On positive real zeros of theta and *L*-functions associated with real, even and primitive characters

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Abstract. Let D range over the positive fundamental discriminants. Let $\theta(t, \chi_D)$, $t > 0$, denote the theta function associated with the real, even and primitive Dirichlet character of conductor D. On the one hand, we prove that there are infinitely many positive discriminants D for which $\theta(t, \chi_D)$ has at least one positive real zero. On the other hand, we prove that for a given positive real number t_0, there are at least $\gg X/\log^{13/2} X$ positive fundamental discriminants $D \leq X$ for which $\theta(t_0, \chi_D) \neq 0$.

1. Introduction

Let χ_D range over the real, even and primitive Dirichlet characters of conductors $D > 1$. Hence, D is a positive fundamental discriminant, i.e. either $D = d \equiv 1 \pmod 4$ is squarefree, or $D = 4d$ with $d \equiv 2$ or $3 \pmod 4$ squarefree. Conversely for such a D there is exactly one real, even and primitive Dirichlet character of conductor D: it is given by the Kronecker's symbol $\chi_D(n) = \left(\frac{D}{n}\right)$. Let

$$\theta(t, \chi_D) := \sum_{n \geq 1} \chi_D(n) e^{-\pi n^2 t/D} \quad (t > 0)$$

be its associated theta series which is used, as in [Dav, Chapter 9], to prove the functional equation of the associated Dirichlet *L*-series

$$L(s, \chi_D) := \sum_{n \geq 1} \frac{\chi_D(n)}{n^s} \quad (\Re(s) > 0).$$

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In fact, one first proves that

$$\theta(1/t, \chi_D) = \sqrt{t}\theta(t, \chi_D) \quad (t > 0), \tag{1}$$

which in using

$$(D/\pi)^{s/2}\Gamma(s/2)L(s, \chi_D) = \int_0^\infty \theta(t, \chi_D)t^{s/2}\frac{dt}{t} \quad (\Re(s) > 1)$$

$$= \int_1^\infty \theta(t, \chi_D)(t^{s/2} + t^{(1-s)/2})\frac{dt}{t} \tag{2}$$

yields the entire continuation of $L(s, \chi_D)$ to the complex plane together with the following functional equation:

$$(D/\pi)^{s/2}\Gamma(s/2)L(s, \chi_D) = (D/\pi)^{(1-s)/2}\Gamma((1-s)/2)L(1-s, \chi_D). \tag{3}$$

By (2), if $\theta(t, \chi_D)$ has no positive real zero, then $L(s, \chi_D) > 0$ for $s > 0$. However, in Theorem 1 we will prove that $\theta(t, \chi_D)$ has at least one positive real zero for infinitely many positive fundamental discriminants D's, but in Theorem 2 we prove that for a given $t_0 > 0$ there are infinitely many D's for which $\theta(t_0, \chi_D) \neq 0$. Conversely, in Theorem 4 we give an easy to check sufficient condition for $\theta(t, \chi_D)$ to have no positive real zero, and we will give several values of D for which this is indeed the case. Finally, following [Ros], we will prove in Theorem 6 that if $\theta(t, \chi_D)$ has at most one real zero greater than or equal to 1 and if $L(1/2, \chi_D) > 0$, then $L(s, \chi_D) > 0$ for $s > 0$ (an example of such a D being $D = 53$).

1.1. Real zeros of $\theta(t, \chi_D)$. Since $\lim_{t \to +\infty} e^{\pi t/D}\theta(t, \chi_D) = 1$, it follows that $\theta(t, \chi_D) > 0$ for t large enough. In fact, for $t \geq 1/6$, we have

$$e^t\theta(Dt/\pi, \chi_D) \geq 1 - \sum_{n \geq 2} e^{-(n^2-1)t} \geq 1 - \sum_{n \geq 2} e^{-(n^2-1)/6} > 0.$$

Hence, by (1), for $t \geq D/6\pi$ and $0 < t \leq 6\pi/D$ we have $\theta(t, \chi) > 0$. In particular, if $D < 6\pi$, i.e. if $D \in \{5, 8, 12, 13, 17\}$, then $\theta(t, \chi) > 0$ for $t > 0$. However, since $\theta(1, \chi_{53}) = -0.11074\cdots < 0$, there exists $t_0 > 1$ such that $\theta(t_0, \chi_{53}) = 0$. More generally, there are infinitely many positive fundamental discriminants D for which $\theta(t, \chi_D)$ has at least one positive real zero:

Theorem 1. *If $d > 1$ is a square free integer in any one of the two arithmetic progressions $\{53 + 120k; \ k \geq 0\}$ or $\{77 + 120k; \ k \geq 0\}$, then $\theta(t, \chi_d)$ has at least one positive real zero.*

PROOF. We could adapt [H]. We give a more explicit argument. Choose D such that $\chi_D(n) = -1$ for $n \in E := \{2, 3, 5\}$, i.e. choose D in any one of the two considered arithmetic progressions. Then, for $\alpha > 0$ we have

$$\theta(\alpha D/\pi, \chi_D) = \sum_{n \geq 1} \chi_D(n) e^{-n^2 \alpha} \leq f_E(\alpha) := -2 \sum_{n \in E} e^{-n^2 \alpha} + \sum_{n \geq 1} e^{-n^2 \alpha}.$$

Now, we have $f_E(0.07) = -0.0746 \cdots < 0$, which yields the desired result. □

The main result of this paper is the following converse result:

Theorem 2. *Fix $t_0 > 0$ and an arithmetic progression $\{a + 4kb; \ k \geq 0\}$, with $1 \leq a < 4b$, $a \equiv 1 \pmod 4$ and $\gcd(a, b) = 1$. Set*

$$C_{a,b} := \frac{1}{4b} \prod_{\substack{p \geq 3 \\ \gcd(p,b)=1}} \left(1 - \frac{1}{p^2}\right) > 0. \tag{4}$$

Then, $\theta(t_0, \chi_D) \neq 0$ for at least $\gg X/\log^{13/2} X$ of the $N(X) \sim C_{a,b} X$ positive fundamental discriminants $D \leq X$ of this arithmetic progression.

PROOF. See Section 4. □

This should be compared with the results obtained in [Lou99] where, by studying their second and fourth moments, we proved that $\theta(1, \chi_p) \neq 0$ for at least $\gg p/\log p$ of the $(p-1)/2$ odd Dirichlet characters mod p a prime number. See also [LM], for the explicit asymptotics for the second and fourth moments of $\theta(1, \chi)$ when χ ranges either over the $(p-1)/2$ odd Dirichlet characters mod p or over the $(p-3)/2$ not trivial even Dirichlet characters mod p. A slight modification of the proofs in [LM] yields similar results for any $t_0 > 0$, so that we have $\theta(t_0, \chi_p) \neq 0$ for at least $\gg p/\log p$ of the $(p-1)/2$ odd Dirichlet characters mod p a prime number and for at least $\gg p/\log p$ of the $(p-3)/2$ non trivial even Dirichlet characters mod p a prime number. However, in [CZ] it is proved that it may (seldom) happen that $\theta(1, \chi) = 0$ for complex characters (their two (up to complex conjugation) examples of such characters are primitive characters of composite conductors).

1.2. Real zeros of $\theta(t, \chi_D)$ and the sign of $L(s, \chi_D)$ for $s > 0$. Let us first recall S. CHOWLA's elementary sufficient condition for an L-series to be positive on positive real numbers (see [Chow]). Let χ be a (not necessarily primitive) non

principal real character mod $f > 1$. Define inductively $X_0(n) = \chi(n)$, $n \geq 0$, and $X_{r+1}(n) = \sum_{k=0}^{n} X_r(k)$, $n \geq 0$. Then,

$$f(t, \chi) := \sum_{n \geq 0} \chi(n) e^{-nt} = (1 - e^{-t})^r \left(\sum_{n \geq 0} X_r(n) e^{-nt} \right) \quad (t > 0). \qquad (5)$$

Since

$$\Gamma(s) L(s, \chi) = \int_0^\infty f(t, \chi) t^{s-1} dt \quad (\Re(s) > 0),$$

we deduce that if there exists $r \geq 0$ such that $X_r(n) \geq 0$ for all $n \geq 0$, then $L(s, \chi) > 0$ for $s > 0$. However, H. HEILBRONN proved in [H] that there are characters for which no such r exists. In fact, we have the following more explicit result (see [BPW]):

Theorem 3. *Let χ be a (not necessarily primitive) non principal real character mod $f > 1$ and assume that $\chi(2) = \chi(3) = \chi(5) = \chi(7) = \chi(11) = -1$, then there does not exist any $r \geq 0$ such that $X_r(n) \geq 0$ for all $n \geq 0$.*

PROOF. By (5), if such an r exists then $f(t, \chi) > 0$ for $t > 0$. Set $E = \{2, 3, 5, 7, 8, 11, 12\}$. Then $\chi(n) = -1$ for $n \in E$ and

$$f(t, \chi) \leq \sum_{n \geq 1} e^{-nt} - 2 \sum_{n \in E} e^{-nt} = \frac{e^{-t}}{1 - e^{-t}} \left(1 - 2(1 - e^{-t}) \sum_{n \in E} e^{-(n-1)t} \right).$$

Hence, if $P_E(x) := 1 - 2(1 - x)(\sum_{n \in E} x^{n-1}) < 0$ for some $x \in (0, 1)$, then $f(t, \chi) < 0$ for $t = -\log x$. Since $P_E(3/4) < 0$, the desired result follows. $\qquad \square$

In [Lou03] we proved that no such r exists if $L(1, \chi)$ is small enough, say if $L(1, \chi) \leq 1 - \log 2$ (see also [Lou04] for improvements), and by [CE] that there are infinitely many such characters.

If $t \mapsto \theta(t, \chi_D)$ has no real zero in $(1, +\infty)$, then $\theta(t, \chi_D) > 0$ for $t \geq 1$, hence $\theta(t, \chi_D) > 0$ for $t > 0$ by (1), and $L(s, \chi_D) > 0$ for $s \geq 1/2$, by (2), hence $L(s, \chi_D) > 0$ for $s > 0$, by (3). We give a simple sufficient condition for this to happen (notice that if χ mod f is even, then $X_1(f - 2) = -1$.):

Theorem 4. *Let $D > 0$ be a fundamental discriminant. Assume that $X_1(n) := \sum_{k=1}^{n} \chi_D(k) \geq 1$ for $1 \leq n \leq N_D := \min\{N \geq 2; \ N^2 / \log N \geq D/\pi\}$. Then $\theta(t, \chi_D) > 0$ for $t \geq 1$ and $L(s, \chi_D) > 0$ for $s > 0$. Notice that N_D is asymptotic to $\sqrt{\frac{D}{2\pi} \log\left(\frac{D}{2\pi}\right)}$.*

PROOF. Set $\beta(t, \chi_D) := \theta(Dt/\pi, \chi_D) = \sum_{n \geq 1} \chi_D(n) e^{-n^2 t}$. By (1), we only have to prove that $\beta(t, \chi_D) > 0$ for $t \geq \pi/D$. Now, we have

$$\beta(t, \chi_D) = \sum_{n=1}^{N-1} X_1(n)(e^{-n^2 t} - e^{-(n+1)^2 t}) + X_1(N) e^{-N^2 t} + \sum_{n>N} \chi_D(n) e^{-n^2 t}$$

$$\geq \sum_{n=1}^{N-1} (e^{-n^2 t} - e^{-(n+1)^2 t}) + e^{-N^2 t} - \sum_{n>N} e^{-n^2 t} = e^{-t} - \sum_{n>N} e^{-n^2 t}.$$

The desired result follows from the last assertion in Lemma 5. $\square$

Lemma 5. *For $N \geq 1$, set $f_N(t) := e^{-t} - \sum_{n>N} e^{-n^2 t}$, $t > 0$. Then, $f_N(t)$ has exactly one positive real zero t_N, and t_N is asymptotic to $N^{-2} \log N$ as $N \to \infty$. Moreover, $f_N(N^{-2} \log N) > 0$.*

PROOF. Since $t \mapsto e^t f_N(t)$ increases with $t > 0$, it follows that $f_N(t)$ has exactly one positive real zero t_N. Now,

$$\sum_{n>N} e^{-(n/N)^2 \log N} \leq \int_N^{\infty} e^{-(t/N)^2 \log N} dt$$

$$= \frac{N}{2\sqrt{\log N}} \int_{\log N}^{\infty} e^{-t} \frac{dt}{\sqrt{t}} \leq \frac{N}{2 \log N} \int_{\log N}^{\infty} e^{-t} dt = \frac{1}{2 \log N}$$

yields $f_N(N^{-2} \log N) \geq e^{-N^{-2} \log N} - \frac{1}{2 \log N} > 0$ for $N \geq 2$. It remains to prove that for any given $\alpha \in (0,1)$ we have $f_N(\alpha N^{-2} \log N) < 0$ for $N \geq N_\alpha$ large enough, which follows from

$$\sum_{n \geq N} e^{-\alpha(n/N)^2 \log N} \geq \int_N^{\infty} e^{-\alpha(t/N)^2 \log N} dt$$

$$= \frac{N}{2\sqrt{\alpha \log N}} \int_{\alpha \log N}^{\infty} e^{-t} \frac{dt}{\sqrt{t}} \geq \frac{2}{3} \frac{N^{1-\alpha}}{2\alpha \log N}$$

(use $I(X) := \int_X^{\infty} e^{-t} \frac{dt}{\sqrt{t}} = \frac{e^{-X}}{\sqrt{X}} - \int_X^{\infty} e^{-t} \frac{dt}{2t\sqrt{t}} \geq \frac{e^{-X}}{\sqrt{X}} - \frac{1}{2} I(X)$ for $X \geq 1$), which proves that for a given $\alpha \in (0,1)$ we have $\lim_{N \to \infty} f_N(\alpha N^{-2} \log N) = -\infty$. $\square$

For example, for $D = 17$ we have $X_1(n) \geq 1$ for $1 \leq n \leq N_D = 2$ and $L(s, \chi_{17}) > 0$ for $s > 0$. For various bounds B's, we computed the following Table, where $N_1(B)$ denotes the number of positive fundamental discriminants less than or equal to a prescribed bound B and $N_2(B)$ denotes the number of

such discriminants for which Theorem 4 applies:

B	$N_1(B)$	$N_2(B)$	$\%(= 100N_2(B)/N_1(B))$
10^2	30	16	53.33
10^3	302	124	41.06
10^4	3043	1025	33.68
10^5	30394	8798	28.95
10^6	303957	76670	25.12
10^7	3039653	682332	22.45
10^8	30396324	6165194	20.28

According to [BM, Corollary 3], $N_2(B)/N_1(B)$ tends to 0 as B tends to $+\infty$.

It has been checked numerically that $\zeta_K(s) < 0$ for $0 < s < 1$ for all the imaginary quadratic number fields K of conductors $f_K \leq 3 \cdot 10^8$ (see [Wat]), which amounts to saying that $L(s, \chi) > 0$ for $0 < s < 1$ for all the odd and quadratic Dirichlet characters χ mod f with $f \leq 3 \cdot 10^8$. In contrast, for non trivial, even and quadratic characters there is no known efficient algorithm to check the same result up to that large moduli. In fact, it has only been checked that $L(s, \chi) > 0$ for $0 < s < 1$ for all the non trivial, even and quadratic Dirichlet characters χ mod f with $f \leq 2 \cdot 10^5$ (see [Chua]).

Theorem 6. *Let D be a positive fundamental discriminant. If $\theta(t, \chi_D)$ has exactly one real zero in $[1, +\infty)$ and if $L(\frac{1}{2}, \chi_D) > 0$, then $L(s, \chi_D) > 0$ for $s > 0$.*

PROOF. Our proof is similar to that of [Ros, Lemma 4, page 511], where the case of negative fundamental discriminants is dealt with (by working on the Dedekind zeta function of the imaginary quadratic field of negative discriminant). Let $a \geq 1$ be this zero. We may assume that a is a zero of odd multiplicity, which implies that $\theta(t, \chi_D)$ changes signs at $t = a$. We may assume that $s \geq 1/2$, by (3). Then,

$$\frac{\partial}{\partial s}\left(\frac{t^{s/2} + t^{(1-s)/2}}{a^{s/2} + a^{(1-s)/2}}\right)$$
$$= \frac{(\log t - \log a)(1 - (at)^{\frac{1}{2}-s}) + (\log t + \log a)(a^{\frac{1}{2}-s} - t^{\frac{1}{2}-s})}{2(at)^{-s/2}(a^{s/2} + a^{(1-s)/2})^2}$$

is negative for $t < a$ and positive for $t > a$. Since $\lim_{t \to +\infty} e^{\pi t/D}\theta(t, \chi_D) = 1$, we have that $\theta(t, \chi_D) < 0$ for $1 \leq t < a$ and $\theta(t, \chi_D) > 0$ for $t > a$. Hence,

$$\frac{\partial}{\partial s}\left(\frac{t^{s/2} + t^{(1-s)/2}}{a^{s/2} + a^{(1-s)/2}}\theta(t, \chi_D)\right) > 0$$

for $s \geq 1/2$ and $t \neq a$. By (2), we deduce that

$$s \mapsto \frac{(D/\pi)^{s/2}\Gamma(s/2)L(s,\chi_D)}{a^{s/2} + a^{(1-s)/2}}$$

increases with $s \geq 1/2$ and the desired result follows.   □

For example, a plot of their graphs leads us to suspect that if χ_{53} is the even, real and primitive Dirichlet character mod 53, then $\theta(t, \chi_{53})$ has exactly 1 real zero in $[1, +\infty)$, whereas $\theta(\chi_{197}, t)$ has exactly 2 such zeros.

2. The mean value of $\theta(t_0, \chi_D)$

For a given $s \in (1/2, 1]$, R. AYOUB evaluated in [Ay] the mean value of $L(s, \chi_D)$ over the fundamental discriminants. Our goal in this section is, for a given $t_0 > 0$, to evaluate the mean value of $\theta(t_0, \chi_D)$ over the fundamental discriminants.

We fix a and b with $1 \leq a < 4b$, $a \equiv 1 \pmod 4$ and $\gcd(a, b) = 1$. We let $D \equiv 1 \pmod 4$ range over the positive fundamental discriminants of the arithmetic progression $\{a + 4kb; \ k \geq 0\}$. Hence, $D = d$ ranges over the squarefree integers greater than one of this arithmetic progression. The number $N_{a,b}(X)$ of positive fundamental discriminants $D \leq X$ in this arithmetic progression is asymptotic to $C_{a,b}X$, where $C_{a,b} > 0$ is defined in (4). Throughout this paper, we let

$$\sum_{\substack{1 < D \leq X \\ D \equiv a \pmod{4b}}}^{*}$$

denote sums over the positive fundamental discriminants $D \leq X$ in such arithmetic progressions, i.e. sums over the square-free integers $d \equiv a \pmod{4b}$ with $1 < d \leq X$.

Theorem 7. *Fix $t_0 > 0$ and an arithmetic progression $\{a + 4kb; \ k \geq 0\}$, with $1 \leq a < 4b$, $a \equiv 1 \pmod 4$ and $\gcd(a, b) = 1$. It holds that*

$$S_1(t_0, X) := \sum_{\substack{1 < D \leq X \\ D \equiv a \pmod{4b}}}^{*} \theta(t_0, \chi_D) \sim \frac{C_1}{t_0^{1/4}} X^{5/4},$$

where

$$C_1 = \frac{\Gamma(1/4)}{5\pi^{1/4}} C_b \quad \text{with} \quad C_b := \frac{1}{4b} \prod_{\substack{p \geq 2 \\ \gcd(p, 4b) = 1}} \left(1 - \frac{2}{p^2} + \frac{1}{p^3} \right).$$

Hence, $\theta(t_0, \chi_D) > 0$ for at least $\gg X^{3/4}$ of the $N(X) \sim C_{a,b} X$ positive fundamental discriminants $D \leq X$ of this arithmetic progression.

PROOF. Let us first prove the last assertion. Let $N_{>0}(X)$ denote the number of such fundamental discriminants. Since

$$|\theta(t_0, \chi_D)| \leq \sum_{n \geq 1} e^{-\pi n^2 t_0 / D} \leq \int_0^\infty e^{-\pi u^2 t_0 / D} du = \sqrt{D/4t_0},$$

we have $S_1(t_0, X) \leq N_{>0}(X)\sqrt{X/4t_0}$ and the desired result follows. Let us now prove the first assertion, We have

$$S_1(t_0, X) = \sum_{\substack{1 < D \leq X \\ D \equiv a \pmod{4b}}}^{*} \sum_{n \geq 1} \left(\frac{D}{n}\right) e^{-\pi n^2 t_0 / D}.$$

We split this double sum into two parts. The first one

$$S_1^{sq}(t_0, X) := \sum_{\substack{1 < D \leq X \\ D \equiv a \pmod{4b}}}^{*} \sum_{\substack{m \geq 1 \\ \gcd(m, D) = 1}} e^{-\pi m^4 t_0 / D} \tag{6}$$

ranges over the indices n which are perfect squares, and the second one

$$S_1^{nsq}(t_0, X) := \sum_{\substack{n \geq 1 \\ n \text{ not a square}}} \sum_{\substack{1 < D \leq X \\ D \equiv a \pmod{4b}}}^{*} \left(\frac{D}{n}\right) e^{-\pi n^2 t_0 / D} \tag{7}$$

ranges over the indices n which are not perfect squares. Using Propositions 10 and 17, the desired result follows. $\square$

3. The mean square value of $\theta(t_0, \chi_D)$

Theorem 8. *Fix $t_0 > 0$. Let $\sum_{1 < D \leq X}^{*}$ denote sums over the positive fundamental discriminants $D \leq X$. It holds that*

$$S_2(t_0, X) := \sum_{1 < D \leq X}^{*} \theta(t_0, \chi_D)^2 \ll X^{3/2} \log^{13/2} X.$$

PROOF. We have

$$S_2(t_0, X) = \sum_{1 < D \leq X}^* \sum_{a \geq 1} \sum_{b \geq 1} \left(\frac{D}{ab}\right) e^{-\pi(a^2+b^2)/D}.$$

We split this double sum into two parts. The first one is over the indices a and b for which ab is a perfect square, which amounts to asking that $a = da'^2$ and $b = db'^2$ with $d \geq 1$ and $\gcd(a', b') = 1$, hence can be written

$$S_2^{sq}(t_0, X) := \sum_{1 < D \leq X}^* \sum_{\substack{d \geq 1 \\ \gcd(d,D)=1}} \sum_{\substack{a,b \geq 1,\ \gcd(a,b)=1 \\ \gcd(a,D)=\gcd(b,D)=1}} e^{-\pi d^2(a^4+b^4)t_0/D}. \qquad (8)$$

The second one

$$S_2^{nsq}(t_0, X) := \sum_{1 < D \leq X}^* \sum_{\substack{a,b \geq 1 \\ ab \text{ not a square}}} \left(\frac{D}{ab}\right) e^{-\pi(a^2+b^2)t_0/D} \qquad (9)$$

ranges over the indices a and b for which ab are not perfect squares. Using Propositions 18 and 28, the desired result follows. $\qquad \square$

4. Proof of Theorem 2

We use Theorems 7 and 8 and notice that, by the Cauchy–Schwarz inequality, it holds that $\theta(t_0, \chi_D) \neq 0$ for at least $S_1(t_0, X)^2/S_2(t_0, X)$ of the positive fundamental discriminants $D \leq X$ of this arithmetic progression.

Remark 9. In fact, even though we could not prove it, we expect $S_2^{nsq}(t_0, X)$ to be negligible compared with $S_2^{sq}(t_0, X)$, i.e. we expect $S_2(t_0, X)$ to be asymptotic to $\frac{C_2}{\sqrt{t_0}} X^{3/2} \log X$, with C_2 as in Proposition 18. Hence, we expect that $\theta(t_0, \chi_D) \neq 0$ for at least $\gg X/\log X$ of the $N(X) \sim C_{a,b} X$ positive fundamental discriminants $D \leq X$ of this arithmetic progression.

The remaining of this paper is devoted to proving Theorems 7 and 8, by studying the behaviors of (6) and (7), in Propositions 10 and 17, and by studying the behaviors of (8) and (9), in Propositions 18 and 28.

5. The behavior of $S_1^{sq}(t_0, X)$

By applying Lemmas 11, 12 and 13 below, we will prove:

Proposition 10. *Fix $t_0 > 0$ and an arithmetic progression $\{a + 4kb;\ k \geq 0\}$, with $1 \leq a < 4b$, $a \equiv 1 \pmod 4$ and $\gcd(a, b) = 1$. Then, $S_1^{sq}(t_0, X)$, defined in (6), is asymptotic to $\frac{C_1}{t_0^{1/4}} X^{5/4}$, where $C_1 > 0$ is given in Theorem 7.*

Lemma 11. *Fix $t_0 > 0$ and $\epsilon > 0$. Set*

$$\Phi_D(t) := \sum_{\substack{m \geq 1 \\ \gcd(m, D) = 1}} e^{-\pi m^4 t / D} \quad (t > 0).$$

It holds that

$$\Phi_D(t_0) = \frac{\Gamma(1/4)}{4(\pi t_0)^{1/4}} \frac{\phi(D)}{D} D^{1/4} + O\left(D^{\epsilon}\right).$$

Hence,

$$S_1^{sq}(t_0, X) = \sum_{\substack{1 < D \leq X \\ D \equiv a \pmod{4b}}}^{*} \Phi_D(t_0) = \frac{\Gamma(1/4)}{4(\pi t_0)^{1/4}} \sum_{1 < D \leq X}^{*} \frac{\phi(D)}{D} D^{1/4} + O\left(X^{1+\epsilon}\right).$$

PROOF. We refer the reader to [Mel] or [Rad, Section 27] for properties of Mellin's transforms. The MELLIN's transform of $\Phi_D(t)$ is

$$\Psi_D(s) := M\Phi_D(s) = \int_0^{\infty} \Phi_D(t) t^s \frac{dt}{t} = \left(\frac{D}{\pi}\right)^s \left\{ \prod_{p | D} \left(1 - \frac{1}{p^{4s}}\right) \right\} \zeta(4s) \Gamma(s).$$

It has a simple pole at $s = 1/4$, and

$$\Phi_D(t_0) = M^{-1} \Psi_D(t_0) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \Psi_D(s) t_0^{-s} ds \quad (\alpha > 1).$$

By moving the vertical line of integration $\Re(s) = \alpha$ to the left to the vertical line $\Re(s) = \epsilon/2$, $\epsilon \in (0, 1/4)$, we pick up one residue, at $s = 1/4$, and we obtain

$$\Phi_D(t_0) = \frac{\Gamma(1/4)}{4(\pi t_0)^{1/4}} \frac{\phi(D)}{D} D^{1/4} + O\left(D^{\epsilon/2} \prod_{p | D} \left(1 + \frac{1}{p^{2\epsilon}}\right)\right).$$

Since

$$\omega(D) := \sum_{p | D} 1 \leq (1 + o(1)) \frac{\log D}{\log \log D}$$

(use $D \geq \omega(D)! \geq (\omega(D)/e)^{\omega(D)}$), we have

$$\prod_{p|D}(1 + p^{-2\epsilon}) \leq 2^{\omega(D)} = D^{(\log 2)\frac{\omega(D)}{\log D}} = D^{o(1)},$$

and the desired result follows. □

Lemma 12. *Let C_b be as in Theorem 7. It holds that*

$$F(X) := \sum_{\substack{1 < D \leq X \\ D \equiv a \pmod{4b}}}^{*} \frac{\phi(D)}{D} \sim C_b X.$$

PROOF. Set $a_d = \phi(d)/d$ if $1 \leq d \equiv a \pmod{4b}$ and d is square-free, and $a_d = 0$ otherwise. Then, set $F(s) = \sum_{d \geq 1} a_d d^{-s}$, a Dirichlet series with nonnegative coefficients. It suffices to prove that $F(s)$ admits an analytic continuation to $\Re(s) > 1/2$, with only one pole, a simple pole of residue C_b at $s = 1$ (e.g., see [Lang, Chapter 15, Section 3]). We have

$$F(s) = \frac{1}{\phi(4b)} \sum_{\chi \bmod 4b} \overline{\chi(a)} F(s, \chi),$$

where

$$F(s, \chi) := \sum_{d \geq 1} \mu^2(d) \chi(d) \frac{\phi(d)}{d} \frac{1}{d^s} = \prod_{p \geq 2} \left(1 + \left(1 - \frac{1}{p}\right) \frac{\chi(p)}{p^s}\right) = \Pi_\chi(s) L(s, \chi),$$

where

$$\Pi_\chi(s) := \prod_{p \geq 2} \left(1 + \left(1 - \frac{1}{p}\right) \frac{\chi(p)}{p^s}\right)\left(1 - \frac{\chi(p)}{p^s}\right)$$

is absolutely convergent and holomorphic for $\Re(s) > 1/2$. If χ is not trivial, then $F(s, \chi)$ is holomorphic for $\Re(s) > 1/2$. Now, assume that χ is the trivial character mod $4b$. Then, $L(s, \chi) = \left\{\prod_{p|4b}\left(1 - \frac{1}{p}\right)\right\}\zeta(s)$ is meromorphic for $\Re(s) > 1/2$, with only one pole, a simple pole at $s = 1$ of residue $\frac{\phi(4b)}{4b}$. Hence, $F(s, \chi)$ is meromorphic for $\Re(s) > 1/2$, with only one pole, a simple pole at $s = 1$ of residue $\frac{\phi(4b)}{4b}\Pi_\chi(1)$, and $F(s)$ is indeed meromorphic for $\Re(s) > 1/2$, with only one pole, a simple pole at $s = 1$ of residue $\frac{1}{4b}\Pi_\chi(1) = C_b$. □

Lemma 13. *If $F(X) = \sum_{1 \leq n \leq X} f(n)$ is asymptotic to cX for some $c \neq 0$, then $S(X) := \sum_{1 \leq n \leq X} f(n)n^\alpha \log^\beta n$ is asymptotic to $\frac{c}{\alpha+1}X^{\alpha+1}\log^\beta X$ for $\alpha > -1$.*

PROOF. We may assume that X is a positive integer. Setting $g(n) := f(n) - c$, it suffices to prove that if $F(X) = o(X)$ then $S(X) = o(X^{\alpha+1}\log^\beta X)$, which follows from $S(X) = \sum_{1 \leq n \leq X-1} F(n)(n^\alpha \log^\beta n - (n+1)^\alpha \log^\beta(n+1)) + F(X)X^\alpha \log^\beta X$. □

6. The behavior of $S_1^{nsq}(t_0, X)$

Lemma 14. *Let ψ be a non trivial Dirichlet character mod $f > 1$. Then,*

$$S(\psi, Y) := \sum_{1 \leq m \leq Y} \mu^2(m)\psi(m) \ll \sqrt{Y\sqrt{f}\log f},$$

where the implied constants do not depend neither on f nor on ψ.

PROOF. Since $\mu(m)^2 = \sum_{d^2|m} \mu(d)$, we obtain

$$S(\psi, Y) = \sum_{1 \leq d \leq Y^{1/2}} \mu(d)\psi(d^2) \sum_{1 \leq m \leq Y/d^2} \psi(m).$$

Since

$$\sum_{1 \leq m \leq Z} \psi(n) \ll B_f := \sqrt{f}\log f$$

(e.g., see [Apo, Th. 13.15]), for any $H \geq 1$ we have

$$|S(\psi, Y)| \ll \sum_{1 \leq d \leq H} B_f + \sum_{H < d \leq Y^{1/2}} \frac{Y}{d^2} \leq HB_f + \frac{Y}{H}.$$

By choosing $H = [\sqrt{Y/B_f}]$, we get the desired upper bound. $\qquad\square$

Lemma 15. *Fix an arithmetic progression $\{a+4kb; \ k \geq 0\}$, with $1 \leq a < 4b$, $a \equiv 1 \pmod 4$ and $\gcd(a,b) = 1$. It holds that*

$$S(n, Y) := \sum_{\substack{1 < D \leq Y \\ D \equiv a \pmod{4b}}}^{*} \chi_D(n) \ll \sqrt{Y\sqrt{n}\log n}$$

for any $Y > 1$ and any $n > 1$ which not a perfect square.

PROOF. Write $n = 2^l m$, with $m \geq 1$ odd and $l \geq 0$. Assume that n is not a perfect square. Let ψ_8 be the character mod 8 defined by $\psi_8(k) = (-1)^{(k^2-1)/8}$ if k is odd and $\psi_8(k) = 0$ otherwise. Let ψ_m be the character mod m defined by $\psi_m(k) = \left(\frac{k}{m}\right)$ (Jacobi's symbol). If $1 \leq D \equiv 1 \pmod 4$ is a fundamental discriminant, then

$$\chi_D(n) = (-1)^{l(D^2-1)/8} \left(\frac{D}{m}\right) = \psi_{8m}(D),$$

where $\psi_{8m} = \psi_8^l \psi_m$ is a Dirichlet character mod $8m$. Moreover, since either l is odd or m is not a perfect square, ψ_{8m} is not trivial. Now,

$$S(n,Y) = \sum_{\substack{d \equiv a \pmod{4b} \\ 1 < d \leq Y}}^{*} \mu^2(d)\psi_{8m}(d)$$

$$= \frac{1}{\phi(4b)} \sum_{\chi \bmod 4b} \overline{\chi(a)} \sum_{1 < d \leq Y} \mu^2(d)\chi(d)\psi_{8m}(d).$$

Since $d \mapsto \chi(d)\psi_{8m}(d)$ is a non trivial Dirichlet character mod $f = 8bm \leq 8bn$, the result follows, by Lemma 14. □

Lemma 16. *Let α, β and γ be three given positive real numbers. As $d \geq 2$ ranges over the positive integers, we have*

$$\sum_{n \geq 1} (\log n)^\alpha n^\beta e^{-\gamma n^2/d} \ll d^{(\beta+1)/2}(\log d)^\alpha.$$

PROOF. Write $n = a + b[\sqrt{d}]$ with $1 \leq a \leq [\sqrt{d}]$ and $b \geq 0$. Then,

$$n \leq (b+1)\sqrt{d}, \quad n^2/d \geq b^2[\sqrt{d}]^2/d \geq b^2/4$$

and

$$(\log n)/(\log\sqrt{d}) \leq 1 + (\log(b+1))/(\log(\sqrt{d})) \leq 1 + (\log(b+1))/(\log(\sqrt{2})).$$

The desired bound follows. □

Proposition 17. *Fix $t_0 > 0$ and an arithmetic progression $\{a+4kb; \ k \geq 0\}$, with $1 \leq a < 4b$, $a \equiv 1 \pmod 4$ and $\gcd(a,b) = 1$. It holds that*

$$S_1^{nsq}(t_0, X) = O\left(X^{9/8}\sqrt{\log X}\right).$$

PROOF. We may assume that $X > 1$ is a positive integer. We have

$$S_1^{nsq}(t_0, X) := \sum_{\substack{n \geq 1 \\ n \text{ not a square}}} \sum_{1 < d \leq X} (S(n,d) - S(n,d-1))e^{-\frac{\pi n^2 t_0}{d}}$$

$$= \sum_{\substack{n \geq 1 \\ n \text{ not a square}}} v\left(S(n,X)e^{-\frac{\pi n^2 t_0}{X}} + \sum_{1 < d \leq X-1} S(n,d)\left(e^{-\frac{\pi n^2 t_0}{d}} - e^{-\frac{\pi n^2 t_0}{d+1}}\right)\right)$$

$$= \sum_{\substack{n \geq 1 \\ n \text{ not a square}}} S(n,X)e^{-\frac{\pi n^2 t_0}{X}} + \sum_{1 < d \leq X-1} \sum_{\substack{n \geq 1 \\ n \text{ not a square}}} \frac{\pi n^2 t_0}{(d+\theta)^2} S(n,d)e^{-\frac{\pi n^2 t_0}{d+\theta}},$$

where $\theta \in (0,1)$ depends on n and d. Hence, by Lemma 15, we obtain

$$S_1^{nsq}(t_0, X) \ll \sqrt{X} \sum_{n>1} n^{1/4} \sqrt{\log n} e^{-\frac{\pi n^2 t_0}{X}} + \sum_{2<d\leq X} \sum_{n>1} \frac{n^{9/4}}{d^{3/2}} \sqrt{\log n} e^{-\frac{\pi n^2 t_0}{d}}.$$

Summing over $n \geq 2$ and using Lemma 16, we obtain the desired bound. $\qquad \square$

7. The behavior of $S_2^{sq}(t_0, X)$

By applying Lemmas 13, 21 and 23, we will prove:

Proposition 18. *Fix $t_0 > 0$. Then, $S_2^{sq}(t_0, X)$, defined in (8), is asymptotic to $\frac{C_2}{\sqrt{t_0}} X^{3/2} \log X$, where*

$$C_2 := \frac{13\Gamma(1/4)^2}{96\pi^{5/2}} C \quad with \quad C := \frac{1}{4} \prod_{p\geq 3} \left(1 - \frac{4}{p(p+1)} + \frac{3}{p^2(p+1)} - \frac{1}{p^3(p+1)}\right).$$

Lemma 19. *Set $\kappa := \frac{\Gamma^2(1/4)}{16\sqrt{\pi}}$. For $\delta_1, \delta_2 > 0$, set*

$$H_{\delta_1,\delta_2}(s) := \sum_{\substack{a\geq 1 \\ b\geq 1}} \frac{1}{(a^4\delta_1^4 + b^4\delta_2^4)^s} \quad (\Re(s) > 1/2).$$

Then, $h_{\delta_1,\delta_2}(s) := \pi^{-s}\Gamma(s) H_{\delta_1,\delta_2}(s)$ admits a meromorphic continuation to the vertical half-plane $\Re(s) > 1/4$, with a unique pole, a simple pole at $s = 1/2$, and in this half-plane we have:

$$h_{\delta_1,\delta_2}(s) = \frac{\kappa}{\delta_1\delta_2} \frac{1}{s - \frac{1}{2}} + c_0(\delta_1, \delta_2) + O\left(s - \frac{1}{2}\right),$$

with $c_0(\delta_1, \delta_2) \ll (1 + \log\delta_1 + \log\delta_2)/\delta_1\delta_2$.

PROOF. Since

$$H_{\delta_1,\delta_2}(s) := \sum_{\substack{a\geq 1 \\ b\geq 1}} \frac{1}{(a^4\delta_1^4 + b^4\delta_2^4)^s} = -\frac{\zeta(4s)}{2\delta_1^{4s}} + \frac{1}{2} \sum_{\substack{a\geq 1 \\ b\in\mathbf{Z}}} \frac{1}{(a^4\delta_1^4 + b^4\delta_2^4)^s}$$

and

$$\pi^{-s}\Gamma(s)\alpha^{-s} = \int_0^\infty e^{-\alpha\pi t} t^s \frac{dt}{t} \quad (\alpha > 0),$$

we have

$$h_{\delta_1,\delta_2}(s) := \pi^{-s}\Gamma(s)H_{\delta_1,\delta_2}(s) = -\frac{\Gamma(s)\zeta(4s)}{2\pi^s\delta_1^{4s}} + \frac{1}{2}A_{\delta_1,\delta_2}(s),$$

where

$$A_{\delta_1,\delta_2}(s) := \sum_{a\geq 1}\int_0^\infty e^{-\pi ta^4\delta_1^4}\left(\sum_{b\in\mathbf{Z}}e^{-\pi tb^4\delta_2^4}\right)t^s\frac{dt}{t}.$$

Applying Poisson summation formula

$$\sum_{b\in\mathbf{Z}}f(\alpha b) = \frac{1}{\alpha}\sum_{b\in\mathbf{Z}}\hat{f}(b/\alpha)$$

with $f(x) = e^{-\pi x^4}$ and $\alpha = t^{1/4}\delta_2 > 0$, where

$$\hat{f}(x) := \int_{-\infty}^\infty f(y)e^{-2i\pi xy}dy,$$

and noticing that $\hat{f}(-b/\alpha) = \hat{f}(b/\alpha)$ (f is even) and that

$$\hat{f}(0) = \int_{-\infty}^\infty e^{-\pi y^4}dy = \frac{\Gamma(1/4)}{2\pi^{1/4}},$$

we obtain

$$\sum_{b\in\mathbf{Z}}e^{-\pi tb^4\delta_2^4} = \frac{\Gamma(1/4)}{2(\pi t)^{1/4}\delta_2} + \frac{2}{t^{1/4}\delta_2}\sum_{b\geq 1}\hat{f}(b/t^{1/4}\delta_2).$$

Hence, we have

$$h_{\delta_1,\delta_2}(s) = -\frac{\Gamma(s)\zeta(4s)}{2\pi^s\delta_1^{4s}} + \frac{\Gamma(1/4)\Gamma(s-1/4)\zeta(4s-1)}{4\pi^s\delta_1^{4s-1}\delta_2} + B_{\delta_1,\delta_2}(s), \qquad (10)$$

where

$$B_{\delta_1,\delta_2}(s) := \frac{1}{\delta_2}\sum_{a\geq 1}\int_0^\infty e^{-\pi ta^4\delta_1^4}\left(\sum_{b\in\mathbf{Z}}\hat{f}(b/t^{1/4}\delta_2)\right)t^{s-1/4}\frac{dt}{t}.$$

Now, there exists $c > 0$ and $c' > 0$ such that $|\hat{f}(x)| \leq c\exp(-c'x^{4/3})$ for $x > 0$ (see [CKK]). Since for $\alpha > 0$ and $\beta > 0$ we have

$$S(\alpha,\beta) := \sum_{n\geq 1}e^{-\beta n^\alpha} \leq \int_0^\infty e^{-\beta x^\alpha}dx = \frac{C_\alpha}{\beta^{1/\alpha}},$$

where $C_\alpha = \Gamma(1/\alpha)/\alpha$, we have

$$\sum_{b\geq 1}\left|\hat{f}(b/t^{1/4}\delta_2)\right| \leq cS\left(\frac{4}{3},\frac{c'}{t^{1/3}\delta_2^{4/3}}\right) \ll t^{1/4}\delta_2$$

and

$$|B_{\delta_1,\delta_2}(s)| \ll \sum_{a\geq 1}\int_0^\infty e^{-\pi t a^4 \delta_1^4}t^\sigma\frac{dt}{t} = \frac{\zeta(4\sigma)\Gamma(\sigma)}{\pi^\sigma\delta_1^{4\sigma}}\quad (\sigma := \Re(s)).$$

Using $(\Gamma'/\Gamma)(1/4) = -(\gamma + 3\log 2 + \pi/2)$, the desired result follows from (10), with

$$c_0(\delta_1,\delta_2) := -\frac{\kappa}{\delta_1\delta_2}\left(\log(8\pi) + \frac{\pi}{2} - 3\gamma + 4\log\delta_1\right) - \frac{\pi^2}{12\delta_1^2} + B_{\delta_1,\delta_2}\left(\frac{1}{2}\right).$$

Let us finally prove the last assertion. Since $h_{\delta_1,\delta_2}(s) = h_{\delta_2,\delta_1}(s)$, we have $c_0(\delta_1,\delta_2) = c_0(\delta_2,\delta_1)$. Hence, we may assume that $\delta_2 \leq \delta_1$, in which case the last assertion is clear, for $B_{\delta_1,\delta_2}(\frac{1}{2}) \ll \frac{1}{\delta_1^2} \leq \frac{1}{\delta_1\delta_2}$. Since our upper bound in this last assertion remains unchanged when we exchange δ_1 and δ_2, we can drop our restriction $\delta_2 \leq \delta_1$. $\qquad\square$

Lemma 20. *Fix $t_0 > 0$. Set $\kappa := \frac{\Gamma^2(1/4)}{16\sqrt{\pi}}$ and*

$$H_D(s) := \sum_{\substack{a,b\geq 1\\ \gcd(a,D)=\gcd(b,D)=1}}\frac{1}{(a^4+b^4)^s} = \sum_{\delta_1|D}\sum_{\delta_2|D}\mu(\delta_1)\mu(\delta_2)H_{\delta_1,\delta_2}(s)\quad (\Re(s)>1/2).$$

Then

$$s \mapsto F_D(s) := \left(\frac{D}{\pi t_0}\right)^s\left\{\prod_{p|D}\left(1+\frac{1}{p^{2s}}\right)\right\}^{-1}\frac{\zeta(2s)}{\zeta(4s)}H_D(s)\Gamma(s)$$

admits a meromorphic continuation to the vertical half-plane $\Re(s) > 1/4$, with a unique pole, at $s = 1/2$, a double pole whose residue $c_{-1}(D)$ satisfies

$$c_{-1}(D) = \frac{3\kappa}{\pi^2\sqrt{t_0}}\left\{\prod_{p|D}\frac{(1-1/p)^2}{1+1/p}\right\}\sqrt{D}\log D + O\left(\sqrt{D}(\log\log D)^3\right).$$

PROOF. We have

$$F_D(s) = \frac{3\kappa}{\pi^2}\sqrt{\frac{D}{t_0}}\left\{\prod_{p|D}\left(1+\frac{1}{p}\right)\right\}^{-1}\frac{G_D(s)}{\left(s-\frac{1}{2}\right)^2},$$

where

$$G_D(s) := \left(\frac{D}{t_0}\right)^{s-1/2}$$
$$\times \left\{ \prod_{p|D} \frac{1+\frac{1}{p}}{1+\frac{1}{p^{2s}}} \right\} \frac{2(s-\frac{1}{2})\zeta(2s)}{\zeta(4s)/\zeta(2)} \sum_{\delta_1|D} \sum_{\delta_2|D} \mu(\delta_1)\mu(\delta_2) \frac{s-\frac{1}{2}}{\kappa} h_{\delta_1,\delta_2}(s).$$

Setting $s = \frac{1}{2} + \epsilon$, we have:

$$\left(\frac{D}{t_0}\right)^{s-1/2} = 1 + \epsilon \log(D/t_0) + O(\epsilon^2),$$

$$\prod_{p|D} \frac{1+\frac{1}{p}}{1+\frac{1}{p^{2s}}} = 1 + 2\left(\sum_{p|D} \frac{\log p}{p+1}\right)\epsilon + O(\epsilon^2)$$

$$\frac{2(s-\frac{1}{2})\zeta(2s)}{\zeta(4s)/\zeta(2)} = 1 + a\epsilon + O(\epsilon^2),$$

$$\frac{s-\frac{1}{2}}{\kappa} h_{\delta_1,\delta_2}(s) = \frac{1}{\delta_1\delta_2} + \frac{c_0(\delta_1,\delta_2)}{\kappa}\epsilon + O(\epsilon^2).$$

Using

$$\sum_{p|D} \frac{\log p}{p} \ll \log\log D$$

we obtain

$$\left(\frac{D}{t_0}\right)^{s-1/2} \left\{ \prod_{p|D} \frac{1+\frac{1}{p}}{1+\frac{1}{p^{2s}}} \right\} \frac{2(s-\frac{1}{2})\zeta(2s)}{\zeta(4s)/\zeta(2)} = 1 + (\log D + O(\log\log D))\,\epsilon + O(\epsilon^2).$$

Using

$$\sum_{\delta|D} \frac{|\mu(\delta)|}{\delta} \ll \log\log D,$$

$$\sum_{\delta|D} \frac{|\mu(\delta)|}{\delta} \log\delta = \sum_{\delta|D} \frac{|\mu(\delta)|}{\delta} \sum_{p|\delta} \log p = \sum_{p|D} \frac{\log p}{p} \sum_{\delta|D/p} \frac{|\mu(p\delta)|}{\delta} \ll (\log\log D)^2$$

and the last assertion of Lemma 19, we obtain

$$\sum_{\delta_1|D} \sum_{\delta_2|D} \mu(\delta_1)\mu(\delta_2) \frac{s-\frac{1}{2}}{\kappa} h_{\delta_1,\delta_2}(s) = c_0(D) + c_1(D)\epsilon + O(\epsilon^2)$$

where

$$c_0(D) = \sum_{\delta_1|D} \sum_{\delta_2|D} \frac{\mu(\delta_1)\mu(\delta_2)}{\delta_1\delta_2} = \prod_{p|D}(1 - \frac{1}{p})^2,$$

and

$$c_1(D) = O\left((\log\log D)^3\right).$$

The desired result follows.   □

Lemma 21 (Compare with Lemma 11). *Fix $t_0 > 0$ and $\epsilon > 0$. Set*

$$\Phi_D(t) := \sum_{\substack{d \geq 1 \\ \gcd(d,D)=1}} \sum_{\substack{a,b\geq 1 \ \gcd(a,b)=1 \\ \gcd(a,D)=\gcd(b,D)=1}} e^{-\pi d^2(a^4+b^4)t/D} \quad (t > 0).$$

It holds that

$$\Phi_D(t_0) = \frac{3\kappa}{\pi^2\sqrt{t_0}} \left\{ \prod_{p|D} \frac{(1-1/p)^2}{1+1/p} \right\} \sqrt{D}\log D + O(\sqrt{D}(\log\log D)^3).$$

Hence,

$$S_2^{sq}(t_0, X) = \sum_{1<D\leq X}^{*} \Phi_D(t_0)$$

$$= \frac{3\kappa}{\pi^2\sqrt{t_0}} \sum_{1<D\leq X}^{*} \left\{ \prod_{p|D} \frac{(1-1/p)^2}{1+1/p} \right\} \sqrt{D}\log D + O\left(X^{3/2}(\log\log X)^3\right).$$

PROOF. The Mellin's transform of

$$\Phi_D(t) = \sum_{\substack{d \geq 1 \\ \gcd(d,D)=1}} \sum_{\substack{\delta \geq 1 \\ \gcd(\delta,D)=1}} \mu(\delta) \sum_{\substack{a,b\geq 1 \\ \gcd(a,D)=\gcd(b,D)=1}} e^{-\pi d^2\delta^4(a^4+b^4)t/D},$$

is

$$\Psi_D(s) = \int_0^\infty \Phi_D(t)t^s \frac{dt}{t}$$

$$= \sum_{\substack{d \geq 1 \\ \gcd(d,D)=1}} \sum_{\substack{\delta \geq 1 \\ \gcd(\delta,D)=1}} \mu(\delta) \sum_{\substack{a,b\geq 1 \\ \gcd(a,D)=\gcd(b,D)=1}} \left(\frac{D}{\pi d^2\delta^4(a^4+b^4)}\right)^s \Gamma(s)$$

$$= \left(\frac{D}{\pi}\right)^s \left\{ \prod_{p|D}\left(1+\frac{1}{p^{2s}}\right) \right\}^{-1} \frac{\zeta(2s)}{\zeta(4s)} H_D(s)\Gamma(s).$$

Hence,

$$\Phi_D(t_0) = M^{-1}\Psi_D(t_0) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Psi_D(s)t_0^{-s}ds = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}F_D(s)ds \quad (\alpha > 1),$$

with $F_D(s)$ defined in Lemma 20. As in the proof of Lemma 11, by moving the vertical line of integration $\Re(s) = \alpha$ to the left to the vertical line $\Re(s) = 1/4+\epsilon/2$, $\epsilon \in (0, 1/4)$, we pick up one residue, at $s = 1/2$, a double pole, and using Lemma 20, we obtain the desired result. $\qquad\square$

Lemma 22. *Set $a = \pm 1$. Let $C > 0$ be as in Proposition 18. It holds that*

$$T_a(X) := \sum_{\substack{1 < D \le X \\ D \equiv a \pmod 4}} \mu^2(D)\left\{\prod_{p|D}\frac{(1-1/p)^2}{1+1/p}\right\} \sim CX.$$

PROOF. Let χ_4 the non trivial character mod 4. We have

$$T_a(X) = \sum_{\substack{1 < d \le X \\ \gcd(d,2)=1}} \mu^2(d)\frac{1 + a\chi_4(d)}{2}\prod_{p|d}\frac{(1-1/p)^2}{1+1/p}.$$

Set $b_{a,d} = \prod_{p|d}\frac{(1-1/p)^2}{1+1/p}$ if $d \equiv a \pmod 4$ and d is square-free, and $b_{a,d} = 0$ otherwise. Then, set $F_a(s) = \sum_{d \ge 1} b_{a,d}d^{-s}$, a Dirichlet series with nonnegative coefficients. It suffices to prove that $F_a(s)$ admits an analytic continuation to $\Re(s) > 1/2$, with only one pole, a simple pole of residue C at $s = 1$. We have

$$F_a(s) = F(s) + aG(s),$$

where

$$F(s) := \frac{1}{2}\sum_{\substack{d \ge 1 \\ \gcd(d,2)=1}}\frac{\mu^2(d)}{d^s}\prod_{p|d}\frac{(1-1/p)^2}{1+1/p} = \frac{1}{2}\prod_{p \ge 3}\left(1 + \frac{(1-\frac{1}{p})^2}{1+\frac{1}{p}}\frac{1}{p^s}\right) = \Pi(s)\zeta(s)$$

with

$$\Pi(s) := \frac{1}{2}\left(1 - \frac{1}{2^s}\right)\prod_{p \ge 3}\left(1 - \frac{1}{p+1}\left(\frac{3}{p^s} - \frac{1}{p^{s+1}} + \frac{1}{p^{2s-1}} + \frac{1}{p^{2s+1}} - \frac{2}{p^{2s}}\right)\right)$$

absolutely convergent and holomorphic for $\Re(s) > 1/2$, and where

$$G(s) := \frac{1}{2}\sum_{\substack{d \ge 1 \\ gcd(d,2)=1}}\frac{\mu^2(d)\chi_4(d)}{d^s}\prod_{p|d}\frac{(1-1/p)^2}{1+1/p} = \Pi_{\chi_4}(s)L(s, \chi_4),$$

with

$$\Pi_{\chi_4}(s) := \frac{1}{2} \prod_{p \geq 3} \left(1 - \frac{1}{p+1} \left(\frac{3\chi_4(p)}{p^s} - \frac{\chi_4(p)}{p^{s+1}} + \frac{1}{p^{2s-1}} + \frac{1}{p^{2s+1}} - \frac{2}{p^{2s}} \right) \right)$$

absolutely convergent and holomorphic for $\Re(s) > 1/2$. Hence $G(s)$ is holomorphic for $\Re(s) > 1/2$ and gives no contribution into the sum. $F(s)$ is meromorphic for $\Re(s) > 1/2$, with only one pole, a simple pole at $s = 1$ of residue C and so is $F_a(s)$. We conclude using Wiener–Ikehara tauberian theorem [MV, Chapter 8, Corollary 8.8]. □

Lemma 23. *It holds that*

$$T(X) := \sum_{1 < D \leq X}^{*} \left\{ \prod_{p | D} \frac{(1 - 1/p)^2}{1 + 1/p} \right\} \sim \frac{13C}{12} X$$

PROOF. We split the sum into 4 sums: $D \equiv 1 \pmod 4$, $D = 4D'$ with $D' \equiv -1 \pmod 4$ and $D = 8D''$ with $D'' \equiv \pm 1 \pmod 4$. Hence we obtain

$$T(X) = T_1(X) + \frac{1}{6} T_{-1}\left(\frac{X}{4}\right) + \frac{1}{6} T_1\left(\frac{X}{8}\right) + \frac{1}{6} T_{-1}\left(\frac{X}{8}\right),$$

which gives the result using lemma 22. □

8. The behavior of $S_2^{nsq}(t_0, X)$

Lemma 24 (See [Jut1] and [Jut2]). *Set* $S(n, X) := \sum_{1 < D \leq X}^{*} \chi_D(n)$. *It holds that*

$$\sum_{\substack{n \leq N \\ n \ not \ a \ square}} S(n, X)^2 \ll NX \log^{10} N.$$

Lemma 25. *For* $n \geq 1$ *an integer and for* $D > 0$ *and* $t_0 > 0$, *we have*

$$g_{D,t_0}(n) := \sum_{b | n} e^{-\frac{\pi t_0}{D}\left(n^2/b^2 + b^2\right)} \leq \tau(n) e^{-2\pi n t_0 / D}$$

and

$$0 < g_{D+1,t_0}(n) - g_{D,t_0}(n) \leq \frac{2\tau(n)}{eD} e^{-\pi n t_0 / 2D}.$$

PROOF. The first assertion is straightforward, using $\alpha := \pi t_0(n^2/b^2 + b^2) \geq 2\pi n t_0$. For the second assertion, for some $\theta \in (D, D+1)$ we have

$$0 \leq e^{-\frac{\alpha}{D+1}} - e^{-\frac{\alpha}{D}} = \frac{\alpha}{\theta^2}e^{-\frac{\alpha}{\theta}} = \left(\frac{\alpha}{2\theta}e^{-\frac{\alpha}{2\theta}}\right)\frac{2}{\theta}e^{-\frac{\alpha}{2\theta}} \leq \frac{1}{e}\frac{2}{D}e^{-\frac{\alpha}{4D}} \leq \frac{2}{eD}e^{-\frac{\pi n t_0}{2D}}.$$

We get the result by summing over b. $\qquad\square$

Lemma 26. *Let α, β and γ be positive real numbers. Let a_n a sequence of real numbers such that $A_n(t) := \sum_{1 \leq n \leq t} a_n \ll t^\alpha \log^\beta t$ for $t \geq 2$. As $d \geq 2$ ranges overs the positive integers, we have*

$$\sum_{n \geq 1} a_n e^{-\gamma n/d} = \frac{\gamma}{d}\int_1^{+\infty} A_n(t)e^{-\gamma t/d}dt \ll d^\alpha \log^\beta d.$$

Lemma 27. *It holds that*

$$S_1 := \sum_{n \text{ not a square}} S(n, X)g_{D, t_0}(n) \ll X^{3/2}\log^{13/2} X$$

and

$$S_2 := \sum_{n \text{ not a square}} \sum_{1 < D \leq X}^* S(n, D)\left(g_{D, t_0}(n) - g_{D+1, t_0}(n)\right) \ll X^{3/2}\log^{13/2} X.$$

PROOF. Using Lemma 24, the Cauchy–Schwarz inequality and Lemmas 25 and 26, we obtain

$$S_1 \leq \left(\sum_{n \text{ not a square}} S(n, X)^2 e^{-2\pi n t_0/X}\right)^{1/2}\left(\sum_{n \geq 1}\tau^2(n)e^{-2\pi n t_0/X}\right)^{1/2}$$
$$\ll (X^2\log^{10} X)^{1/2}(X\log^3 X)^{1/2} \ll X^{3/2}\log^{13/2} X.$$

By Lemmas 24 and 25 and the Cauchy–Schwarz inequality we obtain

$$S_2 \leq \sum_{1 < D \leq X}^* \frac{2}{eD}\sum_{n \text{ not a square}} |S(n, D)|\,\tau(n)e^{-\pi n t_0/2D}$$
$$\leq \sum_{1 < D \leq X}^* \frac{2}{eD}\left(\sum_{n \text{ not a square}} S(n, D)^2 e^{-\pi n t_0/2D}\right)^{1/2}\left(\sum_{n \geq 1}\tau^2(n)e^{-\pi n t_0/2D}\right)^{1/2}$$
$$\ll \sum_{1 < D \leq X}^* \frac{1}{D}(D^2\log^{10} D)^{1/2}(D\log^3 D)^{1/2} \ll X^{3/2}\log^{13/2} X,$$

as claimed. $\qquad\square$

Proposition 28. *Fix $t_0 > 0$. It holds that*

$$S_2^{nsq}(t_0, X) = O\left(X^{3/2}\log^{13/2} X\right).$$

PROOF. We may assume that $X > 1$ is a positive integer. We have

$$S_2^{nsq}(t_0, X) := \sum_{1 < D \leq X}^{*} \sum_{\substack{a,b \geq 1 \\ ab \text{ not a square}}} \left(\frac{D}{ab}\right) e^{-\pi(a^2+b^2)t_0/D}$$

$$= \sum_{n \text{ not a square}} \sum_{1 < D \leq X}^{*} S(n, D) \sum_{b|n} e^{-\frac{\pi t_0}{D}\left(n^2/b^2 + b^2\right)}$$

$$= \sum_{n \text{ not a square}} \sum_{1 < D \leq X}^{*} S(n, D) g_{D,t_0}(n)$$

$$= \sum_{n \text{ not a square}} \left(S(n, X)g_{X,t_0}(n) + \sum_{1 < D \leq X}^{*} S(n, D)\left(g_{D,t_0}(n) - g_{D+1,t_0}(n)\right)\right)$$

Using lemma 27, we obtain the desired bound. □

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