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Sensitivity analysis of the solution map of parametric operator equilibrium problems

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Abstract. The purpose of this paper is to study the parametric operator equilibrium problems. By using new definitions of vector topological pseudomonotonicity we give sufficient conditions for closedness of the solution map. The Hadamard well-posedness of the parametric operator equilibrium problems is also analyzed.

1. Introduction

Domokos and Kolumbán [9] introduced the technique of working with operator solutions instead of scalar or vector variables in field of variational inequalities. Inspired by their work, Kum and Kim [12], [13] developed the scheme of operator variational inequalities from the single-valued case into the multi-valued one. The operator equilibrium problems were studied by Kazmi and Raouf [10], Kum and Kim [14].

The equilibrium theory provides a unified, natural, innovative and general framework for the study of a large variety of problems such as optimization problems, fixed points problems, variational inequalities, Nash equilibria, saddle point problems and complementarity problems as special cases (see [3], [6], [15]). The problems mentioned above often occur in mechanics, physics, finance, economics, network analysis, transportation and elasticity.

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The behavior of the solutions resulted by the change of the problems data is always of major concern. Sensitivity analysis examines the way how the solutions of such problems change when the data of the problems are modified.

Kim, Kum and Lee introduced the parametric form of the generalized operator equilibrium problems in [11]. They analyzed the lower and upper semicontinuity of the solution map. Our aim is to study the closedness of the solution map of parametric operator equilibrium problems.

The paper is organized as follows. The notions of the vector topological pseudomonotonicity are introduced in Section 2. In Section 3 the closedness of the solution map for parametric operator equilibrium problems is proved. The theorems obtained extend the closedness results presented in [4] and [18]. In the final section we investigate the generalized Hadamard well-posedness of parametric operator equilibrium problems. We use similar technique as in [16].

2. Preliminaries

The problem under consideration is the following:

Let X and Y be Hausdorff topological vector spaces, $L(X, Y)$ be the space of all continuous linear operators from X to Y and let $D \subset L(X, Y)$ be a nonempty set. Let P, the set of parameters, be another Hausdorff topological space. Let $C : D \to 2^Y$ be a set-valued mapping such that for each $f \in D$, $C(f)$ is a convex open cone with nonempty interior and $C(f) \neq Y$.

For a given $p \in P$ the parametric operator equilibrium problem $(OEP)_p$ is to find $f_p \in D$ such that

$$F_p(f_p, g) \notin -C(f_p), \quad \forall g \in D,$$

where $F_p : D \times D \to Y$ is a given function.

Let us denote by $S(p)$ the set of the solutions for a fixed p. Suppose that $S(p) \neq \emptyset$, for all $p \in P$. For existence of solutions see [10].

We shall use the following notation. For any subset A of a topological space Y, we denote by A^c the complement of A in Y. Denote by $\overline{A}$ the closure of A, and ∂A the boundary of the set A.

In the following, we will introduce two new definitions of the vector topological pseudomonotonicity. First, the definition of the suprema and the infima of subsets of Y are given. Following [1], for a subset A of Y the suprema of A with respect to an ordering cone C^* is defined by

$$\operatorname{Sup} A = \left\{ y \in \bar{A} : A \cap (y + \operatorname{Int} C^*) = \emptyset \right\}$$

and the infima of A with respect to C^* is defined by

$$\operatorname{Inf} A = \left\{y \in \bar{A} : A \cap (y - \operatorname{Int} C^*) = \emptyset\right\}.$$

For more details see [7].

Let $(y_i)_{i\in I}$ be a net in Y. Let $A_i = \{y_j : j \geq i\}$ for every i in the index set I. The limit inferior and the limit superior of the net (y_i), respectively, are given by

$$\operatorname{Liminf} y_i = \operatorname{Sup}\left(\bigcup_{i\in I} \operatorname{Inf} A_i\right) \qquad \text{and} \qquad \operatorname{Limsup} y_i = \operatorname{Inf}\left(\bigcup_{i\in I} \operatorname{Sup} A_i\right).$$

Theorem 2.1 ([8] Theorem 2.1)**.** *Let $(y_i)_{i\in I}$ be a net in Y convergent to y, and let $A_i = \{y_j : j \geq i\}$.*

i) *If there is an index i_0 such that, for every $i \geq i_0$, there exists $j \geq i$ with $\operatorname{Inf} A_j \neq \emptyset$, then $y \in \operatorname{Liminf} y_i$.*

ii) *If there is an index i_0 such that, for every $i \geq i_0$, there exists $j \geq i$ with $\operatorname{Sup} A_j \neq \emptyset$, then $y \in \operatorname{Limsup} y_i$.*

Now, we can introduce the new definitions of vector topological pseudomonotonicity which generalize the vector topological pseudomonotonicity notions given by Definition 2.1 in [17] and Definition 3 in [18] respectively.

Let us denote by

$$K := \bigcap_{f\in D} C(f).$$

In what follows, $\operatorname{Int} K$ is assumed to be nonempty.

Definition 2.1. A mapping $F : D \times D \to Y$ is called A-vector topologically pseudomonotone if for every $g \in D$, $v \in C(f)$ and for each net $(f_i)_{i\in I}$ in D satisfying $f_i \to f \in D$ with respect to the topology of pointwise convergence and

$$\operatorname{Liminf} F(f_i, f) \cap (-\operatorname{Int} K) = \emptyset,$$

then there is i_0 in the index set I such that

$$\overline{\{F(f_j, g) : j \geq i\}} \subset F(f, g) + v - C(f_i),$$

for all $i \geq i_0$.

Definition 2.2. A mapping $F : D \times D \to Y$ is called B-vector topologically pseudomonotone if for every $g \in D$, $v \in C(f)$ and for each net $(f_i)_{i\in I}$ in D satisfying $f_i \to f \in D$ with respect to the topology of pointwise convergence and

$$\operatorname{Liminf} F(f_i, f) = \emptyset \quad \text{or} \quad \operatorname{Liminf} F(f_i, f) \cap (-\operatorname{Int} K)^c \neq \emptyset$$

then there is i_0 in the index set I such that

$$\overline{\{F(f_j, g) : j \geq i\}} \subset F(f, g) + v - C(f_i),$$

for all $i \geq i_0$.

Remark 2.1. Every B-vector topologically pseudomonotone function is A-vector topologically pseudomonotone.

The inverse relation is not necessarily true.

Example 2.2. Let $X = [0,1]$, $Y = \mathbb{R}^2$ and

$$D = \{u_c : X \to Y : u_c(x) = c(x, x), \forall x \in X \text{ where } c \in X\}.$$

The function $F : D \times D \to Y$ is given by

$$F(u_a, u_b) = f(a, b),$$

where $f : X \times X \to Y$ is defined by

$$f(a,b) = \begin{cases} (a-b, 1-a) & \text{if } a > 0 \\ (b, 1) & \text{if } a = 0. \end{cases}$$

Let $C : D \to 2^Y$ be given by $C(u_c) = C^*$ for every $u_c \in D$, where the cone C^* of $\mathbb{R}^2$ is the third quadrant, i.e.

$$C^* = \left\{(a, b) \in \mathbb{R}^2 : a \leq 0,\ b \leq 0\right\}.$$

The function f is A-vector topologically pseudomonotone, but it is not B-vector topologically pseudomonotone. For proof see [18] Example 7.

The generalization of topological pseudomonotonicity introduced by Brézis [5] was given by Kazmi and Raouf.

Definition 2.3. Let $D \subset L(X, Y)$ be a convex nonempty set. The mapping $F : D \times D \to Y$ is said to be $B - C(f)$-pseudomonotone, if for each net $(f_i)_{i \in I} \subset D$ and $f, g \in D$ such that $f_i \to f$ with respect to the topology of pointwise convergence (w.r.t.p.c. for short) and

$$F(f_i, (1-\lambda)f + \lambda g) \notin -C(f_i), \quad \forall \lambda \in [0,1], \forall f_i,$$

we have

$$F(f, g) \notin -C(f).$$

The relation between A-vector topological pseudomonotonicity and $B{-}C\,(f)$-pseudomonotonicity is given by the following proposition.

Proposition 2.3. *Let X and Y be Hausdorff topological vector spaces, and let $D \subset L(X,Y)$ be a convex nonempty set. Assume that $F : D \times D \to Y$ is A-vector topologically pseudomonotone, then F is $B - C(f)$-pseudomonotone.*

Proof. Let $(f_i)_{i\in I}$ be a net in D such that $f_i \to f$ with respect to the topology of pointwise convergence and $f \in D$. Assume that

$$F\,(f_i, (1-\lambda)f + \lambda g) \notin -C(f_i), \forall \lambda \in [0,1], \quad \forall f_i, \tag{1}$$

where $g \in D$.

If $\lambda = 0$ then we have

$$F\,(f_i, f) \notin -C(f_i), \quad \forall f_i \in I,$$

from where it follows that

$$F\,(f_i, f) \notin - \bigcap_{i\in I} C(f_i), \quad \forall i \in I,$$

therefore

$$F\,(f_i, f) \notin -\operatorname{Int} K.$$

Since $(-\operatorname{Int} K)^c$ is a closed cone, we obtain that

$$\operatorname{Liminf} F\,(f_i, f) \cap (-\operatorname{Int} K) = \emptyset. \tag{2}$$

We have to prove that

$$F(f,g) \notin -C(f).$$

Let suppose the contrary, that $F(f,g) = -v$ where $v \in C\,(f)$. Since (2) holds, from the definition of A-vector topological pseudomonotonicity of the mapping F we obtain that there is i_0 in the index set I such that

$$\overline{\{F\,(f_j, g) : j \geq i\}} \subset F\,(f,g) + v - C\,(f_i) = -C\,(f_i)$$

for all $i \geq i_0$. This is a contradiction with

$$F\,(f_i, g) \notin -C(f_i), \forall f_i,$$

relation obtained by taking $\lambda = 1$ in (1). Hence $F(f,g) \notin -C(f)$. □

3. Closedness of the solution map

In this section the closedness of the solution mapping for parametric operator equilibrium problems is analyzed.

First, we prove the following proposition.

Proposition 3.1. *Let Y be a Hausdorff topological vector space and the cones C_i be convex open and not equal with the space Y, for every index i in the index set I. Let $K := \bigcap_{i\in I} C_i$, such that $\operatorname{Int} K \neq \emptyset$. Then for all $f, g_i, f_i \in Y$, we have*

i) *if $f - g \in -\operatorname{Int} K$ and $f \notin -\operatorname{Int} K$ implies $g \notin -\operatorname{Int} K$;*

ii) *if $f_i - g_i \in -\operatorname{Int} K$ and $g_i \in -C_i$ for every $i \in I$ implies $f_i \in -C_i, \forall i \in I$.*

Proof. i) Assume the contrary, that $g \in -\operatorname{Int} K$. From $f - g \in -\operatorname{Int} K$ it follows that $f \in g - \operatorname{Int} K \subset -\operatorname{Int} K$ contradicting $f \notin -\operatorname{Int} K$.

ii) Since $f_i - g_i \in -\operatorname{Int} K$ we have that $f_i \in g_i - \operatorname{Int} K$. From the definition of cone K we obtain that $f_i \in g_i - \operatorname{Int} K \subset -C_i - C_i = -C_i, \forall i \in I$. □

Next, we prove the following closedness result for $(OEP)_p$.

Theorem 3.2. *Let X and Y be Hausdorff topological vector spaces, and let $D \subset L(X, Y)$ be a nonempty set and $p_0 \in P$ be fixed. Suppose that $S(p) \neq \emptyset$ for each $p \in P$ and the following conditions hold:*

i) *For each net of elements $(p_i, f_{p_i}) \in \operatorname{Graph} S$, if $p_i \to p_0$, $f_{p_i} \to f$ and for every $g \in D$ then*

$$\operatorname{Liminf}\left(F_{p_i}\left(f_{p_i}, g\right) - F_{p_0}\left(f_{p_i}, g\right)\right) \cap (-\operatorname{Int} K) \neq \emptyset;$$

ii) *$F_{p_0} : D \times D \to Y$ is A-vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ is closed at p_0, i.e. for each net of elements $(p_i, f_{p_i}) \in \operatorname{Graph} S$, $p_i \to p_0$ and $f_{p_i} \to f$ imply $(p_0, f) \in \operatorname{Graph} S$.

Proof. Let $(p_i, f_{p_i})_{i\in I}$ be a net of elements $(p_i, f_{p_i}) \in \operatorname{Graph} S$, i.e.

$$F_{p_i}\left(f_{p_i}, g\right) \notin -C(f_{p_i}), \quad \forall g \in D \tag{3}$$

such that $f_{p_i} \to f$ when $p_i \to p_0$. By taking $g = f$, from the assumption i) we obtain that

$$\operatorname{Liminf}\left(F_{p_i}\left(f_{p_i}, f\right) - F_{p_0}\left(f_{p_i}, f\right)\right) \cap (-\operatorname{Int} K) \neq \emptyset.$$

Since $-\operatorname{Int} K$ is an open cone, it follows that there exists a subnet (f_{p_i}), denoted by the same indexes, such that

$$F_{p_i}(f_{p_i}, f) - F_{p_0}(f_{p_i}, f) \in -\operatorname{Int} K \quad \text{for all } i \in I. \tag{4}$$

By replacing g with f in (3) we get

$$F_{p_i}(f_{p_i}, f) \notin -C(f_{p_i}) \quad \text{for all } i \in I,$$

therefore

$$F_{p_i}(f_{p_i}, f) \notin -\operatorname{Int} K \quad \text{for all } i \in I. \tag{5}$$

By using Proposition 3.1 i), from (4) and (5) we obtain that

$$F_{p_0}(f_{p_i}, f) \notin -\operatorname{Int} K, \quad \text{for all } i \in I.$$

Since $(-\operatorname{Int} K)^c$ is closed, it follows

$$\operatorname{Liminf} F_{p_0}(f_{p_i}, f) \cap (-\operatorname{Int} K) = \emptyset.$$

Now, we can apply ii) and we obtain that for every $g \in D$, $v \in C(f)$, there exists $i_0 \in I$ such that

$$\overline{\left\{F_{p_0}\left(f_{p_j}, g\right) : j \geq i\right\}} \subset F_{p_0}(f, g) + v - C(f_{p_i}), \quad \forall i \geq i_0. \tag{6}$$

We have to prove that

$$F_{p_0}(f, g) \notin -C(f), \quad \forall g \in D.$$

Assume the contrary, that there exists $\overline{g} \in D$ such that

$$F_{p_0}(f, \overline{g}) \in -C(f).$$

Let be $F_{p_0}(f, \overline{g}) = -v$ where $v \in C(f)$. From (6) we obtain that there exists $i_0 \in I$ such that

$$\overline{\left\{F_{p_0}\left(f_{p_j}, \overline{g}\right) : j \geq i\right\}} \subset -v + v - C(f_{p_i}) = -C(f_{p_i}), \quad \forall i \geq i_0. \tag{7}$$

By using again the assumption i), it follows that there exists a subnet (f_{p_i}), denoted by the same indexes, for which

$$F_{p_i}(f_{p_i}, \overline{g}) - F_{p_0}(f_{p_i}, \overline{g}) \in -\operatorname{Int} K, \quad \text{for all } i \in I. \tag{8}$$

By using Proposition 3.1 ii), from (7) and (8) it follows that

$$F_{p_i}(f_{p_i}, \overline{g}) \in -C(f_{p_i}), \quad \forall i \geq i_0,$$

contradicting (3). Hence $(p_0, f) \in \operatorname{Graph} S$. □

Remark 3.1. The condition i) can not be replaced by

i') For each net of elements $(p_i, f_{p_i}) \in \operatorname{Graph} S$, if $p_i \to p_0$, $f_{p_i} \to f$ and for every $g \in D$ then

$$\operatorname{Liminf}\left(F_{p_i}\left(f_{p_i}, g\right) - F_{p_0}\left(f_{p_i}, g\right)\right) \cap \left(-\overline{K}\right) \neq \emptyset.$$

The next example shores up this statement.

Example 3.3. Let $P = \mathbb{N} \cup \{\infty\}$, $p_\infty = \infty$. On P we consider the topology induced by the mertic d given by $d(m,n) = |1/m - 1/n|$, $d(n,\infty) = d(\infty,n) = 1/n$, for $m,n \in \mathbb{N}$, and $d(\infty,\infty) = 0$. Let X, Y, D, C and F_∞ be the same as F in Example 2.2. Let the vector functions $F_n : D \times D \to Y$ be given by $F_n(u_a, u_b) = u_a(1) - u_b(1) - (1/n, -2)$, $n \in \mathbb{N}$.

The function F_∞ is A-vector topologically pseudomonotone. From Theorem 2.1 it follows that

$$(0, 1 + 2a - b) \in \operatorname{Liminf}\left(F_n\left(u_{a_n}, u_b\right) - F_\infty\left(u_{a_n}, u_b\right)\right),$$

when $a_n \to a$ and $b \in X$. Since

$$(0, 1 + 2a - b) \in -\overline{K}$$

the assumption i') applies. We have $(n, 1/n) \in \operatorname{Graph} S$ for each $n \in \mathbb{N}$, but $0 \notin S(\infty)$. Hence S is not closed at ∞.

If the subset D is convex then we can replace the A-vector topological pseudomonotonicity with $B - C(f)$-pseudomonotonicity.

Theorem 3.4. *Let X and Y be Hausdorff topological vector spaces, and let $D \subset L(X,Y)$ be a convex nonempty set and $p_0 \in P$ be fixed. Suppose that $S(p) \neq \emptyset$ for each $p \in P$ and the following conditions hold:*

i) *For each net of elements $(p_i, f_{p_i}) \in \operatorname{Graph} S$, if $p_i \to p_0$, $f_{p_i} \to f$ and $g \in D$ then*

$$\operatorname{Liminf}\left(F_{p_i}\left(f_{p_i}, g\right) - F_{p_0}\left(f_{p_i}, g\right)\right) \cap \left(-\operatorname{Int} K\right) \neq \emptyset;$$

ii) *$F_{p_0} : D \times D \to Y$ is $B - C(f)$-pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ is closed at p_0.

Proof. Let $(p_i, f_{p_i})_{i \in I}$ be a net of elements $(p_i, f_{p_i}) \in \operatorname{Graph} S$, i.e.

$$F_{p_i}\left(f_{p_i}, g\right) \notin -C(f_{p_i}), \quad \forall g \in D \tag{9}$$

such that $f_{p_i} \to f$ when $p_i \to p_0$. By taking $g = (1-\lambda)f + \lambda g$, from the assumption i) we obtain that

$$\operatorname{Liminf}\left(F_{p_i}\left(f_{p_i}, (1-\lambda)f + \lambda g\right) - F_{p_0}\left(f_{p_i}, (1-\lambda)f + \lambda g\right)\right) \cap (-\operatorname{Int} K) \neq \emptyset$$

for all $\lambda \in [0,1]$.

Since $-\operatorname{Int} K$ is an open cone, it follows that there exists a subnet (f_{p_i}), denoted by the same indexes, such that

$$\begin{aligned} &F_{p_i}\left(f_{p_i}, (1-\lambda)f + \lambda g\right) \\ &\qquad - F_{p_0}\left(f_{p_i}, (1-\lambda)f + \lambda g\right) \in -\operatorname{Int} K \quad \forall \lambda \in [0,1],\ \forall i \in I. \end{aligned} \tag{10}$$

By replacing g with $(1-\lambda)f + \lambda g$ in (9) we get

$$F_{p_i}\left(f_{p_i}, (1-\lambda)f + \lambda g\right) \notin -C(f_{p_i}) \quad \forall \lambda \in [0,1],\ \forall i \in I. \tag{11}$$

From (11) and (10) we obtain that

$$F_{p_0}\left(f_{p_i}, (1-\lambda)f + \lambda g\right) \notin -C(f_{p_i}), \quad \forall \lambda \in [0,1],\ \forall i \in I.$$

Since F_{p_0} is $B - C(f)$-pseudomonotone we obtain that

$$F_{p_0}(f, g) \notin -C(f), \quad \forall g \in D.$$

Hence $(p_0, f) \in \operatorname{Graph} S$. □

If we replace the assumption i) in Theorem 3.2 with a weaker condition, we have to give a stronger term to assumption ii).

Theorem 3.5. *Let X and Y be Hausdorff topological vector spaces, and let $D \subset L(X,Y)$ be a nonempty set and $p_0 \in P$ be fixed. Suppose that $S(p) \neq \emptyset$ for each $p \in P$ and the following conditions hold:*

i) *For each net of elements $(p_i, f_{p_i}) \in \operatorname{Graph} S$, if $p_i \to p_0$, $f_{p_i} \to f$ and for every $g \in D$ then*

$$\operatorname{Liminf}\left(F_{p_i}(f_{p_i}, g) - F_{p_0}(f_{p_i}, g)\right) \cap \left(-\overline{K}\right) \neq \emptyset;$$

ii) *$F_{p_0} : D \times D \to Y$ is B-vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ is closed at p_0.

Proof. Let $(p_i, f_{p_i})_{i\in I}$ be a net of elements $(p_i, f_{p_i}) \in \text{Graph } S$, i.e.

$$F_{p_i}(f_{p_i}, g) \notin -C(f_{p_i}), \quad \forall g \in D$$

such that $f_{p_i} \to f$ when $p_i \to p_0$. From where it follows that

$$F_{p_i}(f_{p_i}, g) \notin -K. \tag{12}$$

By using assumption i) we obtain that

$$\text{Liminf}\left(F_{p_i}(f_{p_i}, g) - F_{p_0}(f_{p_i}, g)\right) \cap \left(-\overline{K}\right) \neq \emptyset, \quad \forall g \in D. \tag{13}$$

In what follows we will prove that (12) and (13) imply

$$\text{Liminf } F_{p_0}(f_{p_i}, f) = \emptyset \text{ or } \text{Liminf } F_{p_0}(f_{p_i}, f) \cap (-\operatorname{Int} K)^c \neq \emptyset.$$

We distinguish two cases:

Case 1. If $\text{Liminf}\left(F_{p_i}(f_{p_i}, g) - F_{p_0}(f_{p_i}, g)\right) \cap (-\operatorname{Int} K) \neq \emptyset$ in assumption i), similar as in the proof of Theorem 3.2 we obtain that

$$\text{Liminf } F_{p_0}(f_{p_i}, f) \cap (-\operatorname{Int} K) = \emptyset.$$

Indeed, since $-\operatorname{Int} K$ is an open cone, it follows that there exists a subnet, denoted by the same indexes, such that

$$F_{p_i}(f_{p_i}, g) - F_{p_0}(f_{p_i}, g) \in -\operatorname{Int} K, \quad \text{for all } i \in I. \tag{14}$$

By using Proposition 3.1 i), from (12) and (14) we obtain that

$$F_{p_0}(f_{p_i}, g) \notin -\operatorname{Int} K, \quad \text{for all } i \in I.$$

Since $(-\operatorname{Int} K)^c$ is closed, it follows

$$\text{Liminf } F_{p_0}(f_{p_i}, f) \cap (-\operatorname{Int} K) = \emptyset,$$

consequently

$$\text{Liminf } F_{p_0}(f_{p_i}, f) = \emptyset \quad \text{or} \quad \text{Liminf } F_{p_0}(f_{p_i}, f) \cap (-\operatorname{Int} K)^c \neq \emptyset.$$

Case 2. If $\text{Liminf}\left(F_{p_i}(f_{p_i}, g) - F_{p_0}(f_{p_i}, g)\right) \cap (-\operatorname{Int} K) = \emptyset$ in assumption i), then the single interesting case is when

$$F_{p_i}(f_{p_i}, g) - F_{p_0}(f_{p_i}, g) \in (-\operatorname{Int} K)^c, \quad \forall i \in I \tag{15}$$

and

$$F_{p_0}(f_{p_i}, g) \in -\operatorname{Int} K, \quad \forall i \in I. \tag{16}$$

Since $\operatorname{Liminf}(F_{p_i}(f_{p_i}, g) - F_{p_0}(f_{p_i}, g)) \cap (-\operatorname{Int} K) = \emptyset$, from (13) and (15) it follows that, there exists a subnet (f_{p_i}) denoted by the same indexes for which

$$(F_{p_i}(f_{p_i}, g) - F_{p_0}(f_{p_i}, g))_{i \in I} \text{ converges to } -\partial\overline{K}. \tag{17}$$

Indeed, otherwise it must exist $i_0 \in I$ such that

$$\overline{\{F_{p_i}(f_{p_i}, g) - F_{p_0}(f_{p_i}, g) : i \geq i_0\}} \subset \left(-\overline{K}\right)^c,$$

then from the definition of the limit inferior, we obtain that

$$\operatorname{Liminf}(F_{p_i}(f_{p_i}, g) - F_{p_0}(f_{p_i}, g)) \subset \left(-\overline{K}\right)^c,$$

which is in contradiction with assumption i).

From (16) and (17) we obtain that there exists a subnet, denoted by the same indexes, such that

$$(F_{p_0}(f_{p_i}, g))_{i \in I} \text{ converges to an element on } -\partial\overline{K}. \tag{18}$$

To prove this statement, let us suppose the contrary, that

$$\overline{\{F_{p_0}(f_{p_i}, g) : i \in I\}} \subset -\operatorname{Int} K.$$

Then by the convexity of the cone $-K$ and (17) we obtain that

$$F_{p_i}(f_{p_i}, g) \text{ converges to an element in } -\operatorname{Int} K,$$

from where it follows that there exists $i_1 \in I$ such that

$$F_{p_i}(f_{p_i}, g) \in -\operatorname{Int} K, \quad \text{for all } i \geq i_1$$

contradicting (12).

By applying Theorem 2.1 to the subnet in (18) we obtain that

$$\operatorname{Liminf} F_{p_0}(f_{p_i}, g) \cap \left(-\partial\overline{K}\right) \neq \emptyset,$$

or there exists $i_2 \in I$ such that

$$\operatorname{Inf}\{F_{p_0}(f_{p_i}, g) : i \geq i_2\} = \emptyset.$$

This implies that

$$\operatorname{Liminf} F_{p_0}(f_{p_i}, g) \cap (-\operatorname{Int} K)^c \neq \emptyset \quad \text{or} \quad \operatorname{Liminf} F_{p_0}(f_{p_i}, g) = \emptyset.$$

So, in both cases, we can apply ii) and we obtain that for every $g \in D$, $v \in C(f)$, there exists $j_0 \in I$ such that

$$\overline{\{F_{p_0}(f_{p_i}, g) : i \geq j\}} \subset F_{p_0}(f, g) + v - C(f_{p_i}), \quad \forall j \geq j_0. \tag{19}$$

We have to prove that

$$F_{p_0}(f, g) \notin -C(f), \quad \forall g \in D.$$

Assume the contrary, that there exists $\overline{g} \in D$ such that

$$F_{p_0}(f, \overline{g}) \in -C(f).$$

Let be $F_{p_0}(f, \overline{g}) = -v$ where $v \in C(f)$. From (19) we obtain that there exists $j_0 \in I$ such that

$$\overline{\{F_{p_0}(f_{p_i}, \overline{g}) : i \geq j\}} \subset -v + v - C(f_{p_i}) = -C(f_{p_i}), \quad \forall j \geq j_0. \tag{20}$$

By using again the assumption i), it follows that one of the next cases, corresponding to (14) and (17) respectively, holds:
there exists a subnet (f_{p_i}) denoted by the same indexes such that

$$F_{p_i}(f_{p_i}, \overline{g}) - F_{p_0}(f_{p_i}, \overline{g}) \in -\operatorname{Int} K, \quad \forall i \in I \tag{21}$$

or there exists a subnet (f_{p_i}) denoted by the same indexes for which

$$(F_{p_i}(f_{p_i}, \overline{g}) - F_{p_0}(f_{p_i}, \overline{g}))_{i \in I} \text{ converges to an element in } -\partial\overline{K}. \tag{22}$$

By applying Proposition 3.1 ii), from (20) and (21) it follows that there exists $j_1 \in I$ such that

$$F_{p_i}(f_{p_i}, \overline{g}) \in -C(f_{p_i}), \quad i \geq j_1 \geq j_0.$$

From (22) and the definition of the cone K, we obtain that for every $i \in I$

$$(F_{p_i}(f_{p_i}, \overline{g}) - F_{p_0}(f_{p_i}, \overline{g}))_{i \in I} \text{ converges to a point in } -\overline{C(f_{p_i})}. \tag{23}$$

Since $C(f_{p_i})$ is open convex cone for every $i \in I$ and the conditions (20) and (23) hold we get

$$F_{p_i}(f_{p_i}, \overline{g}) \text{ converges to a point in } -C(f_{p_i}).$$

Hence there exists $j_2 \in I$ such that

$$F_{p_i}(f_{p_i}, \overline{g}) \in -C(f_{p_i}), \quad i \geq j_2 \geq j_0.$$

But on the other side $(p_i, f_{p_i}) \in \operatorname{Graph} S$, and

$$F_{p_i}(f_{p_i}, \overline{g}) \notin -C(f_{p_i})$$

which is a contradiction. Hence $(p_0, f) \in \operatorname{Graph} S$. □

Remark 3.2. The Theorem 3.5 extends the Theorem 15 of [18] given for parametric vector equilibrium problems and Theorem 2 of [4] stated for parametric equilibrium problems.

The following examples show how the assumptions of the theorems can be verified.

Example 3.6. Let X, Y, D, P and C be the same as in Example 3.3. Let the vector functions $F_n : D \times D \to Y$ be given by $F_n(u_a, u_b) = u_a(1) - u_b(1) + (3, -1/n)$, $n \in \mathbb{N}$, and the function F_∞ be defined by $F_\infty(u_a, u_b) = u_a(1) - 2u_b(1) + (2, -2)$.

The function F_∞ is A-vector topologically pseudomonotone since it is continuous. Only the assumption i) of Theorem 3.2 has to be verified. Let $a_n, b, a \in [0, 1]$, and $a_n \to a$. From Theorem 2.1 it follows that

$$(1 + b, 2 + b) \in \operatorname{Liminf}(F_n(u_{a_n}, u_b) - F_\infty(u_{a_n}, u_b)).$$

Since $S(n) = [0, 1/n] \neq \emptyset$ for every $n \in N$ and $(1 + b, 2 + b) \in -\operatorname{Int} K$ we obtain that the assumption i) holds. By applying Theorem 3.2 it follows that the function S is closed at ∞.

Example 3.7. Let X, Y, D, P and C be the same as in Example 3.3. Let the functions $F_n : D \times D \to Y$ be given by $F_n(u_a, u_b) = u_a(1) - u_b(1) + (-1/n, 2)$, $n \in \mathbb{N}$ and the function F_∞ be defined by

$$F_\infty(u_a, u_b) = \begin{cases} u_a(1) - u_b(1) + (0, 1) & \text{if } a > 0 \\ -u_b(1) & \text{if } a = 0. \end{cases}$$

The function F_∞ is B-vector topologically pseudomonotone. Indeed, if $a > 0$, the function F_∞ is continuous therefore it is B-vector topologically pseudomonotone. Let us study the case when $a = 0$.

If $a_n \neq 0$, $n \in \mathbb{N}$ we have

$$\operatorname{Liminf} F_\infty\left(u_{a_n}, u_0\right) = \operatorname{Liminf}\left\{(a_n, 1+a_n) : a_n \in \left]0,1\right] \text{ and } a_n \to 0\right\},$$

from Theorem 2.1 it follows that

$$(0,1) \in \operatorname{Liminf} F_\infty\left(u_{a_n}, u_0\right),$$

therefore

$$\operatorname{Liminf} F_\infty\left(u_{a_n}, u_a\right) \cap \left(-\operatorname{Int} K\right)^c \neq \emptyset.$$

We have to prove that there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\overline{\left\{F_\infty\left(u_{a_n}, u_b\right) : n \geq n_0\right\}} \subset F_\infty\left(u_0, u_b\right) + v - K \\ &\qquad \Leftrightarrow \overline{\left\{(a_n - b, 1 + a_n - b) : n \geq n_0\right\}} \subset (-b, -b) + v - K, \end{aligned}$$

where $v \in K$. This is true, since

$$\begin{cases} a_n - b \geq -b \\ 1 + a_n - b \geq -b \end{cases} \qquad \text{for all } a_n, b \in \left]0,1\right].$$

If $a_n = 0$, $n \in N$, then $F_\infty(u_{a_n}, u_0) = F_\infty(u_0, u_0) = (0,0) \in (-\operatorname{Int} K)^c$. Since $F_\infty\left(u_0, u_b\right) \subset F_\infty\left(u_0, u_b\right) + v - K$ we obtain that the function F_∞ is B-vector topologically pseudomonotone.

Now we verify the assumption i). Let $a_n, b \in [0,1]$ and $a_n \to a$. One has

$$\operatorname{Liminf}\left(F_n\left(u_{a_n}, u_b\right) - F_\infty\left(u_{a_n}, u_b\right)\right) \cap \left(-\overline{K}\right) \neq \emptyset$$

that is

$$\begin{aligned} &\operatorname{Liminf}\left\{\left(-1/n, 1\right), n \geq 1\right\} \cap \left(-\overline{K}\right) \neq \emptyset, \quad \text{when } a_n \neq 0; \\ &\operatorname{Liminf}\left\{\left(-1/n, 2\right), n \geq 1\right\} \cap \left(-\overline{K}\right) \neq \emptyset, \quad \text{when } a_n = 0 \end{aligned}$$

which is true, since from Theorem 2.1 we have

$$\begin{aligned} &(0,1) \in \operatorname{Liminf}\left(F_n\left(u_{a_n}, u_b\right) - F_\infty\left(u_{a_n}, u_b\right)\right); \\ &(0,2) \in \operatorname{Liminf}\left(F_n\left(u_{a_n}, u_b\right) - F_\infty\left(u_{a_n}, u_b\right)\right). \end{aligned}$$

By applying Theorem 3.5 it follows that the function S is closed at ∞.

4. Hadamard well-posedness

By using the relations between the notions of closedness, upper semi-continuity and Hadamard well-posedness, we obtain the Hadamard well-posedness of the parametric operator equilibrium problems.

Let X, Y be topological spaces. The map $T : X \to 2^Y$ is said to be *upper semi-continuous* at $u_0 \in domT := \{u \in X | T(u) \neq \emptyset\}$ if for each neighborhood V of $T(u_0)$, there exists a neighborhood U of u_0 such that $T(U) \subset V$.

Closedness and upper semi-continuity of a multifunction are closely related.

Proposition 4.1 ([2] Proposition 1.4.8, 1.4.9)**.** *Let X, Y be Hausdorff topological spaces.*

i) *If $T : X \to 2^Y$ has closed values and is upper semi-continuous then T is closed;*
ii) *If Y is compact and T is closed at $x \in X$ then T is upper semi-continuous at $x \in X$.*

Now we recall the notion of generalized Hadamard well-posedness.

Definition 4.1. The problem $(OEP)_p$ is said to be Hadamard well-posed (briefly H-wp) at $p_0 \in P$ if $S(p_0) = \{f_{p_0}\}$ and for any $f_p \in S(p)$ one has $f_p \to f_{p_0}$, as $p \to p_0$. The problem $(OEP)_p$ is said to be generalized Hadamard well-posed (briefly gH-wp) at $p_0 \in P$ if $S(p_0) \neq \emptyset$ and for any $f_p \in S(p)$, if $p \to p_0$, (f_p) must have a subsequence converging to an element of $S(p_0)$.

With the help of the next result we are able to establish the relationship between upper semi-continuity and Hadamard well-posedness.

Proposition 4.2 ([19] Theorem 2.2)**.** *Let X and Y be Hausdorff topological spaces and $T : X \to 2^Y$ be a set valued map. If T is upper semi-continuous at $x \in X$ and $T(x)$ is compact, then T is gH-wp at x. If more, $T(x) = \{y^*\}$, then T is H-wp at x.*

In the following we prove that the solution map of $(OEP)_p$ has closed value at p_0.

Proposition 4.3. *If D is closed with respect to pointwise convergence (for short w.r.t.p.c) and $F_{p_0} : D \times D \to Y$ is A-vector topologically pseudomonotone, then $S(p_0)$ is closed.*

Proof. Let $S(p_0) \neq \emptyset$ and $f_i \in S(p_0)$, with $f_i \to f$. Since D is closed w.r.t.p.c, we have that $f \in D$. From $f_i \in S(p_0)$ it follows that

$$F_{p_0}(f_i, f) \notin -C(f_i), \quad \forall i \in I,$$

therefore

$$F_{p_0}(f_i, f) \notin -\operatorname{Int} K, \quad \forall i \in I.$$

Since $(-\operatorname{Int} K)^c$ is closed, we get

$$\operatorname{Liminf} F_{p_0}(f_i, f) \cap -\operatorname{Int} K = \emptyset.$$

By using the A-vector topological pseudomonotonicity we obtain that for every $g \in D$ and $v \in C(f)$ there is i_0 in the index set I such that

$$\overline{\{F_{p_0}(f_j, g) : j \geq i\}} \subset F_{p_0}(f, g) + v - C(f_i), \quad \text{for all } i \geq i_0. \tag{24}$$

We have to prove that $f \in S(p_0)$, i.e.

$$F_{p_0}(f, g) \notin -C(f), \quad \forall g \in D.$$

Assume the contrary, that there exists $\overline{g} \in D$ such that

$$F_{p_0}(f, \overline{g}) \in -C(f).$$

Let $F_{p_0}(f, \overline{g}) = -v$ where $v \in C(f)$. From (24) we obtain that

$$\overline{\{F_{p_0}(f_j, \overline{g}) : j \geq i\}} \subset -v + v - C(f_i) = -C(f_i), \quad \forall i \geq i_0$$

which is a contradiction to $f_i \in S(p_0)$. Thus $f \in S(p_0)$. □

It can be easily verified the next proposition.

Proposition 4.4. *If D is convex, closed and $F_{p_0} : D \times D \to Y$ is $B - C(f)$-pseudomonotone, then $S(p_0)$ is closed.*

Now we can formulate the following results.

Corollary 4.5. *Let X and Y be Hausdorff topological vector spaces, and P be a Hausdorff topological space. Let D be a compact subset of $L(X, Y)$. If the hypotheses of Theorem 3.2 are satisfied, then $(OEP)_p$ is generalized Hadamard well-posed at p_0. Furthermore, if $S(p_0) = \{f_{p_0}\}$ (singleton), then $(OEP)_p$ is Hadamard well-posed at p_0.*

Proof. From Theorem 3.2 we obtain that the solution map S is closed at p_0. By using Proposition 4.1 ii) it follows that S is upper semi-continuous at p_0. The set $S(p_0)$ is closed by Proposition 4.3, hence it is compact. The conclusion follows from Proposition 4.2. □

Similarly as Corollary 4.5 we can prove the following result.

Corollary 4.6. *Let X and Y be Hausdorff topological vector spaces, and P be a Hausdorff topological space. Let D be a convex, compact subset of $L(X,Y)$. If the hypotheses of Theorem 3.4 are satisfied, then $(OEP)_p$ is gH-wp at p_0. Furthermore, if $S(p_0) = \{a_{p_0}\}$, then $(OEP)_p$ is H-wp at p_0.*

The following corollary is an immediate consequence of Remark 2.1 and Corollary 4.5.

Corollary 4.7. *Let X and Y be Hausdorff topological vector spaces, and P be a Hausdorff topological space. Let D be a compact subset of $L(X,Y)$. If the hypotheses of Theorem 3.5 are satisfied, then $(OEP)_p$ is gH-wp at p_0. Furthermore, if $S(p_0) = \{f_{p_0}\}$, then $(OEP)_p$ is H-wp at p_0.*

Example 4.8. Since D in Example 3.6 and Example 3.7 is compact, from Corollary 4.5 and Corollary 4.7 it follows that both problems are Hadamard well-posed.

5. Conclusions

In this paper we introduced two new definitions of vector topological pseudomonotonicity. We gave the relations between these notions and the already existing $B - C(f)$ pseudomonotonicity. If the domain of the map is a convex, nonempty set, then every B-vector topologically pseudomonotone map is A-vector topologically pseudomonotone, and every A-vector topologically pseudomonotone map is $B - C(f)$ pseudomonotone.

The main result of this paper gives sufficient conditions for closedness of the solution map for parametric operator equilibrium problems. The theorems obtained extend the closedness results obtained by Bogdan and Kolumbán [4], Salamon and Bogdan [18].

The notions of closedness, upper semi-continuity and Hadamard well-posedness are closely related. If the domain of the given functions is compact, we obtain results for upper semi-continuity of the solution map and the Hadamard well-posedness of the parametric operator equilibrium problems.

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