Publ. Math. Debrecen
83/4 (2013), 617–623
DOI: 10.5486/PMD.2013.5603

Some Ricci-flat Finsler metrics

By ESRA SENGELEN SEVIM (Istanbul), ZHONGMIN SHEN (Indianapolis)
and LILI ZHAO (Shanghai)

Abstract. In this paper, we construct some Ricci-flat Finsler metrics defined by a Riemannian metric and a 1-form.

1. Introduction

Roughly speaking, Riemannian metrics on a manifold are "quadratic" metrics, while Finsler metrics are those without restriction on the quadratic property. The Riemann curvature in Riemannian geometry can be naturally extended to Finsler metrics as a family of linear transformations on tangent spaces. The Ricci curvature is defined as the trace of the Riemann curvature. It is a natural problem to study Finsler metrics $F = F(x, y)$ with isotropic Ricci curvature $\mathrm{Ric} = \mathrm{Ric}(x, y)$, i.e.,

$$\mathrm{Ric} = (n-1)\sigma F^2,$$

where $\sigma = \sigma(x)$ is a scalar function is on the n-dimensional manifold. Such metrics are called *Einstein (Finsler) metrics.*

In this paper, we shall consider Einstein metrics defined by a Riemannian metric α and a 1-form β in the following form

$$F = \alpha\phi\Big(\frac{\beta}{\alpha}\Big), \tag{1.1}$$

Mathematics Subject Classification: 53C60, 53B40.
Key words and phrases: Ricci curvature, Einstein metric and (α, β)-metric.
The second author was supported in part by the Natural Science Foundation, grant (DMS-0810159). The third author was supported in part by the National Natural Science Foundation of China, grant (No.11201295).

where $\phi = \phi(s)$ is a positive smooth function. Finsler metrics defined in (1.1) are called *(α, β)-metrics.* The simplest (α, β)-metrics are Randers metrics defined by $F = \alpha + \beta$. More general (α, β)-metrics are defined by a polynomial

$$F = \alpha \sum_{i=0}^{k} a_i \left(\frac{\beta}{\alpha}\right)^i, \quad (k \geq 2) \tag{1.2}$$

where $a_0 = 1$ and a_i are constants with $a_k \neq 0$. Such metrics are called *polynomial metrics.*

In [1], Bao–Robles find equations on α and β that characterize Randers metrics of constant Ricci curvature. There are many Randers metrics of constant (zero or non-zero) Ricci curvature. However, it has been shown that if a polynomial metric of non-Randers type in (1.2) is of constant Ricci curvature, then it must be Ricci-flat ([3]). Thus one just needs to focus on Ricci-flat (α, β)-metrics, at least for polynomial metrics in (1.2). In [4] and [5], the authors independently obtain equations on α, β and ϕ that characterize Ricci-flat (α, β)-metrics of Douglas type. A natural question arises: are there Ricci-flat (α, β)-metrics of non-Douglas type?

Theorem 1. *Let $F = \alpha\phi(s), s = \beta/\alpha$ be an (α, β)-metric on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta = b_iy^i$ is a 1-form and $\phi = \phi(s)$ is a positive C^∞ function. Suppose that α, β and ϕ satisfy the following conditions:*

(a) $b_{i|j} = \epsilon(b^2 a_{ij} - b_i b_j)$,

(b) $\epsilon_i = -\epsilon^2 b_i$,

(c) ${}^{\alpha}\mathbf{Ric} = -\kappa\epsilon^2(b^2\alpha^2 - \beta^2)$,

(d) ϕ *satisfies*

$$\begin{aligned} &-(b^2 - s^2)(\Xi - (b^2 - s^2)\Psi')^2 \\ &\quad = (n-1)\big\{ -\kappa + 3s\Xi + 2(n-2)b^2\Psi - (b^2 - s^2)^3(\Psi')^2 \\ &\qquad - 2s(b^2 - s^2)\Psi\Xi - (b^2 - s^2)\Xi'[1 - 2(b^2 - s^2)\Psi]\big\}, \end{aligned} \tag{1.3}$$

where $b := \sqrt{a^{ij}b_ib_j}$, $b_{i|j}$ denotes the covariant derivative of b_i with respect to α, κ is a constant, $\epsilon := \epsilon(x)$ is a scalar function, $\epsilon_i := \epsilon_{x^i}$ and

$$\Psi := \frac{Q'}{2\Delta}, \quad \Xi := (n-1)\frac{Q - sQ'}{2\Delta} + (b^2 - s^2)\Psi',$$

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q',$$

then F is Ricci flat.

By Theorem 1 (a), $(b^2)_{|k} = 2a^{ij}b_i b_{j|k} = 0$. Thus $b = \sqrt{a^{ij}b_i b_j}$ is a constant. The equation (1.3) is a fourth order ordinary differential equation in ϕ. According to the ODE theory, the local solution of (1.3) exists nearby $s = 0$ for any given initial conditions. But we are unable to express it in terms of elementary functions and we are unable to show that the solution is defined on an interval containing $[-b, b]$. Thus the (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ defined by ϕ might be singular. In the following example, we construct α and β satisfying Theorem 1(a), (b) and (c). Then for any $\phi = \phi(s)$ satisfying (1.3), we obtain a (possibly singular) Ricci-flat (α, β)-metric.

Example 1. Let $M^n = \mathbb{R}_+ \times N^{n-1}$ be a product manifold. Suppose N admit an Einstein metric $\breve{\alpha}$ with ${}^{\breve{\alpha}}\mathbf{Ric} = (n-2)\lambda\breve{\alpha}$, where λ is a constant. We can construct a warped product metric on M as

$$\alpha := \sqrt{(y^1)^2 + (x^1\breve{\alpha})^2},$$

where (x^1, y^1) is the coordinate of $T\mathbb{R}_+$. Take

$$\beta := y^1, \quad \epsilon = \frac{1}{x^1}$$

we can check $b^2 = 1$ and

$$b_{i|j} = \epsilon(b^2 a_{ij} - b_i b_j), \quad \epsilon_i = -\epsilon^2 b_i, \quad {}^{\alpha}\mathbf{Ric} = (n-2)(\lambda - 1)\epsilon^2(b^2\alpha^2 - \beta^2).$$

Thus if we take $\kappa := (n-2)(1-\lambda)$ and $\phi = \phi(s)$ satisfies (1.3), then F is Ricci flat.

2. Preliminaries

A Finsler metric on a manifold M is a nonnegative scalar function $F = F(x, y)$ on the tangent bundle TM, where x is a point in M and $y \in T_xM$ is a tangent vector at x. In local coordinates, the geodesics of a Finsler metric $F = F(x, y)$ are characterized by

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i := \frac{1}{4}g^{il}(x, y)\Big\{[F^2]_{x^k y^l}(x, y)y^k - [F^2]_{x^l}(x, y)\Big\} \tag{2.1}$$

with g^{ij} the inverse of $g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$. The local functions G^i on TM define a global vector field

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

The vector field G is called the *spray* of F and the local functions $G^i = G^i(x, y)$ are called *spray coefficients* of F.

For any $x \in M$ and $y \in T_xM\backslash\{0\}$, the Riemann curvature $\mathbf{R}_y : T_xM \to T_xM$ is defined by $\mathbf{R}_y(u) = R^i{}_k(x, y)u^k \frac{\partial}{\partial x^i}|_x$, where

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}.$$

Then the Ricci curvature is given by

$$\mathbf{Ric} = 2\frac{\partial G^i}{\partial x^i} - \frac{\partial^2 G^i}{\partial x^m \partial y^i} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^i} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^i}.$$

An (α, β)-metric on a manifold M is a scalar function on TM defined by

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s) > 0$ is a C^∞ function on $(-b_o, b_o)$, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $b(x) := \|\beta_x\|_\alpha < b_o$. It can be shown that for any Riemannian metric α and any 1-form β on M with $b(x) < b_o$, the function $F = \alpha\phi(\beta/\alpha)$ is a (positive definite) Finsler metric if and only if ϕ satisfies

$$\phi(s) - s\phi'(s) + (\rho^2 - s^2)\phi''(s) > 0, \quad (|s| \le \rho < b_o). \tag{2.2}$$

Thus, for a C^∞ positive function $\phi = \phi(s)$ on $(-b_o, b_o)$ satisfying (2.2), if $b(x) := \|\beta_x\|_\alpha < b_o$, the (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ is a regular positive definite Finsler metric.

Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_j := b^i r_{ij}, \qquad s_j := b^i s_{ij},$$

where "|" denotes the covariant derivative with respect to the Levi–Civita connection of α. This covariant derivative can be lifted horizontally to $TM\backslash\{0\}$ and thus one can write $\beta_{|i}$, $(r_j y^j)_{|i}$ and etc. By (2.1), the spray coefficients G^i of F are given by the following Lemma.

Lemma 1 ([2])**.** *For an (α,β)-metric $F=\alpha\phi(s), s=\beta/\alpha$, the spray coefficients of F are given by*

$$G^i = {}^\alpha G^i + \alpha Q s^i{}_0 + \Theta\{r_{00} - 2Q\alpha s_0\}\frac{y^i}{\alpha} + \Psi\{r_{00} - 2Q\alpha s_0\}b^i, \tag{2.3}$$

where ${}^\alpha G^i$ are the spray coefficients of α,

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \frac{Q - sQ'}{2\Delta},$$

$$\Psi := \frac{Q'}{2\Delta}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q', \quad b := \|\beta_x\|_\alpha,$$

and $s^i{}_j := a^{ik}s_{kj}$. The index "0" means contracting with y, for example, $s^i{}_0 := s^i{}_j y^j$, $s_0 := s_i y^i$, $r_{00} := r_{ij}y^i y^j$.

Note that

$$\Delta = \frac{\phi\{\phi - s\phi' + (b^2 - s^2)\phi''\}}{(\phi - s\phi')^2}.$$

If (2.2) holds, then $\Delta = \Delta(s) > 0$ for s with $|s| \le b < b_o$.

3. Proof of Theorem 1.1

In this section we are going to prove Theorem 1.1. First we compute the Ricci curvature of the (α,β)-metric under a special condition (see (3.1) below).

Lemma 2. *Let $F = \alpha\phi(\beta/\alpha)$ be an (α,β)-metric on an n-dimensional manifold M. Suppose that $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$ satisfy the following equation*

$$b_{i|j} = \epsilon(b^2 a_{ij} - b_i b_j), \tag{3.1}$$

where $\epsilon = \epsilon(x)$ is a scalar function with $\epsilon_{x^i} = -\epsilon^2 b_i$. Then $b := \|\beta_x\|_\alpha =$ constant and the Ricci curvature of F is given by

$$\mathbf{Ric} = {}^\alpha\mathbf{Ric} + \epsilon^2\alpha^2\Gamma, \tag{3.2}$$

where

$$\begin{aligned}\Gamma := {} & (n-1)(b^2 - s^2)\big\{2s\Theta + 2b^2\Psi + (b^2 - s^2)\Theta^2 - (b^2 - s^2)\Theta' - 2s(b^2 - s^2)\Theta\Psi \\ & + 2(b^2 - s^2)^2\Theta'\Psi\big\} + (b^2 - s^2)^2\big\{4s\Psi' - (b^2 - s^2)\Psi'' - 6s(b^2 - s^2)\Psi\Psi' \\ & - (b^2 - s^2)^2\Psi'\Psi' + 2(b^2 - s^2)^2\Psi\Psi''\big\} \\ & - (b^2 - s^2)\Big\{2b^2\Psi - s((n-1)\Theta + (b^2 - s^2)\Psi')\big\}.\end{aligned}$$

Proof. Equation (3.1) is equivalent to the following equations

$$r_{ij} = \epsilon(b^2 a_{ij} - b_i b_j), \quad s_{ij} = 0. \tag{3.3}$$

Noticing $(b^2)_{|i} = 2b^j b_{j|i} = 2b^j r_{ji} = 0$, we have b is a constant. By Lemma 2.1, the spray coefficients of F can be rewritten as

$$G^i := {}^\alpha G^i + T^i,$$

where

$$T^i = \Theta r_{00} \frac{y^i}{\alpha} + \Psi r_{00} b^i, \quad r_{00} = \epsilon(b^2\alpha^2 - \beta^2).$$

It is well-known([3]) that the flag curvature tensor can be written as

$$R^i{}_k = {}^\alpha R^i{}_k + H^i{}_k,$$

where

$$H^i{}_k := 2T^i{}_{|k} - T^i{}_{|j\cdot k} y^j + 2T^j T^i{}_{\cdot j\cdot k} - T^i{}_{\cdot j} T^j{}_{\cdot k},$$

and "$\cdot$" means the vertical covariant derivative. Then

$$\mathbf{Ric} = {}^\alpha\mathbf{Ric} + H^i{}_i,$$

where

$$H^i{}_i := 2T^i{}_{|i} - T^i{}_{|j\cdot i} y^j + 2T^j T^i{}_{\cdot j\cdot i} - T^i{}_{\cdot j} T^j{}_{\cdot i}. \tag{3.4}$$

To compute the Ricci curvature, we need

$$r_{00|k} y^k = -3\epsilon^2 s\alpha^3(b^2 - s^2), \quad r_{00|k} b^k = -\epsilon^2 b^2 \alpha^2 (b^2 - s^2),$$

$$r_{00\cdot k} y^k = 2\epsilon\alpha^2(b^2 - s^2), \quad r_{00\cdot k} b^k = 0.$$

We can also easily get

$$s_{|k} y^k = \epsilon\alpha(b^2 - s^2), \quad s_{|k} b^k = 0, \quad s_{\cdot k} y^k = 0, \quad s_{\cdot k} b^k = \frac{1}{\alpha}(b^2 - s^2).$$

Using the above identities, we get

$$\begin{aligned}
T^i{}_{|i} &= (b^2 - s^2)\epsilon^2\alpha^2\big\{(b^2 - s^2)\Theta' - 3s\Theta + (n-2)b^2\Psi\big\},\\
T^i{}_{|j\cdot i} y^j &= (b^2 - s^2)\epsilon^2\alpha^2\big\{(n+1)(b^2 - s^2)\Theta' - 3(n+1)s\Theta\\
&\quad + (b^2 - s^2)^2\Psi'' - 5s(b^2 - s^2)\Psi'\big\},\\
T^j T^i{}_{\cdot j\cdot i} &= \epsilon^2\alpha^2(b^2 - s^2)^2\big\{(n+1)\Theta^2 - (n+1)s\Theta\Psi + (n+1)(b^2 - s^2)\Theta'\Psi\\
&\quad + (b^2 - s^2)\Theta\Psi' - 3s(b^2 - s^2)\Psi\Psi' + (b^2 - s^2)^2\Psi\Psi''\big\},\\
T^i{}_{\cdot j} T^j{}_{\cdot i} &= \epsilon^2\alpha^2(b^2 - s^2)^2\big\{(n+3)\Theta^2 + 2(b^2 - s^2)\Theta\Psi' + 4(b^2 - s^2)\Theta'\Psi\\
&\quad + (b^2 - s^2)^2\Psi'\Psi' - 4s\Theta\Psi\big\}.
\end{aligned}$$

Plugging them into (3.4) we obtain (3.2). □

We now prove Theorem 1.1. By assumption on ${}^{\alpha}\mathbf{Ric}$, we have

$$ {}^{\alpha}\mathbf{Ric} = -\kappa\epsilon^2(b^2\alpha^2 - \beta^2). $$

By (3.2), the Ricci curvature is given by

$$ \mathbf{Ric} = -\epsilon^2\alpha^2\kappa(b^2 - s^2) + \epsilon^2\alpha^2\Gamma, $$

Thus $\mathbf{Ric} = 0$ if and only if

$$ -\kappa(b^2 - s^2) + \Gamma = 0. \tag{3.5} $$

We can rewrite (3.5) as (1.3). □

References

[1] D. Bao and C. Robles, On Ricci curvature and flag curvature in Finsler geometry, In: "A Sampler of Finsler Geometry, MSRI series, *Cambridge University Press*, 2004.

[2] B. Li, Y. Shen and Z. Shen, On a class of Douglas metrics, *Studia Sci. Math. Hungar.* **46**(3) (2009), 355–365.

[3] X. Cheng, Z. Shen and Y. Tian, A class of Einstein (α, β)-metrics, *Israel J. Math.* **192**(1) (2012), 221–249.

[4] X. Cheng and Y. Tian, Ricci-flat Douglas (α, β)-metrics, *Differ. Geom. Appl.* **30**(1) (2012), 20–32.

[5] E. S. Sevim, Z. Shen and L. Zhao, On a class of Ricci-flat Douglas metrics, *Internat. J. Math.* **23**(6) (2012), DOI: 10.1142/S0129167X12500462.

ESRA SENGELEN SEVIM
DEPARTMENT OF MATHEMATICS
ISTANBUL BILGI UNIVERSITY
34440, KURTULUSDERESI
CAD. NO: 1 DOLAPDERE/BEYOGLU
ISTANBUL
TURKEY

E-mail: esra.sengelen@bilgi.edu.tr

ZHONGMIN SHEN
DEPARTMENT OF MATHEMATICAL
SCIENCES
INDIANA UNIVERSITY
PURDUE UNIVERSITY
INDIANAPOLIS
402 N. BLACKFORD STREET
INDIANAPOLIS, IN 46202-3216
USA

E-mail: zshen@math.iupui.edu

LILI ZHAO
DEPARTMENT OF MATHEMATICS
SHANGHAI JIAOTONG UNIVERSITY
DONG CHUAN ROAD 800
SHANGHAI 200240
P.R. CHINA

E-mail: zhaolili@sjtu.edu.cn

(Received July 27, 2012; revised January 13, 2013)