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Arithmetic progressions and Pellian equations

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Abstract. We consider arithmetic progressions on Pellian equations $x^2-d\,y^2=m$, i.e. integral solutions such that the y-coordinates are in arithmetic progression. We construct explicit infinite families of d,m for which there exists a five-term arithmetic progression in the y-coordinate, and we prove the existence of infinitely many pairs d, m parametrized by points of an elliptic curve of positive rank for which the corresponding Pellian equations have solutions whose y-component form a six-term arithmetic progression. Then we exhibit many six-term progressions whose elements are the y-components of solutions for an equation of the form $x^2-d\,y^2=m$ with small coefficients d, m and also several particular seven-term examples. Finally we show a procedure for searching five-term arithmetic progressions for which there exist a couple of pairs (d_1,m_1) and (d_2,m_2) for which the progression is a solution of the associated Pellian equations. These results extend and complement recent results of Dujella, Pethő and Tadić, and Pethő and Ziegler.

1. Introduction

The existence of arithmetic progressions in sets of relevance in the theory of numbers is a classical problem studied by many authors. Probably the most famous among them is the problem of primes in arithmetic progressions, solved by B. Green and T. Tao in [GT]. Bremner considered in [Br1] the existence of

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arithmetic progressions on elliptic curves and constructed elliptic curves with 8 rational points (x, y) whose x-components are in arithmetic progression. Bremner, Silverman and Tzanakis [BST] showed that the elliptic curve $y^2 = x(x^2 - n^2)$ of rank 1 does not have non-trivial integral arithmetic progressions. Campbell [Ca] found an infinite family of elliptic curves with 9 integral points in arithmetic progression; later on Ulas [U1] improved this result to an infinite family with arithmetic progressions of 12 points. Finally MacLeod [ML] got new families of 12 terms and some examples of progressions with 14 terms. Further examples of similar problems and results can be found in [Al], [Br2], [U2].

The case of Pellian equations $x^2 - d\,y^2 = m$ has been studied in the the papers of Dujella, Pethő and Tadić [DPT] and Pethő and Ziegler [PZ]. Dujella, Pethő and Tadić [DPT] have shown that for any four-term arithmetic progression, except $\{0, 1, 2, 3\}$ and $\{-3, -2, -1, 0\}$, there exist infinitely many pairs d, m, with d non-square and $\gcd(d, m)$ square-free, such that the terms of the given progression are y-components of solutions of the equations

$$x^2 - d\,y^2 = m.$$

They also exhibit several examples of six-term progressions and an example of a seven-term progression. Pethő and Ziegler [PZ] have shown that in the case of five-term arithmetic progressions (with different absolute values) there exist at most a finite number of pairs d, m, with d non-square and $\gcd(d, m)$ square-free, such that the elements of the given progression are y-components of solutions. Recently, Bérczes and Ziegler [BZ] have considered similar problems for geometric progressions on Pellian equations.

In this note we look for arithmetic progressions of length N whose terms are the y coordinates of solutions of Pellian equations, that is, we look for integers m, d, a, $\Delta > 0$ and $N > 0$ as large as possible such that

$$d(a + j\,\Delta)^2 + m = \square, \quad 0 \le j \le N, \tag{1}$$

where $\square$ denotes any perfect square; a is the first term of the arithmetic progression and Δ the difference. Furthermore, we require that $\gcd(a, \Delta) = 1$, $\gcd(d, m)$ is square-free, and $|a + i\,\Delta| \ne |a + j\,\Delta|$ if $i \ne j$. The reason to study only the y-component is that a three term arithmetic progression can appear only a finite number of times as the x-component of a Pellian equation (see [PZ]).

We construct, by two different methods, explicit infinite families of pairs d, m for which there exist five-term arithmetic progressions. We also prove the existence of infinitely many pairs d, m parametrized by points of an elliptic curve of rank 3 for which the corresponding Pellian equation has a six-term solution. We also show the existence of infinitely many 5-term arithmetic progressions for which there exist a couple of essentially different pairs (d_1, m_1) and (d_2, m_2) for which the members of the progression are solutions of the associated Pellian equations. Finally we give examples of six and seven term progressions, and of five term progressions which are solutions of two different equations.

2. The first method: direct search

We look for solutions imposing directly that they must satisfy equation (1). To make things somewhat simpler, we divide by Δ^2, let $\alpha = a/\Delta$, and look for solutions in $\mathbb{Q}$ with $\Delta = 1$. We begin with 3-term solutions:

$$d(\alpha + j)^2 + m = x_j^2, \quad j = 0, 1, 2.$$

Solving for m, d and α we get

$$\begin{aligned} d &= \frac{x_0^2 - 2\,x_1^2 + x_2^2}{2}, \\ m &= \frac{-x_0^4 + 8\,x_0^2\,x_1^2 - 16\,x_1^4 + 2\,x_0^2\,x_2^2 + 8\,x_1^2\,x_2^2 - x_2^4}{8(x_0^2 - 2\,x_1^2 + x_2^2)}, \\ \alpha &= \frac{-3\,x_0^2 + 4\,x_1^2 - x_2^2}{2(x_0^2 - 2\,x_1^2 + x_2^2)}. \end{aligned} \tag{2}$$

2.1. Four-term progressions. With the above values of m, d and α, we impose that $\alpha + 3$ is also a solution in the following way:

$$d(\alpha + 3)^2 + m = x_0^2 - 3\,x_1^2 + 3\,x_2^2 = (x_0 + x_3)^2$$

(we denote the fourth square by $(x_0 + x_3)^2$, and not x_3^2, to simplify further notation). From here we get

$$x_0 = \frac{-3\,x_1^2 + 3\,x_2^2 - x_3^2}{2\,x_3}.$$

Substituting the above value of x_0 in (2) we obtain

$$\begin{aligned} d &= 2(9\,x_2^4 - 18\,x_2^2\,x_1^2 + 9\,x_1^4 - 2\,x_2^2\,x_3^2 - 2\,x_1^2\,x_3^2 + x_3^4), \\ m &= -2(3\,x_2 - 3\,x_1 - x_3)(x_2 - x_1 - x_3)(x_2 + x_1 - x_3)(3\,x_2 + 3\,x_1 - x_3) \\ &\quad \times (3\,x_2 - 3\,x_1 + x_3)(x_2 - x_1 + x_3)(x_2 + x_1 + x_3)(3\,x_2 + 3\,x_1 + x_3) \\ &\quad \times (9\,x_2^4 - 18\,x_2^2\,x_1^2 + 9\,x_1^4 - 2\,x_2^2\,x_3^2 - 2\,x_1^2\,x_3^2 + x_3^4), \\ \alpha &= -27\,x_2^4 + 54\,x_2^2\,x_1^2 - 27\,x_1^4 + 14\,x_2^2\,x_3^2 - 2\,x_1^2\,x_3^2 - 3\,x_3^4. \end{aligned} \tag{3}$$

2.2. Five-term progressions. Next we force $\alpha + 4$ as another solution, which gives the quartic equation

$$27\,x_2^4 - 54\,x_1^2\,x_2^2 + 27\,x_1^4 + 6\,x_2^2\,x_3^2 - 14\,x_1^2\,x_3^2 + 3\,x_3^4 = \Box. \tag{4}$$

This admits the parametric solution

$$\begin{cases} x_1 = 24\,u\,v, \\ x_2 = -23\,u^2 - v^2, \\ x_3 = 23(v-u)(v+u). \end{cases}$$

The corresponding values for d, m, a and Δ once simplified and taking off common factors are:

$$\begin{aligned} d &= 279841\,u^4 - 153410\,u^2\,v^2 + 34849\,v^4, \\ m &= 576(23\,u - 13\,v)(23\,u - 11\,v)(23\,u - 5\,v)v^2(23\,u + 5\,v) \\ &\quad \times (23\,u + 11\,v)(23\,u + 13\,v)(279841\,u^4 - 153410\,u^2\,v^2 + 34849\,v^4), \\ a &= -279841\,u^4 + 89930\,u^2\,v^2 - 52009\,v^4, \\ \Delta &= 279841\,u^4 - 153410\,u^2\,v^2 + 34849\,v^4. \end{aligned} \tag{5}$$

An alternative way to parametrize the quartic equation (4) is to take $x_2 = x_1 + g\,x_3$. Then the left-hand side of (4) becomes

$$x_3^2\big(-8\,x_1^2 + 108\,g^2\,x_1^2 + (12\,g\,x_1 + 108\,g^3\,x_1)x_3 + (3 + 6\,g^2 + 27\,g^4)x_3^2\big).$$

It is enough to choose particular values of g for which $3 + 6\,g^2 + 27\,g^4$ is a perfect square and then parametrize the inner conic in x_3. This can be achieved because the quartic $h^2 = 3 + 6\,g^2 + 27\,g^4$ is equivalent to the elliptic curve $y^2 = x^3 + 60\,x^2 + 864\,x$ whose rank is equal to 1. Moreover, for any g such that $-8 + 108\,g^2$ is a square, we can parametrize the inner conic in x_1. This is possible because $-8 + 108\,g^2 = h^2$ has a particular solution, for instance $g = 1$, $h = 10$.

2.3. Six-term progressions. Finally, the condition that

$$d(a + 5\,\Delta)^2 + m = \Box,$$

where m, d, a and Δ are given in (5), yields the quartic equation

$$279841\,u^4 - 166106\,u^2\,v^2 + 26269\,v^4 = \Box.$$

The corresponding curve is birationally equivalent to the elliptic curve of rank 3

$$y^2 = x^3 + 157\,x^2 - 405\,x,$$

giving infinitely many values of the parameters u, v, for which the Pellian equation defined by the parameters in (5) has a six-term solution.

3. The second method: adjusting polynomials

It is based on the fact that for any monic polynomial $P_{2n} \in \mathbb{Q}[z]$ of degree $2\,n$ there exist a monic polynomial $Q_n \in \mathbb{Q}[z]$ of degree n and $R_{n-1} \in \mathbb{Q}[z]$ of degree $n-1$ such that $P_{2n} = Q_n^2 - R_{n-1}$. If $z \in \mathbb{Q}$ is a root of P_{2n}, then $R_{n-1}(z) = \Box$. This idea has been used in [F], [ACP] to construct elliptic curves of high rank.

3.1. Adjusting with polynomials of degree 6. Consider the polynomial of degree 6

$$P(z) = (z - a - u\,\Delta)\prod_{j=0}^{4}(z - a - j\,\Delta),$$

where u is a free parameter. Then

$$P(z) = (q(z))^2 - m - A\,z - d\,z^2,$$

where $q(x)$ is a degree 3 polynomial and

$$\begin{aligned}
m &= 2^{-8}\Delta^4\big(256\,a^2 + 256\,a\,\Delta\,u + 320\,a^2\,u^2 + 1600\,a\,\Delta\,u^2 + 1600\,\Delta^2\,u^2 \\
&\quad - 160a^2\,u^3 - 800\,a\,\Delta\,u^3 - 800\,\Delta^2\,u^3 + 20\,a^2\,u^4 + 120\,a\,\Delta\,u^4 \\
&\quad + 180\,\Delta^2\,u^4 - 4\,a\,\Delta\,u^5 - 20\,\Delta^2\,u^5 + \Delta^2\,u^6\big), \\
A &= 2^{-8}\Delta^4\big(-512\,a - 256\,\Delta\,u - 640\,a\,u^2 - 1600\,\Delta\,u^2 + 320\,a\,u^3 \\
&\quad + 800\,\Delta\,u^3 - 40\,a\,u^4 - 120\,\Delta\,u^4 + 4\,\Delta\,u^5\big), \\
d &= 2^{-8}\Delta^4\big(256 + 320\,u^2 - 160\,u^3 + 20\,u^4\big).
\end{aligned}$$

Now we make $A = 0$ with

$$\Delta = \frac{2\,a(64 + 80\,u^2 - 40\,u^3 + 5\,u^4)}{u(-64 - 400\,u + 200\,u^2 - 30\,u^3 + u^4)}.$$

After simplifying we get the following uniparametric family of five-term solutions

$$\begin{aligned}
d &= 64 + 80\,u^2 - 40\,u^3 + 5\,u^4, \\
m &= 4(-8 + u)(-6 + u)(-4 + u)^2(-2 + u)^2u^2(2 + u)(4 + u) \\
&\quad \times (64 + 80\,u^2 - 40\,u^3 + 5\,u^4), \\
a &= u(-64 - 400\,u + 200\,u^2 - 30\,u^3 + u^4), \\
\Delta &= 2(64 + 80\,u^2 - 40\,u^3 + 5\,u^4). \qquad (6)
\end{aligned}$$

3.2. Adjusting with polynomials of degree 4. Given $0 \leq k \leq 4$, we consider the polynomial of degree 4 defined by

$$P_k(z) = \prod_{\substack{0 \leq j \leq 4 \\ j \neq k}} (z - a - j\,\Delta).$$

It can be written as

$$P_k(z) = \bigl(q_k(z)\bigr)^2 - m_k - d_k\, z^2,$$

where q_k is a monic polynomial of degree 2 and d_k, m_k are rational functions of a and Δ. The equation $m_k + d_k\, y^2 = \Box$ has $\{a + j\,\Delta : 0 \leq j \leq 4,\ j \neq k\}$ as solutions. In order to complete the five-term progression we have to impose the missing term, $y = a + k\,\Delta$, as a new solution. This produces a quadratic equation, whose solutions can be parametrized and give a family of Pellian equations with five-term solutions. When $k = 0$, $k = 2$ or $k = 4$, these families turn out to be trivial.

When $k = 1$ we have

$$\begin{aligned} m_1 &= \frac{\Delta^3(15\,a^3 + 166\,a^2\Delta + 552\,a\Delta^2 + 576\,\Delta^3)}{(4\,a + 9\,\Delta)^2}, \\ d_1 &= -\frac{15\,\Delta^3}{4(4\,a + 9\,\Delta)}. \end{aligned}$$

Then $m_1 + d_1(a + \Delta)^2 = \Box$ if

$$409\,a^2 + 1878\,a\,\Delta + 2169\,\Delta^2 = \Box.$$

Parametrizing this conic and eliminating denominators and superfluous squares we arrive to the following family

$$\begin{aligned} d_1(U, W) &= -15(3\,U - W)(27\,U + W)(153\,U^2 + 32\,U\,W - W^2), \\ m_1(U, W) &= 32(18\,U - W)(33\,U - W)(9\,U - W)(7\,U + W)(12\,U + W) \\ &\qquad \times (9\,U + 2\,W)(153\,U^2 + 32\,U\,W - W^2), \\ a_1(U, W) &= 2(162\,U^2 + 39\,U\,W - W^2), \\ \Delta_1(U, W) &= -153\,U^2 - 32\,U\,W + W^2, \end{aligned} \tag{7}$$

satisfying $d_1\,(a_1 + j\,\Delta_1)^2 + m_1 = \Box$ for $0 \leq j \leq 4$.

The condition $d_1\,(a_1 + 5\,\Delta_1)^2 + m_1 = \Box$ translates into the quartic

$$\Box = -699111\,U^4 + 242028\,U^3\,W + 89046\,U^2\,W^2 - 468\,U\,W^3 - 71\,W^4. \tag{8}$$

Since it has a rational point (e.g. with $U/W = 1/3$), it is birationally equivalent to the elliptic curve $y^2 = x^3 + 12\,x^2 - 180\,x$ whose rank is 2, so again we find infinitely many pairs (d_1, m_1) having six-term solutions for the corresponding Pellian equation $x^2 - d_1\,y^2 = m_1$.

A similar construction can be made for $k = 3$. In this case we get the following family:

$$\begin{aligned}
d_3(U,W) &= -(3\,U - W)(U - W)(85\,U^2 - 36\,U\,W - W^2),\\
m_3(U,W) &= 32\,U(11\,U - 5\,W)(5\,U - 3\,W)(7\,U - W)(4\,U - W)(2\,U - W)\\
&\quad \times (85\,U^2 - 36\,U\,W - W^2),\\
a_3(U,W) &= 2(80\,U^2 - 39\,U\,W + W^2),\\
\Delta_3(U,W) &= -85\,U^2 + 36\,U\,W + W^2. \qquad (9)
\end{aligned}$$

The condition $d_3\,(a_3 + 5\,\Delta_3)^2 + m_3 = \square$ gives the quartic

$$\square = -1319\,U^4 + 2396\,U^3\,W - 930\,U^2\,W^2 - 52\,U\,W^3 + 49\,W^4. \qquad (10)$$

Since it has a rational point (e.g. with $U = 0$), it is birationally equivalent to the elliptic curve $y^2 = x^3 + 27\,x^2 - 360\,x$ of rank 3. Thus, as in the preceding case, infinitely many Pellian equations with six-term progressions as solutions can be derived from it.

4. Five-term progressions for several equations

It is shown in [PZ] that for each five-term progression (with different absolute values) there are at most finitely many $d, m \in \mathbb{Z}$ such that d is not a square, $\gcd(d,m)$ is square-free and such that these five numbers are y-components of solutions to $x^2 - d\,y^2 = m$. In this section we use the two five-term families given by (7) and (9) in order to get examples of five-term arithmetic progressions having at least two essentially different pairs (d,m) such that these are solutions of the corresponding Pellian equations.

Based on the expressions of Δ_1 and Δ_3, we look for values of the parameters u, v, v', w such that $\Delta_1(u + 2\,v, w + 16\,u + 39\,v) = \Delta_3(u - 2\,v', w - 18\,u - 39\,v')$. We have

$$\Delta_1(u + 2\,v, w + 16\,u + 39\,v) = -409\,u^2 - 1636\,u\,v - 1587\,v^2 + 14\,v\,w + w^2,$$

$$\Delta_3(u - 2\,v', w - 18\,u - 39\,v') = -409\,u^2 + 1636\,u\,v' + 3989\,v'^2 - 150\,v'\,w + w^2.$$

Thus we get

$$w=\frac{1636(v+v')u+(1587\,v^2+3989\,v'^2)}{2(7\,v+75\,v')}.$$

With this value of w we let $v'=\lambda v$, $\lambda\in\mathbb{Q}$, and impose $a_1(9u+18v,w+16u+39v)=a_3(u-2v',w-18u-39v')$, which is equivalent to the second degree equation in v

$$\begin{aligned}&-904904\,\lambda^2\,u^2-4437872\lambda\,u^2+1793880\,u^2+2030908\,\lambda^3\,u\,v\\&-11116828\,\lambda^2\,u\,v-9004588\,\lambda\,u\,v+4069164\,u\,v+2400871\,\lambda^4\,v^2\\&-2522100\,\lambda^3\,v^2-11857862\,\lambda^2\,v^2-4554900\,\lambda\,v^2+2306007\,v^2=0.\end{aligned}$$

The discriminant of this equation with respect to v is given by

$$(75\,\lambda+7)^2(1111\,\lambda^2-2402\,\lambda+1111)(128161\,\lambda^2+136506\,\lambda+12969)u^2.$$

So for a rational solution to exist, this expression must be a square, which happens when

$$(1111\,\lambda^2-2402\,\lambda+1111)(128161\,\lambda^2+136506\,\lambda+12969)=\square.$$

But since the polynomial on the left hand side assumes a square value for $\lambda=-1$, this quartic curve is birationally equivalent to the elliptic curve $y^2=x^3-34682\,x^2+293420281\,x$ whose rank is 2, so there are infinitely many solutions $\lambda\in\mathbb{Q}$ for which the Pellian equations corresponding to the pairs (d_1,m_1) and (d_3,m_3) admit a common five-term progression as solutions.

In some cases the values of (d_1,m_1) and (d_3,m_3) are essentially the same. In fact, this happens exactly when λ is a rational zero of the resultant of the polynomials a_1-a_3 and $d_1\,m_3-d_3\,m_1$, and these zeros are -1, $-7/75$, $27/41$ and $699/457$. Therefore, there are infinitely many $\lambda\in\mathbb{Q}$ which produce different pairs (d_1,m_1) and (d_3,m_3). We show next that this construction produces infinitely many (essentially) different pairs (a,Δ) which give five-term progressions satisfying Pellian equations for two different pairs (d_1,m_1) and (d_3,m_3).

Let $\lambda'\in\mathbb{Q}$ produce the pair (a',Δ'), and certain pairs (d_1,m_1) and (d_3,m_3). We are interested in the question how many other rational λ can produce the (essentially) identical pair (a',Δ'). Let $z=a'/\Delta'$ be given. Then we seek for $\lambda\in\mathbb{Q}$ which satisfy the system $a_1-a_3=0$ and $a_1-z\,\Delta_1=0$. After eliminating the denominators, we look at the resultant of the polynomials a_1-a_3 and $a_1-z\,\Delta_1$ (as polynomials in v). The condition that the resultant is equal to 0 gives a (non-zero) polynomial of degree 8 in λ, the leading coefficient being

$$8642970851449\,z^2+34571883405796\,z-2625169872622968,$$

so it has at most 8 rational solutions λ. Assume that our procedure gives only finitely many different pairs (a, Δ). Then infinitely many λ produce the (essentially) identical pair (a, Δ). However, this contradicts what we proved above that at most 8 λ can produce the same pair (a, Δ). For instance, take $z = -36/41$. Then the resultant is equal to 0 for $\lambda = 11/57$, $57/11$, $297/791$, $-755/143$. It can be shown that these four numbers correspond to four 2-torsion points on the elliptic curve induced by the condition $a_1 - a_3 = 0$. The smallest values (λ, u, v) for which the pairs (d_1, m_1) and (d_3, m_3) are different are given in Table 1.

λ	u	v
11/57	-46	57
57/11	46	11
319/157	-2134	785
177/95	-36777	26030
-9669/2257	1989770	760609
4073/1095	-4875791	1158510

Table 1. Small values of (λ, u, v)

5. Examples

In this section we present some of the results found in our search for arithmetic progressions in solutions of Pellian equations. The search was carried out by looking for solutions of the quartics (8) and (10), and then computing the parameters of the Pellian equations and of the corresponding arithmetic progressions using the formulas given in (7) and (9). The computations made use of MWRANK [C], *Mathematica*® [M] and PARI [P].

5.1. Six-term progressions. Table 2 shows some examples of six-term progressions having small coefficients. We have chosen $|d| \leq 5000$.

d	m	a	Δ
-3416	100096425	-164	61
-2526	65857566775	853	842
-1704	16643051425	-3065	71
-1245	375701326	-295	166
-1091	91408016700	-1913	2182
-1055	27120272256	-5058	211
-10	46046	-67	24
291	2533111350	-3559	1746
631	1115071650	-1335	1262
709	933540300	-3181	1418
795	14889206101	-5711	3392
1065	4548544	-118	71
1171	8967108150	-4525	2342
1731	3934187950	-1571	2308
2226	4296914050	61	424
2370	12731719	-271	158
2905	45752256	-97	83
3095	37309738466	-5689	3714
3865	10250944704	-802	773
4195	33151804686	-1297	1678
4249	3269059200	-1329	607
4249	-62546296725	-4273	9712
4299	14559494950	-4513	2866

Table 2. Six-term examples

5.2. Seven-term progressions. In [DPT] an example of a seven-term progression was shown. It is included in Table 3 jointly with another five examples that we have found in our search.

5.3. A particular symmetric progression of six terms. Consider the 6-term progression symmetric around the origin $\{-5,-3,-1,1,3,5\}$, corresponding to values $a=-5$, $\Delta=2$, and let

$$d = -(u-5\,v)(u-v)v(u+v),$$
$$m = (u-3\,v)(u-2\,v)(u+3\,v)(u+7\,v).$$

d	m	a	Δ
-1245	375701326	-461	166
37569	27833977600	-5956	1789
1115646	-747027030131525	-185275	53126
231235	5329956420362574	-294919	92494
505561	12382891041664000	-856524	216669
18529039	18265211513829127697850	-43776175	37058078

Table 3. Seven-term examples

A simple computation shows that $d(a+j\,\Delta)^2+m=\square$ for $0\leq j\leq 5$. So there exist infinitely many non-equivalent pairs (d,m) for which the corresponding Pellian equation has the same six-term progression as solution. In order to get an extra solution both in the left and in the right one has to impose that

$$u^4-44\,u^3\,v+222\,u^2\,v^2+4\,u\,v^3-119\,v^4=\square,$$

and this quartic is equivalent to the elliptic curve

$$y^2=x^3+288\,x^2+11520\,x$$

whose rank is 1. Thus, the 8-term progression $\{-7,-5,-3,-1,1,3,5,7\}$ is a solution of infinitely many Pellian equations. This gives an affirmative answer to a question posed in Section 8 of [PZ]. The smallest two 8-term solutions that appear in this way correspond to the values $(d,m)=(-105,5434)$ and $(d,m)=(570570,4406791)$.

5.4. Five-term solutions for more than one equation. In [DPT] various examples are shown each having a couple of pairs (d,m) of which they are a solution. In Table 4 we show several additional examples, one of them having three pairs (d,m). The first one and the two last ones were found by the procedure of Section 4.

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a	Δ	d	m
-36	41	87945	160389376
		984	1026025
		-615	10506496
-97	134	1474	70385175
		1005	6170164
-157	97	208065	848087296
		81480	-111536711
-174	277	1008280	55523430369
		-831	887286400
-453	218	-545	111945834
		2289	59230600
-471	362	41811	1406035150
		1810	143643591
-514	355	10153	-254454912
		-242607	201349747456
-494932	209067	1367646625	18094425353599558656
		-179887867255	44134212595620130210304
-180106988	106894461	198348195265985	382967154942745378789721222976
		-43046790856584695	2636877642611872714844692076611584

Table 4. Five-term solutions for more than one equation

References

[ACP] J. Aguirre, F. Castañeda and J. C. Peral, High rank elliptic curves with torsion group $\mathbb{Z}/2\mathbb{Z}$, *Math. Comp.* **73**, no. 245 (2006), 323–331.

[Al] D. Allison, On square values of quadratics, *Math. Proc. Cambridge Philos. Soc.* **99**, no. 3 (1986), 381–383.

[BZ] A. Bérczes and V. Ziegler, On geometric progressions on Pell equations and Lucas sequences, *Glas. Mat. Ser. III* **48** (2013), 1–22.

[Br1] A. Bremner, On arithmetic progressions on elliptic curves, *Experiment. Math.* **8**, no. 4 (1999), 409–413.

[Br2] A. Bremner, On square values of quadratics, *Acta Arith.* **108**, no. 2 (2003), 95–11.

[BST] A. Bremner, J. Silverman and N. Tzanakis, Integral points in arithmetic progressions on $y^2 = x(x^2 - n^2)$, *J. Number Theory* **80** (2000), 187–208.

[Ca] G. Campbell, A note on arithmetic progressions on elliptic curves, *J. Integer Seq.* **6** (2003), Article 03.1.3, 5 pp. (electronic).

[C] J. Cremona, Algorithms for Modular Elliptic Curves, *Cambridge University Press, Cambridge*, 1997.

[D] A. Dujella, High rank elliptic curves with prescribed torsion, `http://web.math.hr/~duje/tors/tors.html`.

[DPT] A. Dujella, A. Pethő and P. Tadić, On arithmetic progressions on Pellian equations, *Acta Math. Hungar.* **120**, no. 1–2 (2008), 29–38.

[F] S. Fermigier, Construction of high-rank elliptic curves over $\mathbb{Q}$ and $\mathbb{Q}(t)$ with non trivial 2-torsion (extended abstract), in Algorithmic Number Theory (Talence, 1996), *Springer, Berlin*, 1996.

[GT] B. Green and T. Tao, The primes contain arbitrarily long arithmetic progressions, *Ann. of Math.* **167**, no. 2 (2008), 481–547.

[ML] A. MacLeod, 14-term arithmetic progressions on quartic elliptic curves, *J. Integer Seq.* **9** (2006), Article 06.1.2, 4 pp. (electronic).

[M] Wolfram Research, Inc., Mathematica, Version 8.0.1, Champaign, IL 2008.

[P] PARI/GP, version 2.4.0, Bordeaux, 2008, `http://pari.math.u-bordeaux.fr`.

[PZ] A. Pethő and V. Ziegler, Arithmetic progressions on Pell equations, *J. Number Theory* **128** (2008), 1389–1409.

[Si] J. H. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, *Springer – Verlag, New York*, 1994.

[U1] M. Ulas, A note on arithmetic progressions on quartic elliptic curves, *J. Integer Seq.* **8** (2005), Article 05.3.1, 5 pp. (electronic).

[U2] M. Ulas, Rational points in arithmetic progressions on $y^2 = x^n + k$, *Canad. Math. Bull.* **55**, no. 1 (2012), 193–207.

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