments (Theorem 4.10). The guiding example is that in every group, the ternary commutator operation can be expressed by using binary commutators; for example (see [16]) for normal subgroups N_1, N_2, N_3 of a group, we have $[\![N_1, N_2, N_3]\!] = [\![N_1, [\![N_2, N_3]\!]]\!] \cdot [\![N_2, [\![N_1, N_3]\!]]\!] \cdot [\![N_3, [\![N_1, N_2]\!]]\!]$ (one of the three terms on the right hand side can actually be omitted by the *three subgroups lemma*, but this lemma does not generalize to binary commutators on arbitrary expanded groups). Indeed, for groups, all higher commutator operations can be computed similarly from the binary commutator operation. The main tool will be a version of *commutator calculus* and its detailed discussion in [2].

3. Expressions using commutators

Since the commutator of given ideals is computed using absorbing polynomials, we need to study some properties of polynomial maps. For example, it is obvious that in a group $\mathbf{G}$, the function $\varphi : (x_1, x_2, x_3) \mapsto [x_1, [x_2, x_3]]$ satisfies $\varphi(N_1 \times N_2 \times N_3) \subseteq [\![N_1, [\![N_2, N_3]\!]]\!]$ for all normal subgroups N_1, N_2, N_3 of $\mathbf{G}$. In this section, we establish similar results for higher commutators in expanded groups. To be more specific, let us consider an expanded group $\mathbf{V} = (V, \cdot, ^{-1}, g_1)$, where g_1 is a ternary absorbing operation on $\mathbf{V}$. Let A_1, A_2, A_3 be ideals of $\mathbf{V}$, and let $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$. Then the purpose of Lemma 3.1 is to show that $g_1(a_1, a_2, a_2)^{-1} \cdot a_3^{-1} \cdot g_1(a_1, a_2, a_2) \cdot a_3$ lies in the ideal $[\![[\![A_1, A_2, A_2]\!], A_3]\!]$ of $\mathbf{V}$. Lemma 3.2 guarantees $[\![[\![A_1, A_2, A_2]\!], A_3]\!] \subseteq [\![A_1, A_2, A_3]\!]$. Lemma 3.3 yields that omitting certain ideals from our "commutator expression", we obtain the same or a larger ideal, hence $[\![[\![A_1, A_2, A_2]\!], A_3]\!] \subseteq [\![[\![A_2, A_2]\!], A_3]\!]$. Finally, Lemma 3.4 will yield an upper bound for $[\![[\![A_1, A_2, A_2]\!], A_3]\!]$ that contains each A_i at most once, namely $[\![[\![A_1, A_2]\!], A_3]\!]$.

Except for Lemma 3.1, these properties of the higher commutator operation are almost immediate consequences of the eight properties of higher commutators, (HC1) to (HC8), which were established in [5] and some of which we list here for easier reference. Actually, for all $n \in \mathbb{N}$ and for all ideals $I_1, \dots, I_n$, $J_1, \dots, J_n$ of an expanded group $\mathbf{V}$, we have:

(HC2) If $I_1 \leq J_1, \dots, I_n \leq J_n$, then $[\![I_1, \dots, I_n]\!] \leq [\![J_1, \dots, J_n]\!]$;

(HC3) $[\![I_1, \dots, I_n]\!] \leq [\![I_2, \dots, I_n]\!]$;

(HC4) If $\pi \in S_n$, then $[\![I_1, \dots, I_n]\!] = [\![I_{\pi(1)}, \dots, I_{\pi(n)}]\!]$;

(HC8) $[\![I_1, \dots, I_k, [\![I_{k+1}, \dots, I_n]\!]]\!] \leq [\![I_1, \dots, I_n]\!]$ for $k \in \{0, \dots, n-1\}$

As consequences, we obtain the following inequalities.

(HC3') If $k, i_1, \dots, i_k \in \{1, \dots, n\}$ and $i_1, \dots, i_k$ are all distinct, then

$$[\![I_1, \dots, I_n]\!] \leq [\![I_{i_1}, \dots, I_{i_k}]\!].$$

(HC8') If $j \in \{0, \dots, n-1\}$ and $k \in \{j+1, \dots, n\}$, then

$$[\![I_1, \dots, I_j, [\![I_{j+1}, \dots, I_k]\!], I_{k+1}, \dots, I_n]\!] \leq [\![I_1, \dots, I_n]\!].$$

These properties have been proved for the commutator operations of general Mal'cev algebras in [5]. Now using [5, Corollary 6.12], they follow from the corresponding properties established in [5]. Using our definition of the commutator operations through absorbing polynomials, (HC2), (HC3), and (HC4) can be seen directly from this definition.

In order to express what we mean by a "commutator expression", we let $\mathcal{F}$ be the language with operation symbols $\mathsf{f}_1, \mathsf{f}_2, \dots$, where each f_i has arity i ($i \in \mathbb{N}$). We will often abbreviate a term $\mathsf{f}_k(\mathsf{t}_1, \dots, \mathsf{t}_k)$ by $[\mathsf{t}_1, \dots, \mathsf{t}_k]$. For an expanded group $\mathbf{V} = (V, \cdot, {}^{-1}, 1, g_1, g_2, \dots)$, we define the language $\mathcal{F}_{\mathbf{V}}$ as the language that contains all operation symbols in $\mathcal{F}$, and the following symbols that are added:

- one nullary symbol $\mathsf{c}(v)$ for each $v \in V$,
- an r-ary symbol g for each r-ary nonconstant fundamental operation g of $\mathbf{V}$ except for the binary group multiplication $\cdot$ and the group inverse operation ${}^{-1}$.

Finally, the *language of* $\mathbf{V}$ will, as usually, consist of the function symbols $\{\cdot, {}^{-1}, 1, \mathsf{g}_1, \mathsf{g}_2, \dots\}$ of $\mathbf{V}$.

We will now define two algebras with language $\mathcal{F}_{\mathbf{V}}$. The first one is $\mathbf{V}'$ with universe V, $\mathsf{c}(v)^{\mathbf{V}'} := v$ for each $v \in V$, and $\mathsf{g}_i^{\mathbf{V}'} := g_i$ for all nonconstant fundamental operations g_i of $\mathbf{V}$. Furthermore, $\mathsf{f}_1^{\mathbf{V}'}(x) := x$ for all $x \in V$,

$\mathsf{f}_2^{\mathbf{V}'}(x,y) := x^{-1}y^{-1}xy$ for all $x, y \in V$, and $\mathsf{f}_l^{\mathbf{V}'}(x_1,\ldots,x_l) := 1$ for $l \geq 3$ and $x_1,\ldots,x_l \in V$.

The other algebra is $\mathbf{I}(\mathbf{V})$ whose universe is the set $I(\mathbf{V})$ of all ideals of $\mathbf{V}$. We define $\mathsf{c}(v)^{\mathbf{I}(\mathbf{V})} := V$ for all $v \in V$, $\mathsf{f}_l^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_l) := [\![A_1,\ldots,A_l]\!]$ for each $l \in \mathbb{N}$ and $\mathsf{g}_i^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_r) := [\![A_1,\ldots,A_r]\!]$ for each i, where $r \in \mathbb{N}$ is the arity of g_i. For a term t in the language $\mathcal{F}_{\mathbf{V}}$ over the variables $\{\mathsf{x}_1,\ldots,\mathsf{x}_n\}$ we define the *length* of t by $l(\mathsf{x}_i) = l(\mathsf{c}(v)) = 1$ and $l(\mathsf{h}(\mathsf{t}_1,\ldots,\mathsf{t}_k)) := 1+\sum_{i=1}^{k} l(\mathsf{t}_\mathsf{i})$, where h is a k-ary operation symbol of $\mathcal{F}_{\mathbf{V}}$.

Lemma 3.1. *Let $\mathbf{V} = (V,\cdot,^{-1},1,g_1,g_2,\ldots)$ be an expanded group such that each g_i is an absorbing operation of arity at least 1, let t be a term in the language $\mathcal{F}_{\mathbf{V}}$ over the variables $\{\mathsf{x}_1,\ldots,\mathsf{x}_n\}$, let $A_1,\ldots,A_n$ be ideals of $\mathbf{V}$, and let $a_1 \in A_1,\ldots,a_n \in A_n$. Then we have $\mathsf{t}^{\mathbf{V}'}(a_1,\ldots,a_n) \in \mathsf{t}^{\mathbf{I}(\mathbf{V})}\,(A_1,\ldots,A_n)$.*

Proof. We proceed by induction on the length of t. If $\mathsf{t} = \mathsf{c}(v)$, then $\mathsf{t}^{\mathbf{V}'}(a_1,\ldots,a_n) = v \in V = \mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n)$. If $\mathsf{t} = \mathsf{x}_i$, then $\mathsf{t}^{\mathbf{V}'}(a_1,\ldots,a_n) = a_i \in A_i = \mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n)$. For the induction step, we fix a term t of length at least 2. We distinguish the following cases.

- *Case* $\mathsf{t} = \mathsf{g}_m(\mathsf{t}_1,\ldots,\mathsf{t}_k)$: Then we have $\mathsf{t}_i^{\mathbf{V}'}(a_1,\ldots,a_n) \in \mathsf{t}_i^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n)$ for all $i \in \{1,\ldots,k\}$ by the induction hypothesis. Since g_m is absorbing, we have

$$g_m(\mathsf{t}_1^{\mathbf{V}'}(a_1,\ldots,a_n),\ldots,\mathsf{t}_k^{\mathbf{V}'}(a_1,\ldots,a_n)) \\ \in [\![\mathsf{t}_1^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n),\ldots,\mathsf{t}_k^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n)]\!],$$

  which can be rewritten as

$$\mathsf{t}^{\mathbf{V}'}(a_1,\ldots,a_n) \in \mathsf{g}_m^{\mathbf{I}(\mathbf{V})}(\mathsf{t}_1^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n),\ldots,\mathsf{t}_k^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n)) \\ = \mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n).$$

- *Case* $\mathsf{t} = \mathsf{f}_1(\mathsf{t}_1)$: Then we have $\mathsf{t}_1^{\mathbf{V}'}(a_1,\ldots,a_n) \in \mathsf{t}_1^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n)$ by the induction hypothesis. Hence, $\mathsf{f}_1^{\mathbf{V}'}(\mathsf{t}_1^{\mathbf{V}'}(a_1,\ldots,a_n)) = \mathsf{t}_1^{\mathbf{V}'}(a_1,\ldots,a_n) \in \mathsf{t}_1^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n) = \mathsf{f}_1^{\mathbf{I}(\mathbf{V})}(\mathsf{t}_1^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n))$.
- *Case* $\mathsf{t} = \mathsf{f}_2(\mathsf{t}_1,\mathsf{t}_2)$: Then, we have $\mathsf{t}_i^{\mathbf{V}'}(a_1,\ldots,a_n) \in \mathsf{t}_i^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n)$ for $i \in \{1,2\}$, by the induction hypothesis. Using that $\mathsf{f}_2^{\mathbf{V}'}$ is an absorbing polynomial of $\mathbf{V}$, we obtain

$$\begin{aligned} \mathsf{t}^{\mathbf{V}'}(a_1,\ldots,a_n) &= \mathsf{f}_2^{\mathbf{V}'}(\mathsf{t}_1^{\mathbf{V}'}(a_1,\ldots,a_n),\mathsf{t}_2^{\mathbf{V}'}(a_1,\ldots,a_n)) \\ &\in [\![\mathsf{t}_1^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n),\mathsf{t}_2^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n)]\!] \\ &= \mathsf{f}_2^{\mathbf{I}(\mathbf{V})}(\mathsf{t}_1^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n),\mathsf{t}_2^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n)) = \mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1,\ldots,A_n). \end{aligned}$$

- *Case* $\mathsf{t} = \mathsf{f}_k(\mathsf{t}_1, \ldots, \mathsf{t}_k)$ *for* $k \geq 3$: Then, we have $\mathsf{t}^{\mathbf{V}'}(a_1, \ldots, a_n) = 1 \in \mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n)$. □

For each term t in the languages $\mathcal{F}$ or $\mathcal{F}_{\mathbf{V}}$, we denote the set of those variables that occur in t by $\mathrm{Var}(\mathsf{t})$.

Lemma 3.2. *Let* $\mathbf{V}$ *be an expanded group, let* $n \in \mathbb{N}$, *let* t *be a term in the language* $\mathcal{F}$ *over the variables* $\{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$ *such that all of these variables occur in* t, *i.e., such that* $\mathrm{Var}(\mathsf{t}) = \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$. *Then for all ideals* $A_1, \ldots, A_n$ *of* $\mathbf{V}$ *we have*

$$\mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n) \leq [\![A_1, \ldots, A_n]\!]. \tag{3.1}$$

Proof. We proceed by induction on the length of t. If $\mathsf{t} = \mathsf{x}_1$, then the left hand side of (3.1) is equal to the right hand side. For the induction step, we let t be an $\mathcal{F}$-term and assume that (3.1) holds for all ideals $A_1, \ldots, A_n$ and all terms of length smaller than the length of t. Let $k \in \mathbb{N}$ be such that $\mathsf{t} = \mathsf{f}_k(\mathsf{t}_1, \ldots, \mathsf{t}_k)$. Now we compute $\mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n) = [\![t_1^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n), \ldots, t_k^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n)]\!]$. By the induction hypothesis, each $t_j^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n)$ is $\leq [\![A_{i_{j,1}}, \ldots, A_{i_{j,m_j}}]\!]$, where $\mathrm{Var}(\mathsf{t}_j) = \{\mathsf{x}_{i_{j,1}}, \ldots, \mathsf{x}_{i_{j,m_j}}\}$. Hence using monotonicity (HC2), we obtain $\mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n) \leq [\![[\![A_{i_{1,1}}, \ldots, A_{i_{1,m_1}}]\!], [\![A_{i_{2,1}}, \ldots, A_{i_{2,m_2}}]\!], \ldots, [\![A_{i_{k,1}}, \ldots, A_{i_{k,m_k}}]\!]]\!]$. Flattening the right hand side by repeated application of (HC8'), omitting repeated occurrences of each A_j by using (HC3'), and finally sorting the A_j's by using (HC4), we obtain the result. □

According to Lemma 3.2 we have that $[\![[\![A_1, A_2]\!], [\![A_3, A_3]\!]]\!] \leq [\![A_1, A_2, A_3]\!]$. Sometimes it will be desirable that the commutators that occur in the upper bound on the right hand side do not have larger arity than those on the left hand side. An upper bound for $[\![[\![A_1, A_2]\!], [\![A_3, A_3]\!]]\!]$ of the desired form would be $[\![[\![A_1, A_2]\!], A_3]\!]$. Such an upper bound can be found using the following lemmas. The first lemma tells how to drop unwanted variables, and the second one gets rid of repeated occurrences.

Lemma 3.3. *Let* $\mathbf{V}$ *be an expanded group, let* $k \geq 2$, *and let* t *be a term in the language* $\{\mathsf{f}_1, \ldots, \mathsf{f}_k\}$. *If* S *is a nonempty subset of* $\mathrm{Var}(\mathsf{t})$, *then there exists a term* s *in the language* $\{\mathsf{f}_1, \ldots, \mathsf{f}_k\}$ *such that* $\mathrm{Var}(\mathsf{s}) = S$ *and*

$$\mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n) \leq \mathsf{s}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n) \tag{3.2}$$

for all $A_1, \ldots, A_n \in I(\mathbf{V})$. *Furthermore, each variable in* S *has the same number of occurrences in* s *and* t.

Proof. If $S = \mathrm{Var}(\mathsf{t})$, then we choose $\mathsf{s} := \mathsf{t}$. Hence we will now assume that S is a proper nonempty subset of $\mathrm{Var}(\mathsf{t})$; in this case t is not a variable. Since we can drop the variables in $\mathrm{Var}(\mathsf{t}) \setminus S$ one by one, it is sufficient to show that for each term t in the language $\{\mathsf{f}_1, \ldots, \mathsf{f}_k\}$ that contains at least two different variables, and for each variable $\mathsf{x} \in \mathrm{Var}(\mathsf{t})$, there exists a term s in the language $\{\mathsf{f}_1, \ldots, \mathsf{f}_k\}$ such that s satisfies the inequality (3.2) and $\mathrm{Var}(\mathsf{s}) = \mathrm{Var}(\mathsf{t}) \setminus \{\mathsf{x}\}$. We prove this statement by induction on the length of t. For $\mathsf{t} = \mathsf{f}_2(\mathsf{x}_1, \mathsf{x}_2)$, let x_2 be the variable that we want to omit. In this case, we define $\mathsf{s} := \mathsf{x}_1$. Clearly, we have $\mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1, A_2) = [\![A_1, A_2]\!] \leq A_1 = \mathsf{s}^{\mathbf{I}(\mathbf{V})}(A_1, A_2)$ by (HC3'). For the induction step, let t be a term in the language $\{\mathsf{f}_1, \ldots, \mathsf{f}_k\}$ with $|\mathrm{Var}(\mathsf{t})| \geq 2$, and let $\mathsf{x} \in \mathrm{Var}(\mathsf{t})$ be the variable that we want to omit. Since t contains at least two variables, we have $\mathsf{t} = \mathsf{f}_l(\mathsf{t}_1, \ldots, \mathsf{t}_l)$ for an $l \in \{1, \ldots, k\}$. Now we group the terms t_i into three classes: those t_i's that do not contain x (they will be copied), those t_i's that contain the variable x, but no other variables (they will be omitted), and those t_i's that contain x and at least one other variable (they will be processed by induction). Doing this formally, we let $i_1 < \cdots < i_r$ be the distinct elements of the set $\{i \mid \mathsf{x} \notin \mathrm{Var}(\mathsf{t}_i)\}$, we define $T := \{i \mid \mathrm{Var}(\mathsf{t}_i) = \{\mathsf{x}\}\}$, and we let $j_1 < \cdots < j_p$ be the distinct elements of the set $\{1, \ldots, l\} \setminus (T \cup \{i_1, \ldots, i_r\})$. Since t contains at least one variable other than x, we have $1 \leq r + p \leq k$. Now for all $i \in \{j_1, \ldots, j_p\}$, we use the induction hypothesis to find a term s_i in the language $\{\mathsf{f}_1, \ldots, \mathsf{f}_k\}$ with $\mathrm{Var}(\mathsf{s}_i) = \mathrm{Var}(\mathsf{t}_i) \setminus \{\mathsf{x}\}$ and

$$\mathsf{t}_i^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n) \leq \mathsf{s}_i^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n)$$

for all ideals $A_1, \ldots, A_n$ of $\mathbf{V}$ and the additional property that all variables in $\mathrm{Var}(\mathsf{t}) \setminus \{\mathsf{x}\}$ have the same number of occurrences in t_i and s_i. Then we define $\mathsf{s} := \mathsf{f}_{r+p}(\mathsf{t}_{i_1}, \ldots, \mathsf{t}_{i_r}, \mathsf{s}_{j_1}, \ldots, \mathsf{s}_{j_p})$. The term s contains only operation symbols from $\{\mathsf{f}_1, \ldots, \mathsf{f}_k\}$, we have $\mathrm{Var}(\mathsf{s}) = \mathrm{Var}(\mathsf{t}) \setminus \{\mathsf{x}\}$, all variables from $\mathrm{Var}(\mathsf{t}) \setminus \{\mathsf{x}\}$ occur in s as often as they do in t, and by (HC3'), (HC4) and (HC2) we have

$$\begin{aligned}
&\mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n) \\
&\leq [\![\mathsf{t}_{i_1}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n), \ldots, \mathsf{t}_{i_r}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n), \mathsf{t}_{j_1}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n), \ldots, \mathsf{t}_{j_p}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n)]\!] \\
&\leq [\![\mathsf{t}_{i_1}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n), \ldots, \mathsf{t}_{i_r}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n), \mathsf{s}_{j_1}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n), \ldots, \mathsf{s}_{j_p}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n)]\!] \\
&= \mathsf{s}^{\mathbf{I}(\mathbf{V})}(A_1, \ldots, A_n). \qquad (3.3)
\end{aligned}$$

□

Lemma 3.4. *Let $\mathbf{V}$ be an expanded group, let $n \in \mathbb{N}$, and let t be a term in the language $\mathcal{F}_{\mathbf{V}}$ such that $\mathrm{Var}(\mathsf{t}) = \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$. Let $m \in \mathbb{N}$ be the maximal arity of operation symbols that occur in t. Then there is a term s in the language*

$\{\mathsf{f}_1,\dots,\mathsf{f}_m\}$ *that contains each* x_i *exactly once such that*

$$\mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n) \le \mathsf{s}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n) \tag{3.4}$$

for all $A_1,\dots,A_n \in I(\mathbf{V})$.

Proof. We proceed by induction on the length of t. First, we note that the case that t is a nullary operation symbol cannot occur because $|\operatorname{Var}(\mathsf{t})| = n \ge 1$. If $\mathsf{t} = \mathsf{x}_i$ for an $i \in \{1,\dots,n\}$, then we define $\mathsf{s} := \mathsf{t}$. For the induction step, we let t be a term of length at least 2 in the language $\mathcal{F}_\mathbf{V}$ with $\operatorname{Var}(\mathsf{t}) \ne \emptyset$. Let h be its outermost operation symbol and let k be the arity of h; hence $\mathsf{t} = \mathsf{h}(\mathsf{t}_1,\dots,\mathsf{t}_k)$ with $1 \le k \le m$. All k-ary operation symbols in $\mathcal{F}_\mathbf{V}$ induce the function $\mathsf{f}_k^{\mathbf{I}(\mathbf{V})} : (Y_1,\dots,Y_k) \mapsto [\![Y_1,\dots,Y_k]\!]$ on $I(\mathbf{V})$. Let $i_1 < \dots < i_r$ be the elements of the set $\{i \in \{1,\dots,k\} \mid \operatorname{Var}(\mathsf{t}_i) \ne \emptyset\}$. By assumption, $\operatorname{Var}(\mathsf{t}) \ne \emptyset$, and thus $r \ge 1$; also $r \le k$. Then we have

$$\begin{aligned}
\mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n) &= [\![\mathsf{t}_1^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n),\dots,\mathsf{t}_k^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n)]\!] \\
&\le [\![\mathsf{t}_{i_1}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n),\dots,\mathsf{t}_{i_r}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n)]\!] \\
&= \mathsf{f}_r^{\mathbf{I}(\mathbf{V})}(\mathsf{t}_{i_1}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n),\dots,\mathsf{t}_{i_r}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n))
\end{aligned} \tag{3.5}$$

for all $A_1,\dots,A_n \in \mathbf{I}(\mathbf{V})$. By the induction hypothesis, there exist terms $\mathsf{s}_1,\dots,\mathsf{s}_r$ in the language $\{\mathsf{f}_1,\dots,\mathsf{f}_m\}$ such that

$$\mathsf{t}_{i_j}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n) \le \mathsf{s}_j^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n)$$

for all $A_1,\dots,A_n \in I(\mathbf{V})$ and for each $j \in \{1,\dots,r\}$, every variable of t_{i_j} occurs exactly once in s_j. For $j \in \{1,\dots,r\}$ let $I_1 := \operatorname{Var}(\mathsf{s}_1)$ and $I_j := \operatorname{Var}(\mathsf{s}_j) \setminus (\operatorname{Var}(\mathsf{s}_1) \cup \dots \cup \operatorname{Var}(\mathsf{s}_{j-1}))$ for $j \in \{2,\dots,r\}$. Let $j_1 < \dots < j_{r'}$ be the elements of $\{j \in \{1,\dots,r\} \mid I_j \ne \emptyset\}$. Clearly, $I_{j_1} \cup \dots \cup I_{j_{r'}} = \{\mathsf{x}_1,\dots,\mathsf{x}_n\}$. For each $l \in \{1,\dots,r'\}$, Lemma 3.3 yields a p_l in the language $\{\mathsf{f}_1,\dots,\mathsf{f}_m\}$ such that $\operatorname{Var}(\mathsf{p}_l) = I_{j_l}$,

$$\mathsf{s}_{j_l}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n) \le \mathsf{p}_l^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n)$$

for all $A_1,\dots,A_n \in I(\mathbf{V})$, and p_l contains every variable in I_{j_l} exactly once. Continuing the calculations from (3.5) and using (HC2) and (HC3'), we have

$$\begin{aligned}
&\mathsf{f}_r^{\mathbf{I}(\mathbf{V})}(\mathsf{t}_{i_1}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n),\dots,\mathsf{t}_{i_r}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n)) \\
&\qquad \le \mathsf{f}_r^{\mathbf{I}(\mathbf{V})}(\mathsf{s}_1^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n),\dots,\mathsf{s}_r^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n)) \\
&\qquad \le \mathsf{f}_{r'}^{\mathbf{I}(\mathbf{V})}(\mathsf{p}_1^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n),\dots,\mathsf{p}_{r'}^{\mathbf{I}(\mathbf{V})}(A_1,\dots,A_n))
\end{aligned}$$

for all $A_1,\dots,A_n \in I(\mathbf{V})$. Now the term $\mathsf{s} := \mathsf{f}_{r'}(\mathsf{p}_1,\dots,\mathsf{p}_{r'})$ is a term in the languange $\{\mathsf{f}_1,\dots,\mathsf{f}_m\}$, contains every variable in $\{\mathsf{x}_1,\dots,\mathsf{x}_n\}$ exactly once, and satisfies the inequality (3.4). □

4. The commutator calculus

In this section, we will bring every polynomial term of a multilinear expanded group $\mathbf{V}$ into a specified form. Here, by *polynomial term* we mean a term in the language of $\mathbf{V}$ with the constant symbols $\{\mathsf{c}(v) \mid v \in V\}$ added.

The first task is to multiply out. For a binary multilinear operation g on the group $(V, \cdot, ^{-1}, 1)$, we clearly have $g(a \cdot b, c \cdot d) = g(a, c) \cdot g(b, c) \cdot g(a, d) \cdot g(b, d) = g(a, c) \cdot g(a, d) \cdot g(b, c) \cdot g(b, d)$; in Lemma 4.1 we provide a general version of this expansion. First, we define how the product symbol $\prod$ has to be read. The expression $\prod_{i=1}^{n} a_i$ stands for $a_1 \cdot a_2 \cdot \cdots \cdot a_n$. If the index set of the product is a subset of $\mathbb{N}^n$, we order the factors in a lexicographic way. Writing $\underline{m}$ as an abbreviation of $\{1, \ldots, m\}$, we therefore define

$$\prod_{(i_1,\ldots,i_k)\in\underline{m_1}\times\cdots\times\underline{m_k}} a(i_1, \ldots, i_k)$$

$$:= \begin{cases} \displaystyle\prod_{j=1}^{m_1} a(j) & \text{if } k = 1, \\ \displaystyle\prod_{j=1}^{m_1} \left(\prod_{(i_2,\ldots,i_k)\in\underline{m_2}\times\cdots\times\underline{m_k}} a(j, i_2, \ldots, i_k) \right) & \text{if } k > 1. \end{cases}$$

Then we have

Lemma 4.1. *Let $\mathbf{V} = (V, \cdot, ^{-1}, 1, g_1, g_2, \ldots)$ be a multilinear expanded group. For $k, m_1, \ldots, m_k \in \mathbb{N}$ and a k-ary multilinear operation g of V, we have:*

$$g\left(\prod_{j=1}^{m_1} a_{1,j}, \ldots, \prod_{j=1}^{m_k} a_{k,j} \right) = \prod_{(i_1,\ldots,i_k)\in\underline{m_1}\times\cdots\times\underline{m_k}} g(a_{1,i_1}, \ldots, a_{k,i_k}),$$

for all $a_{1,1}, \ldots, a_{1,m_1}, \ldots, a_{k,1}, \ldots, a_{k,m_k} \in V$.

Proof. We use induction on k and the multilinearity of g. $\square$

The next task is to adapt the commutator calculus known from group theory to our setting. We will do this by extending the procedure given in [2, Sections 5 and 6].

Definition 4.2. Let $n \geq 0$, let $\mathbf{V} = (V, \cdot, ^{-1}, 1, g_1, g_2, \ldots)$ be an expanded group and let X be a set of variables. We denote $\{\mathsf{x}^{-1} \mid \mathsf{x} \in X\}$ by X^{-1}, and we define $C(V, X)$ to be the smallest set of terms in the language $\mathcal{F}_{\mathbf{V}}$ such that:

- $\{\mathsf{c}(v) \mid v \in V\} \cup X \cup X^{-1} \subseteq C(V, X)$,

- if $\mathsf{u}_1, \mathsf{u}_2 \in C(V, X)$, then $\mathsf{f}_2(\mathsf{u}_1, \mathsf{u}_2) = [\mathsf{u}_1, \mathsf{u}_2] \in C(V, X)$, and
- if $k \geq 1$, g_i is a k-ary operation from $\mathbf{V}$ and $\mathsf{u}_1, \ldots, \mathsf{u}_k \in C(V, X)$, then $\mathsf{g}_i(\mathsf{u}_1, \ldots, \mathsf{u}_k) \in C(V, X)$.

Let $\mathbf{V}$ be an expanded group, let X be a set of variables, and let A be a subset of X. By $C(V, X, A)$, we abbreviate the set of all terms t in $C(V, X)$ such that $\operatorname{Var}(\mathsf{t}) = A$.

Proposition 4.3. *Let $\mathbf{V}$ be a multilinear expanded group, let $X=\{\mathsf{x}_1, \ldots, \mathsf{x}_k\}$ be a set of variables, let $\mathsf{p} \in C(V, X)$, let $\mathsf{p}^{\mathbf{V}}$ be the k-ary function that p induces on $\mathbf{V}$, let $i \in \{1, \ldots, k\}$ be such that $\mathsf{x}_i \in \operatorname{Var}(\mathsf{p})$, and let $a_1, \ldots, a_k \in V$. Then $\mathsf{p}^{\mathbf{V}}(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_k) = 1$.*

Proof. We prove the statement by induction on the length of p. If $\mathsf{p} = \mathsf{x}_i$ or $\mathsf{p} = \mathsf{x}_i^{-1}$, then the statement is true. For the induction step we distinguish the following cases:

- *Case $\mathsf{p} = [\mathsf{u}, \mathsf{v}]$ for some $\mathsf{u}, \mathsf{v} \in C(V, X)$*: Then $\mathsf{x}_i \in \operatorname{Var}(\mathsf{u})$ or $\mathsf{x}_i \in \operatorname{Var}(\mathsf{v})$, and therefore, by the induction hypothesis, $\mathsf{u}^{\mathbf{V}}(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_k) = 1$ or $\mathsf{v}^{\mathbf{V}}(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_k) = 1$. Thus

$$\begin{aligned}&\mathsf{p}^{\mathbf{V}}(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_k) \\ &\quad = [\mathsf{u}^{\mathbf{V}}(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_k), \mathsf{v}^{\mathbf{V}}(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_k)] = 1.\end{aligned}$$

- *Case $\mathsf{p} = \mathsf{g}(\mathsf{u}_1, \ldots, \mathsf{u}_r)$ for an r-ary operational symbol in the language of $\mathbf{V}$ with $\mathsf{g} \notin \{\cdot, ^{-1}, 1\}$ and $\mathsf{u}_1, \ldots, \mathsf{u}_r \in C(V, X)$*: Then $\mathsf{x}_i \in \operatorname{Var}(\mathsf{u}_1) \cup \cdots \cup \operatorname{Var}(\mathsf{u}_r)$. Let $j \in \{1, \ldots, r\}$ be such that $\mathsf{x}_i \in \operatorname{Var}(\mathsf{u}_j)$. Then by the induction hypothesis, $\mathsf{u}_j^{\mathbf{V}}(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_k) = 1$. Since $\mathsf{g}^{\mathbf{V}}$ is multilinear, it is absorbing, and therefore $\mathsf{p}^{\mathbf{V}}(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_k) = 1$. $\square$

Proposition 4.4. *Let $\mathbf{V} = (V, \cdot, ^{-1}, 1, g_1, g_2, \ldots)$ be a multilinear expanded group, and let $\mathsf{p} \in C(V, X)$. Then there exists a $\mathsf{d} \in C(V, X)$ such that $\mathsf{d}^{\mathbf{V}} = (\mathsf{p}^{-1})^{\mathbf{V}}$.*

Proof. We proceed by induction on the length of p. If $\mathsf{p} \in X$, then $\mathsf{p}^{-1} \in X^{-1} \subseteq C(V, X)$ and we define $\mathsf{d} := \mathsf{p}^{-1}$. If $\mathsf{p} \in X^{-1}$, then there is an $\mathsf{x} \in X \subseteq C(V, X)$ such that $\mathsf{x}^{\mathbf{V}} = (\mathsf{p}^{-1})^{\mathbf{V}}$. If $\mathsf{p} \in \{\mathsf{c}(v) \mid v \in V\}$, then we take $\mathsf{d} := \mathsf{c}(v^{-1})$. For the induction step, we consider the following cases:

- *Case $\mathsf{p} = [\mathsf{u}_1, \mathsf{u}_2]$ for some $\mathsf{u}_1, \mathsf{u}_2 \in C(V, X)$* : Then we have $\mathsf{p}^{-1} = [\mathsf{u}_1, \mathsf{u}_2]^{-1}$. Hence, we obtain $(\mathsf{p}^{-1})^{\mathbf{V}} = ([\mathsf{u}_1, \mathsf{u}_2]^{-1})^{\mathbf{V}} = [\mathsf{u}_2, \mathsf{u}_1]^{\mathbf{V}}$. Clearly, $[\mathsf{u}_2, \mathsf{u}_1] \in C(V, X)$ which completes the induction step in this case.

- *Case* $\mathsf{p} = \mathsf{g}_i(\mathsf{u}_1, \dots, \mathsf{u}_k)$ *for some* $\mathsf{u}_1, \dots, \mathsf{u}_k \in C(V, X)$: By the induction hypothesis, there exists $\mathsf{t} \in C(V, X)$ such that $\mathsf{t}^{\mathbf{V}} = (\mathsf{u}_1^{-1})^{\mathbf{V}}$. Since $\mathsf{g}_i^{\mathbf{V}} = g_i$ is multilinear, it satisfies $(\mathsf{g}_i^{\mathbf{V}}(a_1, a_2, \dots, a_k))^{-1} = \mathsf{g}_i^{\mathbf{V}}(a_1^{-1}, a_2, \dots, a_k)$ for all $a_1, \dots, a_k \in V$. Hence
$$(\mathsf{p}^{-1})^{\mathbf{V}} = (\mathsf{g}_i(\mathsf{u}_1, \dots, \mathsf{u}_k)^{-1})^{\mathbf{V}} = (\mathsf{g}_i(\mathsf{t}, \mathsf{u}_2, \dots, \mathsf{u}_k))^{\mathbf{V}},$$
which finishes the induction step because $\mathsf{g}_i(\mathsf{t}, \mathsf{u}_2, \dots, \mathsf{u}_k) \in C(V, X)$. □

Proposition 4.5. *Let* $\mathbf{V}$ *be a multilinear expanded group, let* $X=\{\mathsf{x}_1, \dots, \mathsf{x}_n\}$, *and let* q *be a polynomial term in the language of* $\mathbf{V} = (V, \cdot, ^{-1}, 1, g_1, g_2, \dots)$. *Then there are* $k \in \mathbb{N}$ *and* $\mathsf{p}_i \in C(V, X)$, $i \in \{1, \dots, k\}$ *such that*
$$\mathsf{s} := \prod_{i=1}^{k} \mathsf{p}_i$$
and q *induce the same function on* $\mathbf{V}$.

Proof. We proceed by induction on the length of q. If $\mathsf{q} \in \{\mathsf{c}(v) \mid v \in V\} \cup X$, then we choose $k = 1$ and define $\mathsf{p}_1 := \mathsf{q}$. If q is a nullary operational symbol in the language of $\mathbf{V}$, we choose $k = 1$ and define $\mathsf{p}_1 := \mathsf{c}(\mathsf{q}^{\mathbf{V}})$. For the induction step, we consider the following cases.

- *Case* $\mathsf{q} = \mathsf{u}_1 \cdot \mathsf{u}_2$ *for some polynomial terms* u_1 *and* u_2: By the induction hypothesis, there are $m, n \in \mathbb{N}$ and $\mathsf{p}_1, \dots, \mathsf{p}_m, \mathsf{d}_1, \dots, \mathsf{d}_n \in C(V, X)$ such that $\mathsf{u}_1^{\mathbf{V}} = \prod_{i=1}^{m} \mathsf{p}_i^{\mathbf{V}}$ and $\mathsf{u}_2^{\mathbf{V}} = \prod_{j=1}^{n} \mathsf{d}_j^{\mathbf{V}}$. Therefore, q and $\mathsf{p}_1 \cdot \dots \cdot \mathsf{p}_m \cdot \mathsf{d}_1 \cdot \dots \cdot \mathsf{d}_n$ induce the same function on $\mathbf{V}$.
- *Case* $\mathsf{q} = \mathsf{u}^{-1}$ *for a polynomial term* u *of* $\mathbf{V}$: By the induction hypothesis, there are $k \in \mathbb{N}$ and $\mathsf{p}_i \in C(V, X)$, $i \in \{1, \dots, k\}$ such that $\mathsf{u}^{\mathbf{V}} = \prod_{i=1}^{k} \mathsf{p}_i^{\mathbf{V}}$. Therefore, $(\mathsf{u}^{-1})^{\mathbf{V}} = \prod_{i=1}^{k} (\mathsf{p}_{k+1-i}^{-1})^{\mathbf{V}}$. By Proposition 4.4 we know that there exist $\mathsf{d}_i \in C(V, X)$, $i \in \{1, \dots, k\}$ such that $\mathsf{d}_i^{\mathbf{V}} = (\mathsf{p}_{k+1-i}^{-1})^{\mathbf{V}}$ for all $i \in \{1, \dots, k\}$. Now q and $\mathsf{d}_1 \cdot \dots \cdot \mathsf{d}_k$ induce the same function on $\mathbf{V}$.
- *Case* $\mathsf{q} = \mathsf{g}_i(\mathsf{u}_1, \dots, \mathsf{u}_k)$ *for a* $k \in \mathbb{N}$ *and polynomial terms* $\mathsf{u}_1, \dots, \mathsf{u}_k$: By the induction hypothesis, there exist $m_1, \dots, m_k \in \mathbb{N}$ and $\mathsf{p}_{1,1}, \dots, \mathsf{p}_{1,m_1}, \dots, \mathsf{p}_{k,1}, \dots, \mathsf{p}_{k,m_k} \in C(V, X)$ such that $\mathsf{u}_i^{\mathbf{V}} = \prod_{j=1}^{m_i} \mathsf{p}_{i,j}^{\mathbf{V}}$ for all $i \in \{1, \dots, k\}$. Then, we obtain:
$$\mathsf{q}^{\mathbf{V}} = g_i\left(\prod_{j=1}^{m_1} \mathsf{p}_{1,j}^{\mathbf{V}}, \dots, \prod_{j=1}^{m_k} \mathsf{p}_{k,j}^{\mathbf{V}}\right) = \prod_{(i_1,\dots,i_k) \in \underline{m_1} \times \cdots \times \underline{m_k}} g_i(\mathsf{p}_{1,i_1}^{\mathbf{V}}, \dots, \mathsf{p}_{k,i_k}^{\mathbf{V}}),$$
by Lemma 4.1. Clearly, $\mathsf{g}_i(\mathsf{p}_{1,i_1}, \dots, \mathsf{p}_{k,i_k}) \in C(V, X)$ for all $(i_1, \dots, i_k) \in \underline{m_1} \times \cdots \times \underline{m_k}$. This finishes the induction step. □

For a multilinear expanded group $\mathbf{V}$ and a nonempty set X of variables, we let $C(V,X)^*$ denote the set of all finite words over the alphabet $C(V,X)$, including the word λ of length 0. For a word $w = w_1 w_2 \dots w_k \in C(V,X)^*$, the function $w^{\mathbf{V}}$ that w induces on $\mathbf{V}$ is defined as the product $\prod_{i=1}^{k} w_i^{\mathbf{V}}$ of the functions induced by the single letters; in other words, the juxtaposition of words is read as their product; the empty word λ is defined to induce a constant function with value 1.

Lemma 4.6. *Let $\mathbf{V}$ be a multilinear expanded group, let $X = \{\mathsf{x}_1, \dots, \mathsf{x}_n\}$, and let $\mathsf{a}, \mathsf{z} \in C(V,X)^*$, $A, B \subseteq \{\mathsf{x}_1, \dots, \mathsf{x}_n\}$, $r \in \mathbb{N}$, $\mathsf{p}_1, \dots, \mathsf{p}_r \in C(V,X,A)$ and $\mathsf{d} \in C(V,X,B)$. Then there exist $r' \in \mathbb{N}$, $\mathsf{p}'_1, \dots, \mathsf{p}'_{r'} \in C(V,X,A)$, $t \in \mathbb{N}_0$ and $\mathsf{e}_1, \dots, \mathsf{e}_t \in C(V,X,A \cup B)$ such that $\mathsf{a}(\prod_{i=1}^{r} \mathsf{p}_i)\mathsf{d}\mathsf{z}$ and $\mathsf{a}\mathsf{d}(\prod_{i=1}^{r'} \mathsf{p}'_i)(\prod_{i=1}^{t} \mathsf{e}_i)\mathsf{z}$ induce the same function on $\mathbf{V}$.*

Proof. We repeat the proof of [2, Lemma 6.2]. □

Lemma 4.7. *Let $\mathbf{V}$ be a multilinear expanded group, let $X = \{\mathsf{x}_1, \dots, \mathsf{x}_n\}$, and let $\mathsf{a}, \mathsf{z} \in C(V,X)^*$, $A, B \subseteq \{\mathsf{x}_1, \dots, \mathsf{x}_n\}$, $r, s \in \mathbb{N}$, $\mathsf{p}_1, \dots, \mathsf{p}_r \in C(V,X,A)$ and $\mathsf{d}_1, \dots, \mathsf{d}_s \in C(V,X,B)$. Then there exist $r' \in \mathbb{N}$, $\mathsf{p}'_1, \dots, \mathsf{p}'_{r'} \in C(V,X,A)$, $t \in \mathbb{N}_0$ and $\mathsf{e}_1, \dots, \mathsf{e}_t \in C(V,X,A \cup B)$ such that $\mathsf{a}(\prod_{i=1}^{r} \mathsf{p}_i)(\prod_{i=1}^{s} \mathsf{d}_i)\mathsf{z}$ and $\mathsf{a}(\prod_{i=1}^{s} \mathsf{d}_i)(\prod_{i=1}^{r'} \mathsf{p}'_i)(\prod_{i=1}^{t} \mathsf{e}_i)\mathsf{z}$ induce the same function on $\mathbf{V}$.*

Proof. We repeat the proof of [2, Lemma 6.3], using Lemma 4.6 instead of [2, Lemma 6.2]. □

For the sequel, we fix a set $X = \{\mathsf{x}_1, \dots, \mathsf{x}_n\}$ of variables and a total order $\leq$ on the subsets of X that is a refinement of the subset relation. For $A_1, A_2 \subseteq X$, we write $A_1 < A_2$ if $A_1 \leq A_2$ and $A_1 \neq A_2$.

Proposition 4.8. *Let $\mathbf{V}$ be a multilinear expanded group, let $X=\{\mathsf{x}_1,\dots,\mathsf{x}_n\}$, and let q be a polynomial term over X in the language of $\mathbf{V}$. Then there are subsets $A_1, \dots, A_k$ of X such that $A_1 < \dots < A_k$, and there are $\mathsf{p}_{i,j}$ such that $\mathsf{p}_{i,j}$ lies in $C(V,X,A_i)$ for all i, j and furthermore*

$$\mathsf{s} := \prod_{i=1}^{k} \prod_{j=1}^{m_i} \mathsf{p}_{i,j}$$

and q induce the same function on $\mathbf{V}$.

Proof. Using Lemma 4.7 instead of [2, Lemma 6.3], we repeat the proof of [2, Lemma 6.5]. □

Lemma 4.9. *Let $\mathbf{V}$ be a multilinear expanded group, let $X = \{\mathsf{x}_1, \dots, \mathsf{x}_n\}$, and let p be an n-ary absorbing polynomial of $\mathbf{V}$. Then, there are $r \geq 0$ and*

$\mathsf{p}_1, \ldots, \mathsf{p}_r \in C(V, X, X)$ *such that each* $\mathsf{p}_i^{\mathbf{V}}$ *is an* n*-ary absorbing polynomial and*

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{r} \mathsf{p}_i^{\mathbf{V}}(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n) \in V^n$.

Proof. Let p be an absorbing n-ary polynomial. In the case that p is constant, we set $r := 0$, so we can assume that p is not constant. By Proposition 4.8 there are subsets $A_1, \ldots, A_s$ of $\{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$ such that $A_1 < \cdots < A_s$, and there are $\mathsf{d}_{i,j}$ such that $\mathsf{d}_{i,j}$ lies in $C(V, X, A_i)$ for all i, j and for

$$\mathsf{p} := \prod_{i=1}^{s} \left(\prod_{j=1}^{m_i} \mathsf{d}_{i,j} \right), \tag{4.1}$$

we have $p = \mathsf{p}^{\mathbf{V}}$. By Proposition 4.3 each $\mathsf{d}_{i,j}^{\mathbf{V}}$ evaluates to 1 if one of the variables in A_i is set to 1.

We will now prove by induction on s that for all products

$$\mathsf{q} = \prod_{i=1}^{s} \left(\prod_{j=1}^{m_i} \mathsf{c}_{i,j} \right) \tag{4.2}$$

with $\mathsf{c}_{i,j} \in C(V, X, A_i)$ and $A_1 < \cdots < A_s$ such that q induces a nonconstant absorbing polynomial, we have $\mathsf{q}^{\mathbf{V}} = (\prod_{j=1}^{m_s} \mathsf{c}_{s,j})^{\mathbf{V}}$, and $A_s = X$. If $s = 1$, then $A_1 = \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$, because p, as a nonconstant absorbing polynomial, depends on all its arguments. For the induction step, we let $s \geq 2$. Then $A_1 \neq X$. We observe that each of the sets A_i $(i = 2, \ldots, s)$ is not a subset of A_1; hence each A_i has an element that is not contained in A_1. In (4.2), we now set all variables that are not elements of A_1 to 1. Since $\mathsf{q}^{\mathbf{V}}$ is absorbing, the left hand side of (4.2) then evaluates to 1. Looking at the right side, for $i \geq 2$, each $\mathsf{c}_{i,j}$ is an element of $C(V, X)$ that contains all variables in A_i; at least one of these is set to 1, and so by Proposition 4.3, $\mathsf{c}_{i,j}$ evaluates to 1 for $i \geq 2$. Thus we obtain from (4.2) that $\prod_{j=1}^{m_1} \mathsf{c}_{1,j}^{\mathbf{V}}$ is identically 1. Hence, we have $\mathsf{q}^{\mathbf{V}} = (\prod_{i=2}^{s}(\prod_{j=1}^{m_i} \mathsf{c}_{i,j}))^{\mathbf{V}}$. Now, the assertion follows from the induction hypothesis.

Therefore, we have $A_s = X$ and $p = (\prod_{j=1}^{m_s} \mathsf{d}_{s,j})^{\mathbf{V}}$. So, we can set $r := m_s$ and $\mathsf{p}_j := \mathsf{d}_{s,j}$. □

The following theorem tells that in a multilinear expanded group of degree m, all higher commutators can be computed from the commutator operations of arity at most m.

Theorem 4.10. *Let $\mathbf{V}$ be a multilinear expanded group of degree m, let $n \geq 2$, and let $A_1, \dots, A_n$ be ideals of $\mathbf{V}$. Let T be the set of those terms in the language $\{\mathsf{f}_2, \dots, \mathsf{f}_m\}$ that contain each of the variables $\mathsf{x}_1, \dots, \mathsf{x}_n$ exactly once. Then $[\![A_1, \dots, A_n]\!]$ is the join of all ideals in the set*

$$\{\mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1, \dots, A_n) \mid \mathsf{t} \in T\}.$$

Proof. Let $B := \bigvee\{\mathsf{t}^{\mathbf{I}(\mathbf{V})}(A_1, \dots, A_n) \mid \mathsf{t} \in T\}$. From Lemma 3.2 we obtain $B \leq [\![A_1, \dots, A_n]\!]$. To prove the opposite inequality, we will show that all of the generators of $[\![A_1, \dots, A_n]\!]$ of the form $p(a_1, \dots, a_n)$ with p an absorbing n-ary polynomial of $\mathbf{V}$ and $a_1 \in A_1, \dots, a_n \in A_n$ are elements of B. By Lemma 4.9 there exist $r \geq 1$ and $\mathsf{p}_1, \dots, \mathsf{p}_r \in C(V, X)$ such that each $\mathsf{p}_i^{\mathbf{V}}$ is an absorbing n-ary polynomial and

$$p(a_1, \dots, a_n) = \prod_{i=1}^{r} \mathsf{p}_i^{\mathbf{V}}(a_1, \dots, a_n).$$

Let $i \in \{1, \dots, r\}$. We will prove that $\mathsf{p}_i^{\mathbf{V}}(a_1, \dots, a_n) \in B$. Using Lemma 3.1 we have $\mathsf{p}_i^{\mathbf{V}}(a_1, \dots, a_n) \in \mathsf{p}_i^{\mathbf{I}(\mathbf{V})}(A_1, \dots, A_n)$. Now we notice that p_i is a term in $C(V, X)$, and therefore, all operation symbols of p_i have arity at most m. By Lemma 3.4, there exists a term s_i' in the language $\{\mathsf{f}_j \mid j \leq m\}$ such that every variable $\mathsf{x}_1, \dots, \mathsf{x}_n$ occurs exactly once in s_i' and $\mathsf{p}_i^{\mathbf{I}(\mathbf{V})}(A_1, \dots, A_n) \leq \mathsf{s}_i'^{\mathbf{I}(\mathbf{V})}(A_1, \dots, A_n)$ by Lemma 3.4. Now replacing all subterms of the form $\mathsf{f}_1(\mathsf{t})$ by t, we obtain a term s_i that induces the same function on $\mathbf{I}(\mathbf{V})$ as s_i', but does not contain f_1. Hence, $\mathsf{p}_i^{\mathbf{V}}(a_1, \dots, a_n) \in \mathsf{s}_i^{\mathbf{I}(\mathbf{V})}(A_1, \dots, A_n)$ and since $\mathsf{s}_i \in T$, we obtain that $\mathsf{p}_i^{\mathbf{V}}(a_1, \dots, a_n) \in B$. Since B, as an ideal, is closed under multiplication, we get $p(a_1, \dots, a_n) \in B$. This completes the proof that all generating elements of $[\![A_1, \dots, A_n]\!]$ belong to B and therefore $[\![A_1, \dots, A_n]\!] \leq B$. ☐

Hence in a multilinear expanded group of degree 2, the 4-ary commutator $[\![A_1, A_2, A_3, A_4]\!]$ is the join of all $[\![A_{\pi(1)}, [\![A_{\pi(2)}, [\![A_{\pi(3)}, A_{\pi(4)}]\!]]\!]]\!]$ and $[\![[\![A_{\pi(1)}, A_{\pi(2)}]\!], [\![A_{\pi(3)}, A_{\pi(4)}]\!]]\!]$ with $\pi \in S_4$. Furthermore, if a multilinear expanded group of degree 2 satisfies the property $[\![[\![Y, Z]\!], X]\!] \leq [\![Y, [\![X, Z]\!]]\!] \vee [\![Z, [\![X, Y]\!]]\!]$ for all ideals X, Y, Z, then for every $n \in \mathbb{N}$ the n-ary commutator can be computed as the join of all $[\![A_{\pi(1)}, [\![A_{\pi(2)}, [\![\dots]\!]]\!]]\!]$ with $\pi \in S_n$; for groups, the above property is a consequence of the three subgroups lemma.

5. Connections between nilpotence and supernilpotence

Let t be a term in the language $\mathcal{F}$. The depth of a term is defined by $d(\mathsf{x}) = 0$ if x is a variable, and $d(\mathsf{f_k}(\mathsf{t}_1, \dots, \mathsf{t}_k)) := 1 + \max\{d(\mathsf{t}_i) \mid i \in \{1\dots, k\}\}$. We will now relate the result of $\mathsf{t}^{\mathbf{I}(\mathbf{V})}(V,\dots,V)$ to the lower central series of $\mathbf{V}$, which is defined by $\gamma_1(\mathbf{V}) := V$, $\gamma_n(\mathbf{V}) := [\![V, \gamma_{n-1}(\mathbf{V})]\!]$ $(n \geq 2)$.

Lemma 5.1. *Let* t *be a term of depth* d *in the language* $\{\mathsf{f}_i \mid i \geq 2\}$ *over the variables* $\{\mathsf{x}_1,\dots,\mathsf{x}_n\}$. *Then we have* $\mathsf{t}^{\mathbf{I}(\mathbf{V})}(V,\dots,V) \leq \gamma_{d+1}(\mathbf{V})$.

Proof. We proceed by induction on d. For $d = 0$, we have $\mathsf{t}^{\mathbf{I}(\mathbf{V})}(V,\dots,V) = V = \gamma_1(\mathbf{V})$. Now assume $d(\mathsf{t}) = d \geq 1$. Then $\mathsf{t}^{\mathbf{I}(\mathbf{V})}(V,\dots,V) = \mathsf{f}_k^{\mathbf{I}(\mathbf{V})}(\mathsf{t}_1(V,\dots,V), \dots, \mathsf{t}_k(V,\dots,V))$. Let i be such that t_i is the term in $\mathsf{t}_1,\dots,\mathsf{t}_k$ with maximal depth, and let j be an element in $\{1,\dots,k\}\setminus\{i\}$. Then we have $\mathsf{f}_k^{\mathbf{I}(\mathbf{V})}(\mathsf{t}_1(V,\dots,V), \dots, \mathsf{t}_k(V,\dots,V)) \leq \mathsf{f}_2^{\mathbf{I}(\mathbf{V})}(\mathsf{t}_i(V,\dots,V), \mathsf{t}_j(V,\dots,V))$. By the induction hypothesis, $\mathsf{t}_i(V,\dots,V) \leq \gamma_d(\mathbf{V})$, and therefore $\mathsf{f}_2^{\mathbf{I}(\mathbf{V})}(\mathsf{t}_i(V,\dots,V), \mathsf{t}_j(V,\dots,V)) =$
$[\![\mathsf{t}_i(V,\dots,V), \mathsf{t}_j(V,\dots,V)]\!] \leq [\![\gamma_d(\mathbf{V}), V]\!] = \gamma_{d+1}(\mathbf{V})$. $\square$

From these results, we will now derive Theorem 1.5.

Proof of Theorem 1.5. Let $n := m^{k-1}+1$. By Theorem 4.10, $[\![\underbrace{V,\dots,V}_{n}]\!]$ can be computed as the join of all $\mathsf{t}^{\mathbf{I}(\mathbf{V})}(V,\dots,V)$, where t is a term in the language $\{\mathsf{f}_2,\dots,\mathsf{f}_m\}$ that contains each of the variables $\mathsf{x}_1,\dots,\mathsf{x}_n$ exactly once. Let t be one such term, and let d be its depth. Then t, seen as a tree, has exactly n leaves. Since this tree is at most m-ary, we obtain $n \leq m^d$, and hence $n-1 < m^d$, which implies $k-1 < d$, and thus $k \leq d$. By Lemma 5.1, $\mathsf{t}^{\mathbf{I}(\mathbf{V})}(V,\dots,V) \leq \gamma_{k+1}(\mathbf{V})$, and $\gamma_{k+1}(\mathbf{V}) = \{1\}$ because $\mathbf{V}$ is nilpotent of class k. Now from Theorem 4.10, we obtain $[\![\underbrace{V,\dots,V}_{n}]\!] = \{1\}$, and therefore $\mathbf{V}$ is m^{k-1}-supernilpotent. $\square$

6. An example of a multilinear expanded group

In this section, we provide examples of expanded groups that prove that the bound in Theorem 1.5 is sharp. To this end, we first construct expanded groups all of whose higher commutators are easy to calculate. In contrast to the previous sections, we will now write groups in additive notation.

Proposition 6.1. *Let* p *be a prime,* $m, n \in \mathbb{N}$ *with* $m \geq 2$, *let* $V := {\mathbb{Z}_p}^n$, *let* $(e_1,\dots,e_n)$ *be the canonical basis of* V *as a vector space over* $\mathbb{Z}_p$, *let* $e_0 := 0$ *and let* $f : {\mathbb{Z}_p}^n \to {\mathbb{Z}_p}^n$ *be the linear mapping defined by* $f(e_i) = e_{i-1}$ *for* $i \in \{1,\dots,n\}$. *Let* $\delta : \{1,\dots,n\} \to \{0,\dots,n\}$ *be such that* $\delta(i) \leq i$ *for all* $i \in \{1,\dots,n\}$. *We define an* m*-ary multilinear operation* g *on* V *by*

$$g(e_{i_1},\dots,e_{i_m}) := e_{\delta(\min(i_1,\dots,i_m))}$$

for all $i_1, \dots, i_m \in \{1, \dots, n\}$. For each $i \in \{0, \dots, n\}$, let E_i be the subgroup of $(V, +)$ generated by $\{e_0, \dots, e_i\}$ (hence $E_0 = \{0\}$ and $E_n = V$). Then we have:

(1) *For all $i \in \{1, \dots, m\}$, $v \in V^m$, $a \in {E_i}^m$, we have $g(v + a) - g(v) \in E_{\delta(i)}$.*

(2) *The expanded group $\mathbf{V} := (V, +, -, 0, f, g)$ has exactly the ideals $E_0, E_1, \dots, E_n$.*

(3) *For all $k \in \{2, \dots, m\}$ and $i_1, \dots, i_k \in \{1, \dots, n\}$, the k-ary commutator operation of $\mathbf{V}$ satisfies $[\![E_{i_1}, \dots, E_{i_k}]\!] = E_{\delta(\min(i_1, \dots, i_k))}$.*

Proof. (1) For each $k \in \{1, \dots, m\}$ we take $\alpha_1^{(k)}, \dots, \alpha_i^{(k)} \in \mathbb{Z}_p$ such that $a_k = \sum_{j=1}^{i} \alpha_j^{(k)} * e_j$. Then $g(v+a) = g(v_1 + \sum_{j=1}^{i} \alpha_j^{(1)} * e_j, \dots, v_m + \sum_{j=1}^{i} \alpha_j^{(m)} * e_j)$. Using multilinearity, we expand the last expression into $(i+1)^m$ summands. One summand is $g(v_1, \dots, v_m)$, each other summand is of the form $g(w_1, \dots, w_{k-1}, \alpha_j^{(k)} * e_j, w_{k+1}, \dots, w_m)$ where $j \leq i$. Now in each of this summands, we write each w_l as $\sum_{r=1}^{n} \beta_r^{(l)} * e_r$ and expand using multilinearity; in this way we obtain n^{m-1} summands each of which lies in $E_{\delta(i)}$. Altogether $g(v + a) - g(v) \in E_{\delta(i)}$, which completes the proof of (1). For proving (2), we let I be an ideal of $\mathbf{V}$; let d be its dimension as a vector space over $\mathbb{Z}_p$. As an ideal of $\mathbf{V}$, I is an f-invariant subspace. Since the restriction $f|_I$ is a nilpotent linear mapping on the d-dimensional space I, its characteristic polynomial is x^d and therefore by the Cayley-Hamilton Theorem, $f^d(I) \subseteq E_0$. Hence $I \subseteq \{x \in V \mid f^d(x) = 0\} = E_d$. Since I and E_d have the same dimension, we obtain $I = E_d$. For proving that each E_i is really an ideal of $\mathbf{V}$, we have to show that for all $i \in \{0, \dots, n\}$, $k \in \mathbb{N}$, $j \in \{1, \dots, k\}$, for all k-ary fundamental operations h of $\mathbf{V}$ and for all $v_1, \dots, v_k \in V$ and $w \in E_i$, we have $h(v_1, \dots, v_{j-1}, v_j + w, v_{j+1}, \dots, v_k) - h(v_1, \dots, v_k) \in E_i$. For $h = f$, this follows from the fact that f is linear and E_i is an f-invariant subspace. For $h = g$, we obtain from item (1) that $g(v_1, \dots, v_{j-1}, v_j + w, v_{j+1}, \dots, v_k) - g(v_1, \dots, v_k)$ lies in $E_{\delta(i)}$. Since $\delta(i) \leq i$, the result follows. For establishing (3), we first consider the polynomial $p(x_1, \dots, x_k) := g(x_1, \dots, x_k, x_k, \dots, x_k)$. Since p is absorbing, we have $p(e_{i_1}, \dots, e_{i_k}) \in [\![E_{i_1}, \dots, E_{i_k}]\!]$, which implies $e_{\delta(\min(i_1, \dots, i_k))} \in [\![E_{i_1}, \dots, E_{i_k}]\!]$. Therefore $E_{\delta(\min(i_1, \dots, i_k))} \leq [\![E_{i_1}, \dots, E_{i_k}]\!]$. For proving the other inclusion, let $i_1, \dots, i_k \in \{0, 1, \dots, n\}$, let j be such that i_j is minimal among $i_1, \dots, i_k$, and let l be an element of $\{1, \dots, k\} \setminus \{j\}$. Then we have $[\![E_{i_1}, \dots, E_{i_k}]\!] \leq [\![E_{i_j}, E_{i_l}]\!]$. We will now show

$$[\![E_{i_j}, E_{i_l}]\!] \leq E_{\delta(i_j)}. \tag{6.1}$$

From [14] (cf. [4, Proposition 5.2], [3, Proposition 2.3 and Lemma 2.4]) we obtain that it is sufficient for (6.1) to show that every fundamental operation of $\mathbf{V}$

preserves the relation $\{(x_1, x_2, x_3, x_4) \in V^4 \mid x_1 - x_2 \in E_{i_j}, x_2 - x_3 \in E_{i_l}, x_1 - x_2 + x_3 - x_4 \in E_{\delta(i_j)}\}$. Let $y \in V^m$, $a \in E_{i_j}^m$, $b \in E_{i_l}^m$ and $z \in V^m$ such that $(y+a) - y + (y+b) - z \in E_{\delta(i_j)}$; we would like to show

$$g(y+a) - g(y) + g(y+b) - g(z) \in E_{\delta(i_j)}. \tag{6.2}$$

From item (1), we obtain $g(y+a) - g(y) \in E_{\delta(i_j)}$. By assumption, we have $(y+a) - y + (y+b) - z = (y+a+b) - z \in E_{\delta(i_j)}$. Since g is congruence preserving, we have $g(y+a+b) - g(z) \in E_{\delta(i_j)}$. Now, using (1) again, we know that $g(y+b) - g(y+a+b) \in E_{\delta(i_j)}$. Therefore, $(g(y+a) - g(y)) + (g(y+b) - g(y+a+b)) + (g(y+a+b) - g(z)) \in E_{\delta(i_j)}$, which implies (6.2). This completes the proof of (6.1); therefore also the $\leq$-inclusion of item (3) is proved. $\square$

The following example proves Proposition 1.6.

Example 6.2. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, and let $\mathbf{V}$ be the multilinear expanded group $\mathbf{V} = ({\mathbb{Z}_p}^n, +, -, 0, f, g)$ with the m-ary multilinear operation g defined by

$$g(e_{i_1}, \dots, e_{i_m}) := e_{\min(i_1,\dots,i_m)-1} \text{ for } i_1, \dots, i_m \in \{1, \dots, n\};$$

this is the operation that is obtained from the construction of Proposition 6.1 using $\delta(i) := i - 1$ for $i \in \{1, \dots, n\}$. Then from item (3) of Proposition 6.1, we can compute the lower central series of $\mathbf{V}$ as $\gamma_i(\mathbf{V}) = E_{n+1-i}$ for $i \in \{1, \dots, n+1\}$. Hence $\mathbf{V}$ is nilpotent of class n. Now consider the terms t_k ($k \in \{1, \dots, n-1\}$) defined by the following recursion; each term t_k will have exactly m^k variables. We define $\mathsf{t}_1 := \mathsf{f}_m(\mathsf{x}_1, \dots, \mathsf{x}_m)$, $\mathsf{t}_k := \mathsf{f}_m(\mathsf{t}_{k-1}(\mathsf{x}_1, \dots, \mathsf{x}_{m^{k-1}}), \mathsf{t}_{k-1}(\mathsf{x}_{m^{k-1}+1}, \dots, \mathsf{x}_{2\,m^{k-1}}), \dots, \mathsf{t}_{k-1}(\mathsf{x}_{m^k - m^{k-1}+1}, \dots, \mathsf{x}_{m^k}))$. From Proposition 6.1, we obtain that $\mathsf{t}_i^{\mathbf{I}(\mathbf{V})} = E_{n-i}$ for $i \in \{1, \dots, n\}$. Setting $i := n-1$, we obtain that $\mathsf{t}_{n-1}^{\mathbf{I}(\mathbf{V})}(V, \dots, V) \neq E_0$, and therefore, by Lemma 3.2, $[\![\underbrace{V, \dots, V}_{m^{n-1}}]\!] \neq E_0$. From this, we see that $\mathbf{V}$ is not $(m^{n-1} - 1)$-supernilpotent. By Theorem 4.10, $\mathbf{V}$ is m^{n-1}-supernilpotent, and therefore $\mathbf{V}$ is supernilpotent of class m^{n-1}. $\square$

Example 6.3. We will now give an example of an expanded group $\mathbf{V} = (V, +, -, 0, f, g_1, g_2)$ such that for $\mathbf{V}_1 := (V, +, -, 0, f, g_1)$ and $\mathbf{V}_2 := (V, +, -, 0, f, g_2)$ we have $[\![V_i, [\![V_i, V_i]\!]]\!] = [\![V_i, V_i, V_i]\!] = 0$ ($i = 1, 2$), but in $\mathbf{V}$, we have $[\![V, [\![V, V]\!]]\!] \neq 0$ and $[\![V, V, V]\!] \neq 0$. This example shows that the operations that preserve the nilpotence class (or supernilpotence class) of a given expanded group need not form a clone. We will construct both examples on ${\mathbb{Z}_p}^3$ by using the construction of Proposition 6.1. The binary operation g_1 is defined by using δ_1 with $\delta_1(3) = \delta_1(2) = 1$ and $\delta_1(1) = 0$; the binary operation

g_2 is constructed setting $\delta_2(3) = 2$, $\delta_2(2) = \delta_2(1) = 0$. From Proposition 6.1, we see that both expanded groups satisfy $[\![E_3, [\![E_3, E_3]\!]]\!] = E_0 = 0$, and hence Theorem 4.10 yields that both $\mathbf{V}_1$ and $\mathbf{V}_2$ satisfy $[\![E_3, E_3, E_3]\!] = 0$. However, in $\mathbf{V}$, the absorbing polynomial g_2 yields $[\![E_3, E_3]\!] \geq E_2$; now g_1 shows $[\![E_3, E_2]\!] \geq E_1$. Altogether, $\mathbf{V}$ satisfies $[\![V, [\![V, V]\!]]\!] \neq 0$, and therefore using (HC8), also $[\![V, V, V]\!] \neq 0$.

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ERHARD AICHINGER
INSTITUT FÜR ALGEBRA
JOHANNES KEPLER UNIVERSITÄT LINZ
ALTENBERGERSTRASSE 69
4040 LINZ
AUSTRIA

E-mail: erhard@algebra.uni-linz.ac.at

NEBOJŠA MUDRINSKI
DEPARTMENT OF MATHEMATICS
AND INFORMATICS
FACULTY OF SCIENCES
UNIVERSITY OF NOVI SAD
TRG DOSITEJA OBRADOVIĆA 4
21000 NOVI SAD
SERBIA

E-mail: nmudrinski@dmi.uns.ac.rs

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