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Irrationality of infinite products

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Abstract. This paper deals with a sufficient condition for the infinite product of infinite series of rational numbers to be an irrational number. The proof is based on an idea of Erdős. As an example we obtain that the number $\prod_{m=1}^{\infty}\left(1+\sum_{n=m}^{\infty}\frac{1}{2^{(n+1)!}+1}\right)$ is irrational.

1. Introduction

In 1975 Erdős [5] proved that if $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers such that $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$ then the number $\sum_{n=1}^{\infty}\frac{1}{a_n}$ is irrational. We prove the following result:

Theorem 1. *Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of positive integers with $\lim_{n\to\infty} a_n^{\frac{1}{n!}} = \infty$. Then the number $\prod_{m=1}^{\infty}\big(1+\sum_{n=0}^{\infty}\frac{1}{a_{n+m}+n}\big)$ is irrational.*

We say that the number y is Liouville if for each n there exist integers p and q such that $0<\left|y-\frac{p}{q}\right|<\frac{p}{q^n}$. The authors do not know if the number $\prod_{m=1}^{\infty}\big(1+\sum_{n=0}^{\infty}\frac{1}{2^{(n+m)!}+n}\big)$ is irrational although we know from another result of Erdős (e.g., [5] page 6, line 9) that the number $\sum_{n=1}^{\infty}\frac{1}{2^{n!}+n}$ is Liouville. We are also not able to find a sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers with $\liminf_{n\to\infty} a_n^{\frac{1}{n!}}>1$ and such that the number $\prod_{m=1}^{\infty}\big(1+\sum_{n=0}^{\infty}\frac{1}{a_{n+m}}\big)$ is rational.

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The irrationality and transcendence of infinite products has a great history. Badea [1] proved that if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that $a_{n+1} > \frac{b_{n+1}}{b_n}a_n^2 + \frac{b_{n+1}(b_n-1)}{b_n}a_n + 1 - b_{n+1}$ holds for every sufficiently large n then the number $\prod_{n=1}^{\infty}\bigl(1+\frac{b_n}{a_n}\bigr)$ is irrational. Using Brun's criterion, Laohakosol and Kuhapatanakul [13]–[15] worked in the spirit of Badea. Some approximations of the numbers $\prod_{n=1}^{\infty}\bigl(1+\frac{z}{q^n}\bigr)$ can be found in the paper of Väänänen [18]. Zhou and Lubinski [20] demostrated some irrationality results regarding $\prod_{j=0}^{\infty}(1 \pm q^{-j}r + q^{-2j}s)$. See also Hančl and Kolouch [7].

In 2000 Zhu [21] proved several criteria for infinite products to be transcendental. Nyblom [16] constructed a certain set of transcendental valued infinite products with the help of second order linear recurrence sequences. Using theta series Kim and Koo [12] described some interesting infinite products. Utilizing a result of Corvaja and Zannier [3] or [4], Corvaja and Hančl [2] established a condition for certain infinite products to be transcendental. Tachiya [17] considered infinite products in several variables of certain algebraic numbers and proved that these products are transcendental numbers. Zhou [19] worked with similar products and obtained some irrationality results. All this shows that metric properties of infinite products is of considerable current interest.

Erdős [5] (e.g., [5] page 6, line 9) proved that if $a = \{a_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers such that $\lim_{n\to\infty}\frac{1}{n}\log\log a_n = \infty$ then the expressible set $E_a = \bigl\{\sum_{n=1}^{\infty}\frac{1}{a_n c_n}, c_n \in \mathbb{N}\bigr\}$ consists only of Liouville numbers. Using this idea of Erdős, Hančl, Nair and Šustek [8] found some necessary conditions for the Lebesgue measure of E_a to be equal to zero. For other applications of the method of Erdős see e.g. [6], [9], [10] or [11]. It seems that this method still has great potential.

The main result of this paper is Theorem 2 which says that certain infinite products of infinite series are irrational numbers. Its proof is complicated but does not require any deep results. Note that the product x in Theorem 2 can contain infinitely many factors that are Liouville numbers.

Let $\mathbb{Z}^+$ be the set of all positive integers. For $n \in \mathbb{Z}^+$ and δ a real number with $0 \le \delta < 1$ let $(n+\delta)! = \prod_{j=1}^{n}(j+\delta)$. Functions $\log x$ and $\ln x$ mean logarithm of x to base 2 and e respectively.

2. Main result

Theorem 2. *Let ε be a positive real number. Assume that $(a_{n,m})_{m,n\ge1}$ and $(b_{n,m})_{m,n\ge1}$ are two infinite matrices of positive integers. Suppose that the* *sequence* $\{a_{n,1}\}_{n=1}^{\infty}$ *is non-decreasing with*

$$\limsup_{n\to\infty} a_{n,1}^{\frac{1}{n!}} = \infty \tag{1}$$

and for all sufficiently large n

$$n^{1+\varepsilon} \le a_{n,1}, \tag{2}$$

$$\sum_{j=1}^{n} \frac{b_{n-j+1,j}}{a_{n-j+1,j}} \le a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1} \tag{3}$$

and

$$\prod_{j=1}^{n} a_{n-j+1,j} \le a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}. \tag{4}$$

Then the number $x = \prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{a_{n,m}}\right)$ *is irrational.*

Example 1. As an immediate consequence of Theorem 2 we obtain that the products of the series

$$\prod_{m=1}^{\infty}\left(1+\sum_{n=m}^{\infty}\frac{1}{2^{(n+1)!}}\right) = \prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{1}{2^{(n+m)!}}\right),$$

$$\prod_{m=1}^{\infty}\left(1+\sum_{n=m}^{\infty}\frac{1}{2^{(n+1)!}+1}\right) = \prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{1}{2^{(n+m)!}+1}\right),$$

$$\prod_{m=1}^{\infty}\left(1+\sum_{n=m}^{\infty}\frac{n}{2^{(n+1)!}+m}\right) = \prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{n+m}{2^{(n+m)!}+m}\right),$$

$$\prod_{m=1}^{\infty}\left(1+\sum_{n=m}^{\infty}\frac{1}{2^{n^n}}\right) = \prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{1}{2^{(n+m)^{n+m}}}\right)$$

and

$$\prod_{m=1}^{\infty}\left(1+\sum_{n=m}^{\infty}\frac{n!}{2^{n^n}+m^n}\right) = \prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{(n+m)!}{2^{(n+m)^{n+m}}+m^{n+m}}\right)$$

are irrational numbers.

Remark 1. Let us note that if we omit finite number of the terms in the sequence of the Erdős theorem then it does not have any influence on the irrationality. On the other side it is more complicate in the case of Theorems 1 and 2 since the products consist of the irrational numbers.

3. Proofs

Theorem 1 is an immediate consequence of Theorem 2 when we set $b_{n,m} = 1$ and $a_{n,m} = a_{n+m-1} + n - 1$ for all $n, m \in \mathbb{Z}^+$. Then

$$\sum_{j=1}^{N} \frac{b_{N-j+1,j}}{a_{N-j+1,j}} = \sum_{j=1}^{N} \frac{1}{a_N + N - j} \le \frac{2N}{a_N + N - 1} \le a_{N,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{N,1}} - 1}$$

and

$$\prod_{j=1}^{N} a_{N-j+1,j} = \prod_{j=1}^{N} (a_N + N - j) \le \prod_{j=1}^{N} (a_N + N - 1) \le a_{N,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{N,1}} + N}$$

hold for every sufficiently large N since $a_{N,1} = a_N + N - 1 > 2^{N!}$.

Lemma 1. *Let the sequence $\{a_{n,1}\}_{n=1}^{\infty}$ satisfy all conditions stated in Theorem 2. Then*

$$\sum_{j=0}^{\infty} a_{n+j,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{n+j,1}} - 1} < a_{n,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}} \tag{5}$$

holds for every sufficiently large n.

Proof. (of Lemma 1)
From (2) and the fact that the sequence $\{a_{n,1}\}_{n=1}^{\infty}$ is non-decreasing we obtain

$$\begin{aligned}
\sum_{j=0}^{\infty} a_{n+j,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{n+j,1}} - 1} &= \sum_{n+j < a_{n,1}^{\frac{1}{1+\varepsilon}}} a_{n+j,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{n+j,1}} - 1} + \sum_{n+j \ge a_{n,1}^{\frac{1}{1+\varepsilon}}} a_{n+j,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{n+j,1}} - 1} \\
&\le a_{n,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{n,1}} - 1} a_{n,1}^{\frac{1}{1+\varepsilon}} + \sum_{n+j \ge a_{n,1}^{\frac{1}{1+\varepsilon}}} a_{n+j,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{n+j,1}} - 1} \\
&\le a_{n,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{n,1}} - 1} a_{n,1}^{\frac{1}{1+\varepsilon}} + \sum_{n+j \ge a_{n,1}^{\frac{1}{1+\varepsilon}}} (n+j)^{(1+\varepsilon)\left(\frac{1}{\log^{3+\varepsilon} \log (n+j)^{1+\varepsilon}} - 1\right)} \\
&\le a_{n,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{n,1}} - 1} a_{n,1}^{\frac{1}{1+\varepsilon}} + \sum_{n+j \ge a_{n,1}^{\frac{1}{1+\varepsilon}}} (n+j)^{-(1+\frac{2\varepsilon}{3})} \le a_{n,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}}. \qquad \square
\end{aligned}$$

Lemma 2. *Let the sequence $\{a_{n,1}\}_{n=1}^{\infty}$ satisfy all conditions stated in Theorem 2 and instead of (2) we have*

$$2^n < a_{n,1} \tag{6}$$

for every sufficiently large n. Then

$$\sum_{j=0}^{\infty} a_{n+j,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n+j,1}}-1} < a_{n,1}^{\frac{1}{\log^{3+\frac{\varepsilon}{2}}\log a_{n,1}}-1} \tag{7}$$

holds for every sufficiently large n.

Proof. (of Lemma 2)
From (6) and the fact that the sequence $\{a_{n,1}\}_{n=1}^{\infty}$ is non-decreasing we obtain

$$\begin{aligned}
\sum_{j=0}^{\infty} a_{n+j,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n+j,1}}-1} &= \sum_{n+j<\log a_{n,1}} a_{n+j,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n+j,1}}-1} + \sum_{n+j\ge\log a_{n,1}} a_{n+j,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n+j,1}}-1}\\
&\le a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1}\log a_{n,1} + \sum_{n+j\ge\log a_{n,1}} a_{n+j,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n+j,1}}-1}\\
&\le a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1}\log a_{n,1} + \sum_{n+j\ge\log a_{n,1}} 2^{(n+j)\left(\frac{1}{\log^{3+\varepsilon}\log 2^{(n+j)}}-1\right)}\\
&\le a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1}\log a_{n,1} + \int_{\log a_{n,1}}^{\infty} 2^{u\left(\frac{1}{\log^{3+\frac{2\varepsilon}{3}}\log 2^{u}}-1\right)}\,du.
\end{aligned} \tag{8}$$

We have for sufficiently large x

$$\begin{aligned}
&\left(-2^{x\left(\frac{1}{\log^{3+\frac{3\varepsilon}{5}}\log 2^{x}}-1\right)}\right)'\\
&= -(\ln 2)2^{x\left(\frac{1}{\log^{3+\frac{3\varepsilon}{5}}\log 2^{x}}-1\right)}\left(\frac{1}{\log^{3+\frac{3\varepsilon}{5}}\log 2^{x}} - 1 - \frac{\frac{3+\frac{3\varepsilon}{5}}{\ln 2}}{\log^{4+\frac{3\varepsilon}{5}}\log 2^{x}}\right)\\
&\ge 2^{x\left(\frac{1}{\log^{3+\frac{2\varepsilon}{3}}\log 2^{x}}-1\right)}.
\end{aligned}$$

Hence

$$\int_{\log a_{n,1}}^{\infty} 2^{u\left(\frac{1}{\log^{3+\frac{2\varepsilon}{3}}\log 2^{u}}-1\right)} \le a_{n,1}^{\frac{1}{\log^{3+\frac{3\varepsilon}{5}}\log a_{n,1}}-1}.$$

From this and (8) we obtain

$$\begin{aligned}
\sum_{j=0}^{\infty} a_{n+j,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n+j,1}}-1} &\le a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1}\log a_{n,1} + \int_{\log a_{n,1}}^{\infty} 2^{u\left(\frac{1}{\log^{3+\frac{2\varepsilon}{3}}\log 2^{u}}-1\right)}du\\
&\le a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1}\log a_{n,1} + a_{n,1}^{\frac{1}{\log^{3+\frac{3\varepsilon}{5}}\log a_{n,1}}-1} \le a_{n,1}^{\frac{1}{\log^{3+\frac{\varepsilon}{2}}\log a_{n,1}}-1}. \qquad \square
\end{aligned}$$

Lemma 3. *Let δ be a real number with $0 \le \delta < 1$ and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive real numbers such that*

$$\lim_{n\to\infty} \sup a_n^{\frac{1}{(n+\delta)!}} = \infty. \tag{9}$$

Then for infinitely many N

$$a_{N+1}^{\frac{1}{(N+1+\delta)!}} > \left(1+\frac{1}{N^2}\right) \max_{k=1,\dots,N} a_k^{\frac{1}{(k+\delta)!}} \tag{10}$$

and

$$a_{N+1} > \left(1+\frac{1}{N^2}\right)^{(N+1+\delta)!} \left(\prod_{n=1}^{N} a_n^n\right)\left(\prod_{n=1}^{N} a_n\right)^{\delta}. \tag{11}$$

Proof. (of Lemma 3)
From (9) we obtain that there exist infinitely many N such that (10) holds otherwise there exists N_0 such that for each $N > N_0$

$$\begin{aligned}
a_N^{\frac{1}{(N+\delta)!}} &\le \left(1+\frac{1}{(N-1)^2}\right) \max_{k=1,\dots,N-1} a_k^{\frac{1}{(k+\delta)!}} \\
&\le \left(1+\frac{1}{(N-1)^2}\right)\left(1+\frac{1}{(N-2)^2}\right) \max_{k=1,\dots,N-2} a_k^{\frac{1}{(k+\delta)!}} < \cdots \\
&\le \left(1+\frac{1}{(N-1)^2}\right)\left(1+\frac{1}{(N-2)^2}\right)\cdots\left(1+\frac{1}{N_0^2}\right) \max_{k=1,\dots,N_0} a_k^{\frac{1}{(k+\delta)!}} \\
&\le 5 \max_{k=1,\dots,N_0} a_k^{\frac{1}{(k+\delta)!}},
\end{aligned}$$

which contradicts (9). From (10) we obtain that for infinitely many N

$$\begin{aligned}
&a_{N+1} > (1+\frac{1}{N^2})^{(N+1+\delta)!}\left(\max_{k=1,\dots,N} a_k^{\frac{1}{(k+\delta)!}}\right)^{(N+1+\delta)!} \\
&> \left(1+\frac{1}{N^2}\right)^{(N+1+\delta)!}\left(\max_{k=1,\dots,N} a_k^{\frac{1}{(k+\delta)!}}\right)^{(N+\delta)(N+\delta)!+(N-1+\delta)(N-1+\delta)!+\cdots+(1+\delta)(1+\delta)!} \\
&> \left(1+\frac{1}{N^2}\right)^{(N+1+\delta)!}\left(\prod_{n=1}^{N} a_n^n\right)\left(\prod_{n=1}^{N} a_n\right)^{\delta}. \qquad \square
\end{aligned}$$

Proof. (of Theorem 2)
Assume that the number x is a positive rational number. Then there exists $(p,q) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $x = \frac{p}{q}$. So for each $(P,Q) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ the number

$\left|qQ(x-\frac{P}{Q})\right| = \left|qQ\left(\frac{p}{q}-\frac{P}{Q}\right)\right| = |pQ-Pq|$ is an integer. To prove our theorem it is enough to find $(P,Q) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that

$$0 < H = \left|qQ\left(x - \frac{P}{Q}\right)\right| < 1. \tag{12}$$

Let N be a sufficiently large positive integer. Set $Q_N = \prod_{m=1}^{N}\prod_{n=1}^{N-m+1} a_{n,m}$ and $P_N = \left(\prod_{m=1}^{N}\prod_{n=1}^{N-m+1} a_{n,m}\right)\prod_{m=1}^{N}\left(1+\sum_{n=1}^{N-m+1}\frac{b_{n,m}}{a_{n,m}}\right)$. Then we have

$$\begin{aligned}
0 < H_N &= \left|qQ_N\left(x - \frac{P_N}{Q_N}\right)\right| \\
&= \left|q\left(\prod_{m=1}^{N}\prod_{n=1}^{N-m+1} a_{n,m}\right)\left(\prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{a_{n,m}}\right) - \prod_{m=1}^{N}\left(1+\sum_{n=1}^{N-m+1}\frac{b_{n,m}}{a_{n,m}}\right)\right)\right| \\
&= qP_N\left(\left(\prod_{m=1}^{N}\left(1+\frac{\sum_{n=N-m+2}^{\infty}\frac{b_{n,m}}{a_{n,m}}}{1+\sum_{n=1}^{N-m+1}\frac{b_{n,m}}{a_{n,m}}}\right)\right)\left(\prod_{m=N+1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{a_{n,m}}\right)\right)-1\right).
\end{aligned}$$

From this and the fact that $x \geq \frac{P_N}{Q_N}$ we obtain that

$$\begin{aligned}
H_N \leq qQ_N x\Bigg(&\left(\prod_{m=1}^{N}\left(1+\frac{\sum_{n=N-m+2}^{\infty}\frac{b_{n,m}}{a_{n,m}}}{1+\sum_{n=1}^{N-m+1}\frac{b_{n,m}}{a_{n,m}}}\right)\right) \\
&\left(\prod_{m=N+1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{a_{n,m}}\right)\right)-1\Bigg).
\end{aligned} \tag{13}$$

The facts that N is sufficiently large and that the series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_{n,m}}{a_{n,m}}$ converges absolutely imply that

$$\sum_{m=1}^{N}\sum_{n=N-m+2}^{\infty}\frac{b_{n,m}}{a_{n,m}} + \sum_{m=N+1}^{\infty}\sum_{n=1}^{\infty}\frac{b_{n,m}}{a_{n,m}} \leq 1. \tag{14}$$

From (13) and (14) we obtain that

$$\begin{aligned}
H_N &\leq qQ_N x\left(\left(\prod_{m=1}^{N}\left(1+\frac{\sum_{n=N-m+2}^{\infty}\frac{b_{n,m}}{a_{n,m}}}{1+\sum_{n=1}^{N-m+1}\frac{b_{n,m}}{a_{n,m}}}\right)\right)\left(\prod_{m=N+1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{a_{n,m}}\right)\right)-1\right) \\
&= qQ_N x\left(\mathrm{e}^{\ln\left(\left(\prod_{m=1}^{N}\left(1+\frac{\sum_{n=N-m+2}^{\infty}\frac{b_{n,m}}{a_{n,m}}}{1+\sum_{n=1}^{N-m+1}\frac{b_{n,m}}{a_{n,m}}}\right)\right)\left(\prod_{m=N+1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{a_{n,m}}\right)\right)\right)}-1\right)
\end{aligned}$$

$$= qQ_Nx\left(\mathrm{e}^{\sum_{m=1}^{N}\ln\left(1+\frac{\sum_{n=N-m+2}^{\infty}\frac{b_{n,m}}{a_{n,m}}}{1+\sum_{n=1}^{N-m+1}\frac{b_{n,m}}{a_{n,m}}}\right)+\sum_{m=N+1}^{\infty}\ln\left(1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{a_{n,m}}\right)}-1\right)$$

$$\leq qQ_Nx\left(\mathrm{e}^{\sum_{m=1}^{N}\sum_{n=N-m+2}^{\infty}\frac{b_{n,m}}{a_{n,m}}+\sum_{m=N+1}^{\infty}\sum_{n=1}^{\infty}\frac{b_{n,m}}{a_{n,m}}}-1\right)$$

This and (14) imply that there exists positive real number K which does not depend on N and such that

$$H_N \leq KqQ_Nx\left(\sum_{m=1}^{N}\sum_{n=N-m+2}^{\infty}\frac{b_{n,m}}{a_{n,m}}+\sum_{m=N+1}^{\infty}\sum_{n=1}^{\infty}\frac{b_{n,m}}{a_{n,m}}\right)$$
$$= KqQ_Nx\sum_{n=N+1}^{\infty}\sum_{j=1}^{n}\frac{b_{n-j+1,j}}{a_{n-j+1,j}}.$$

From this, (3), (4) and the definition of Q_N we obtain that

$$H_N \leq KqQ_Nx\sum_{n=N+1}^{\infty}\sum_{j=1}^{n}\frac{b_{n-j+1,j}}{a_{n-j+1,j}} \leq Kqx\prod_{m=1}^{N}\prod_{n=1}^{N-m+1}a_{n,m}\sum_{n=N+1}^{\infty}a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1}$$
$$\leq Kqx\prod_{n=1}^{N}a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\sum_{n=N+1}^{\infty}a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1}. \tag{15}$$

Now the proof falls into several cases.

1. Let us assume that (6) holds for every sufficiently large n and there is a real number δ with $0<\delta<1$ and such that

$$\lim_{n\to\infty}\sup a_{n,1}^{\frac{1}{(n+\delta)!}} = \infty. \tag{16}$$

This and Lemma 3 imply that there exist infinitely many N such that

$$a_{N+1,1} > \left(1+\frac{1}{N^2}\right)^{(N+1+\delta)!}\left(\prod_{n=1}^{N}a_{n,1}^{n}\right)\left(\prod_{n=1}^{N}a_{n,1}\right)^{\delta}.$$

From this, Lemma 2, (15) and Stirling factorial formula we obtain that for infinitely many sufficiently large N

$$0 < H_N < Kqx\prod_{n=1}^{N}a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\sum_{n=N+1}^{\infty}a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1}$$

$$\leq Kqx\Bigg(\prod_{n=1}^{N} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\Bigg) a_{N+1,1}^{\frac{1}{\log^{3+\frac{\varepsilon}{2}}\log a_{N+1,1}}-1}$$

$$\leq Kqx\Bigg(\prod_{n=1}^{N} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\Bigg) a_{N+1,1}^{\frac{1}{\log^{3+\frac{\varepsilon}{2}}\log\left(\left(1+\frac{1}{N^2}\right)^{(N+1+\delta)!}\left(\prod_{n=1}^{N} a_{n,1}^{n}\right)\left(\prod_{n=1}^{N} a_{n,1}\right)^{\delta}\right)}-1}$$

$$\leq \Bigg(\prod_{n=1}^{N} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\Bigg) a_{N+1,1}^{\frac{1}{N^{3+\frac{\varepsilon}{2}}}-1}$$

$$\leq \Bigg(\prod_{n=1}^{N} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\Bigg)\Bigg(\Big(1+\frac{1}{N^2}\Big)^{(N+1+\delta)!}\Bigg(\prod_{n=1}^{N} a_{n,1}^{n}\Bigg)\Bigg(\prod_{n=1}^{N} a_{n,1}\Bigg)^{\delta}\Bigg)^{\frac{1}{N^{3+\frac{\varepsilon}{2}}}-1}$$

$$= \Bigg(\prod_{n=1}^{N} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+\frac{n+\delta}{N^{3+\frac{\varepsilon}{2}}}-\delta}\Bigg)\Bigg(\Big(1+\frac{1}{N^2}\Big)^{(N+1+\delta)!}\Bigg)^{\frac{1}{N^{3+\frac{\varepsilon}{2}}}-1} \leq \prod_{n=1}^{N} a_{n,1}^{-\frac{\delta}{2}} < 1.$$

So (12) holds when we set $P = P_N$, $Q = Q_N$, and $H = H_N$.

2. Let us assume that (6) holds for every sufficiently large n and there is not a real number δ with $1 > \delta > 0$ and such that (16) holds. From this we see that for every $\delta > 0$

$$a_{n,1} < 2^{(n+\delta)!} \tag{17}$$

holds for every sufficiently large n. Let δ be sufficiently small. Lemma 3 and (1) imply that for infinitely many N

$$a_{N+1,1} > (1+\frac{1}{N^2})^{(N+1)!}\Bigg(\prod_{n=1}^{N} a_{n,1}^{n}\Bigg).$$

This, Lemma 2 and (15) imply that for infinitely many N

$$0 < H_N \leq Kqx\prod_{n=1}^{N} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\sum_{n=N+1}^{\infty} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1}$$

$$\leq Kqx\Bigg(\prod_{n=1}^{N} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\Bigg) a_{N+1,1}^{\frac{1}{\log^{3+\frac{\varepsilon}{2}}\log a_{N+1,1}}-1}$$

$$\leq \Bigg(\prod_{n=1}^{N} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\Bigg) a_{N+1,1}^{\frac{1}{N^{3+\frac{\varepsilon}{2}}}-1}$$

$$\leq \Bigg(\prod_{n=1}^{N} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\Bigg)\Bigg(\Big(1+\frac{1}{N^2}\Big)^{(N+1)!}\Bigg(\prod_{n=1}^{N} a_{n,1}^{n}\Bigg)\Bigg)^{\frac{1}{N^{3+\frac{\varepsilon}{2}}}-1}$$

$$\leq \Bigg(\prod_{n=1}^{N} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+\frac{n}{N^{3+\frac{\varepsilon}{2}}}}\Bigg)\Bigg(\Big(1+\frac{1}{N^2}\Big)^{(N+1)!}\Bigg)^{\frac{1}{N^{3+\frac{\varepsilon}{2}}}-1}.$$

From this, the fact that N is sufficiently large and (17) we obtain that for infinitely many N

$$0 < H_N \leq \left(\prod_{n=1}^{N} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+\frac{n}{N^{3+\frac{\varepsilon}{2}}}}\right)\left(\left(1+\frac{1}{N^2}\right)^{(N+1)!}\right)^{\frac{1}{N^{3+\frac{\varepsilon}{2}}}-1}$$
$$\leq \left(\prod_{n=1}^{N} 2^{(n+\delta)!\left(\frac{1}{\log^{3+\varepsilon}\log 2^{(n+\delta)!}}+\frac{n}{N^{3+\frac{\varepsilon}{2}}}\right)}\right)\left(\left(1+\frac{1}{N^2}\right)^{(N+1)!}\right)^{\frac{2}{N^{3+\frac{\varepsilon}{2}}}-1} < 1.$$

So (12) holds when we set $P = P_N$, $Q = Q_N$, and $H = H_N$.

3. Now let us assume that for infinitely many n

$$a_{n,1} \leq 2^n \tag{18}$$

and there is a real number δ with $0 < \delta < 1$ such that (16) holds. Let A be a sufficiently large positive integer and δ sufficiently small. From (16) we see that there exists n such that

$$a_{n,1}^{\frac{1}{(n+\delta)!}} > A. \tag{19}$$

Let k be the least positive integer satisfying (19) and s be the greatest positive integer less than k such that (18) holds. So

$$a_{k,1} > A^{(k+\delta)!} = 2^{(\log_2 A)(k+\delta)!}. \tag{20}$$

Then there is a positive integer n such that

$$a_{n,1}^{\frac{1}{(n+\delta)!}} > 2. \tag{21}$$

Let t be the least positive integer greater than s such that (21) holds. It follows that for every $r = s, s+1, \dots, t-1$

$$a_{r,1} < 2^{(r+\delta)!} \tag{22}$$

and

$$a_{t,1} > 2^{(t+\delta)!}. \tag{23}$$

Let us note that k, s and t depend on A and if A tends to infinity then also k, s and t tend to infinity. From (18), (22) and the fact that the sequence $\{a_{n,1}\}_{n=1}^{\infty}$ is non-decreasing we obtain that

$$\prod_{n=1}^{t-1} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n} = \left(\prod_{n=1}^{s} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\right)\left(\prod_{n=s+1}^{t-1} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\right)$$

$$\leq \left(\prod_{n=1}^{s} 2^{s\left(\frac{1}{\log^{3+\varepsilon}\log 2^s}+s\right)}\right)\left(\prod_{n=s+1}^{t-1} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\right)$$

$$\leq 2^{s^3}\left(\prod_{n=s+1}^{t-1} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n}\right) \leq 2^{s^3}\left(\prod_{n=s+1}^{t-1} 2^{(n+\delta)!\left(\frac{1}{\log^{3+\varepsilon}\log 2^{(n+\delta)!}}+n\right)}\right)$$

$$\leq 2^{(t+\delta)!-\frac{\delta}{2}(t-1+\delta)!}. \tag{24}$$

Lemma 1, Lemma 2 and (23) imply

$$\sum_{n=t}^{\infty} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1} = \sum_{n=t}^{k-1} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1} + \sum_{n=k}^{\infty} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1}$$

$$\leq a_{t,1}^{\frac{1}{\log^{3+\frac{\varepsilon}{2}}\log a_{t,1}}-1} + a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}} \leq 2^{(t+\delta)!\left(\frac{1}{\log^{3+\frac{\varepsilon}{2}}\log 2^{(t+\delta)!}}-1\right)} + a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}}$$

$$\leq 2^{(t-2+\delta)!-(t+\delta)!} + a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}}.$$

From this, (15), (20) and (24) we obtain

$$0 < H_{t-1} \leq Kqx \prod_{n=1}^{t-1} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n} \sum_{n=t}^{\infty} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1}$$

$$\leq Kqx2^{(t+\delta)!-\frac{\delta}{2}(t-1+\delta)!}\left(2^{(t-2+\delta)!-(t+\delta)!} + a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}}\right)$$

$$\leq Kqx2^{(t+\delta)!-\frac{\delta}{2}(t-1+\delta)!}\left(2^{(t-2+\delta)!-(t+\delta)!} + 2^{-\frac{\varepsilon}{2(1+\varepsilon)}(\log_2 A)(k+\delta)!}\right)$$

$$= Kqx\left(2^{(1-\frac{\delta}{2}(t-1+\delta))(t-2+\delta)!} + 2^{-\frac{\varepsilon}{2(1+\varepsilon)}(\log_2 A)(k+\delta)!+(t+\delta)!-\frac{\delta}{2}(t-1+\delta)!}\right) < 1$$

when we take sufficiently large t and A. So (12) holds when we set $P = P_{t-1}$, $Q = Q_{t-1}$, and $H = H_{t-1}$.

4. Finally let us assume that for infinitely many n inequality (18) holds and there is no real number δ with $1 > \delta > 0$ and such that (16) holds. This implies that for every $\delta > 0$ and sufficiently large n inequality (17) holds. Let δ be sufficiently small and A sufficiently large. From (1) we obtain

$$a_{n,1}^{\frac{1}{n!}} > A \tag{25}$$

for infinitely many n. Let k be the least positive integer satisfying (25). Then

$$a_{k,1} > A^{k!} = 2^{(\log_2 A)k!}. \tag{26}$$

Let s be the greatest positive integer less than k such that (18) holds. From (1) and Lemma 3 we obtain that (10) with $\delta = 0$ holds for infinitely many N. Let t be the least positive integer greater than s satisfying

$$a_{t,1}^{\frac{1}{t!}} > \left(1 + \frac{1}{t^2}\right) \max_{j=s,\ldots,t-1} a_{j,1}^{\frac{1}{j!}} \tag{27}$$

and

$$a_{r,1}^{\frac{1}{r!}} \leq \left(1 + \frac{1}{r^2}\right) \max_{j=s,\ldots,r-1} a_{j,1}^{\frac{1}{j!}} \tag{28}$$

for every $r = s+1, \ldots, t-1$. Inequality (27) and the fact that $a_{r,1} \leq 2^s$ for all $r = 1, 2, \ldots, s$ yield

$$\begin{aligned} a_{t,1} &> \left(\left(1 + \frac{1}{t^2}\right) \max_{j=s,\ldots,t-1} a_{j,1}^{\frac{1}{j!}}\right)^{t!} = \left(1 + \frac{1}{t^2}\right)^{t!} \left(\max_{j=s,\ldots,t-1} a_{j,1}^{\frac{1}{j!}}\right)^{t!} \\ &\geq \left(1 + \frac{1}{t^2}\right)^{t!} \left(\max_{j=s,\ldots,t-1} a_{j,1}^{\frac{1}{j!}}\right)^{(t-1)!(t-1)+(t-2)!(t-2)+\cdots+(s+1)!(s+1)} 2^{s!} \\ &\geq \left(1 + \frac{1}{t^2}\right)^{t!} \left(\prod_{r=1}^{t-1} a_{r,1}^{r}\right) 2^{s!-s^3} \geq \left(1 + \frac{1}{t^2}\right)^{t!} \left(\prod_{r=1}^{t-1} a_{r,1}^{r}\right). \end{aligned} \tag{29}$$

From (28) we obtain

$$\begin{aligned} a_{r,1}^{\frac{1}{r!}} \leq& \left(1 + \frac{1}{r^2}\right) \max_{j=s,\ldots,r-1} a_{j,1}^{\frac{1}{j!}} \leq \left(1 + \frac{1}{r^2}\right)\left(1 + \frac{1}{(r-1)^2}\right) \max_{j=s,\ldots,r-2} a_{j,1}^{\frac{1}{j!}} \\ &\leq \cdots \leq \prod_{j=s+1}^{r} \left(1 + \frac{1}{j^2}\right) a_{s,1}^{\frac{1}{s!}} \leq D, \end{aligned}$$

where $D < 2\prod_{j=1}^{\infty}\left(1 + \frac{1}{j^2}\right)$ is a positive real constant which does not depend on A and k. It follows that

$$a_{r,1} \leq D^{r!} = 2^{(\log_2 D) r!} \tag{30}$$

for every $r = s+1, \ldots, t-1$. From this together with $a_{s,1} < 2^s$ and the fact that the sequence $\{a_{n,1}\}_{n=1}^{\infty}$ is nondecreasing, we obtain

$$\begin{aligned} \prod_{r=1}^{t-1} a_{r,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{r,1}} + r} =& \left(\prod_{r=1}^{s} a_{r,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{r,1}} + r}\right)\left(\prod_{r=s+1}^{t-1} a_{r,1}^{\frac{1}{\log^{3+\varepsilon} \log a_{r,1}} + r}\right) \\ \leq& \left(\prod_{r=1}^{s} 2^{2s^2}\right)\left(\prod_{r=s+1}^{t-1} 2^{(\log_2 D)(r!r+(r-3)!)}\right) \\ \leq& 2^{s^3} 2^{(\log_2 D)(t!+(t-3)!-s!)} \leq 2^{(\log_2 D)(t!+(t-3)!)}. \end{aligned} \tag{31}$$

Similarly

$$\prod_{r=1}^{t-1} a_{r,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{r,1}}} \le 2^{(\log_2 D)(t-3)!} \tag{32}$$

and

$$\prod_{r=1}^{t-1} a_{r,1}^{r} \le 2^{(\log_2 D)t!}. \tag{33}$$

Note that (26), (30) and the definitions of k, t and s yield that $s < t \le k$ and if A tends to infinity then also k, t and s tend to infinity. Lemma 1, Lemma 2, (29) and (33) imply

$$\begin{aligned}
\sum_{n=t}^{\infty} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1} &= \sum_{n=t}^{k-1} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1} + \sum_{n=k}^{\infty} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1} \\
&\le a_{t,1}^{\frac{1}{\log^{3+\frac{\varepsilon}{2}}\log a_{t,1}}-1} + a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}} \\
&\le \left(\left(1+\frac{1}{t^2}\right)^{t!}\left(\prod_{r=1}^{t-1} a_{r,1}^{r}\right)\right)^{\frac{1}{\log^{3+\frac{\varepsilon}{2}}\log\left(\left(1+\frac{1}{t^2}\right)^{t!}\left(\prod_{r=1}^{t-1} a_{r,1}^{r}\right)\right)}-1} + a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}} \\
&\le \left(\left(1+\frac{1}{t^2}\right)^{t!}\left(\prod_{r=1}^{t-1} a_{r,1}^{r}\right)\right)^{\frac{1}{t^{3+\frac{\varepsilon}{2}}}-1} + a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}} \\
&= \left(1+\frac{1}{t^2}\right)^{t!\left(\frac{1}{t^{3+\frac{\varepsilon}{2}}}-1\right)} \left(\prod_{r=1}^{t-1} a_{r,1}^{r}\right)^{\frac{1}{t^{3+\frac{\varepsilon}{2}}}} \left(\prod_{r=1}^{t-1} a_{r,1}^{r}\right)^{-1} + a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}} \\
&\le \left(1+\frac{1}{t^2}\right)^{t!\left(\frac{1}{t^{3+\frac{\varepsilon}{2}}}-1\right)} \left(2^{(\log_2 D)t!}\right)^{\frac{1}{t^{3+\frac{\varepsilon}{2}}}} \left(\prod_{r=1}^{t-1} a_{r,1}^{r}\right)^{-1} + a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}} \\
&\le 2^{-\frac{1}{2}(t-2)!}\left(\prod_{r=1}^{t-1} a_{r,1}^{r}\right)^{-1} + a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}}.
\end{aligned}$$

From this, (15), (31) and (32) we obtain

$$\begin{aligned}
0 < H_{t-1} &\le Kqx \prod_{n=1}^{t-1} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n} \sum_{n=t}^{\infty} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}-1} \\
&\le Kqx \prod_{n=1}^{t-1} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n} \left(2^{-\frac{1}{2}(t-2)!}\left(\prod_{r=1}^{t-1} a_{r,1}^{r}\right)^{-1} + a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}}\right) \\
&= Kqx\left(\left(\prod_{n=1}^{t-1} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}}\right)2^{-\frac{1}{2}(t-2)!} + \prod_{n=1}^{t-1} a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}}+n} a_{k,1}^{-\frac{\varepsilon}{2(1+\varepsilon)}}\right) \\
&\le Kqx\left(2^{(\log_2 D)(t-3)!}2^{-\frac{1}{2}(t-2)!} + 2^{(\log_2 D)(t!+(t-3)!)}2^{-\frac{\varepsilon}{2(1+\varepsilon)}(\log_2 A)k!}\right) < 1.
\end{aligned}$$

So (12) holds when we set $P = P_{t-1}$, $Q = Q_{t-1}$, and $H = H_{t-1}$. $\square$

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References

[1] C. Badea, The irrationality of certain infinite products, *Studia Univ. Babeş-Bolyai Math.* **31**, no. 3 (1986), 3–8.

[2] P. Corvaja and J. Hančl, A transcendence criterion for infinite products, *Atti Acad. Naz. Lincei. Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **18**, no. 3 (2007), 295–303.

[3] P. Corvaja and U. Zannier, On the rational approximation to the powers of an algebraic number: Solution of two problems of Mahler and Mendès France, *Acta Math.* **193** (2004), 175–191.

[4] P. Corvaja and U. Zannier, Some new applications of the subspace theorem, *Comp. Math.* **131** (2002), 319–340.

[5] P. Erdős, Some problems and results on the irrationality of the sum of infinite series, *J. Math. Sci.* **10** (1975), 1–7.

[6] M. Genčev, Evaluation of infinite series involving special products and their algebraic characterization, *Math. Slovaca* **59**, no. 3 (2009), 365–378.

[7] J. Hančl and O. Kolouch, Erdős' method for determining the irrationality of products, *Bull. Aust. Math. Soc.* **84**, no. 3 (2011), 414–424.

[8] J. Hančl and R. Nair and J. Šustek, On the Lebesgue Measure of the Expressible Sets of Certain Sequences, *Indag. Mathem., N.S.* **17**, no. 4 (2006), 567–581.

[9] J. Hančl and P. Rucki and J. Šustek, A Generalization of Sándor's Theorem Using Iterated Logarithms, *Kumamoto J. Math.* **19** (2006), 25–36.

[10] J. Hančl and S. Sobková, Special linearly unrelated sequences, *J. Math. Kyoto Univ.* **46**, no. 1 (2006), 31–45.

[11] J. Hančl and J. Štěpnička and J. Šustek, Linearly Unrelated Sequences and Erdős' Problem, *Ramanujan J.* **17**, no. 3 (2008), 331–342.

[12] D. Kim and J. K. Koo, On the infinite products derived from theta series I, *J. Korean Math. Soc.* **44**, no. 1 (2007), 55–107.

[13] V. Laohakosol and K. Kuhapatanakul, The reverse irrationality criteria of Brun and Badea, *East-West J. Math., Special Vol.* (2008), 217–234.

[14] V. Laohakosol and K. Kuhapatanakul, The irrationality criteria of Brun and Badea are essentially equivalent, Diophantine analysis and related fields – DARF 2007/2008, AIP Conf. Proc., 976, *Amer. Inst. Phys.*, *Melville, NY*, 2008, 144–159.

[15] V. Laohakosol and K. Kuhapatanakul, Irrationality criteria for infinite products, *J. Comb. Number Theory* **1**, no. 1 (2008), 49–57.

[16] M. A. Nyblom, On the construction of a family of transcendental valued infinite products, *Fibonacci Quart.* **42**, no. 4 (2004), 353–358.

[17] Y. Tachiya, Transcendence of the values of infinite products in several variables, *Results Math.* **48**, no. 3-4 (2005), 344–370.

[18] K. Väänänen, On the approximation of certain infinite products, *Math. Scand.* **73**, no. 2 (1993), 197–208.

[19] P. Zhou, On the irrationality of a certain multivariable infinite product, *Quaest. Math.* **29**, no. 3 (2006), 351–365.

[20] P. Zhou and D. S. Lubinski, On the irrationality of $\prod_{j=0}^{\infty}(1 \pm q^{-j}r + q^{-2j}s)$, *Analysis* **17**, no. 2-3 (1997), 129–153.

[21] Y. Ch. Zhu, Transcendence of certain infinite products, *Acta Math. Sinica* **43**, no. 4 (2000), 605–610.

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