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On generalized metric spaces and their associated Finsler spaces I. Fundamental relations

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Dedicated to Professor Lajos Tamássy on his 70th birthday

§0. Introduction

In a previous paper [3], we have investigated a generalized metric space $M_n = (M_T, g_{ij}(x,y))$. Here let us consider the Finsler space $F_n^*(g) = (M_T, F(x,y))$ associated with M_n, where its Finsler metric is given by $F(x,y) := \sqrt{g_{ij}y^i y^j}$.

It is noticed that the metric tensor $g_{ij}(x,y)$ used here is positively homogeneous of degree 0 in y. Sometimes a generalized metric space $M_n = (M_T, g_{ij}(x,y))$ was considered under the supposition that the metric g_{ij} is (a) p-homogeneous, (b) non-homogeneous and (c) irrespective of homogeneity. On the other hand, H. Rund [9] showed, in his book: *The Hamilton-Jacobi theory in the calculus of variations*, that the case (a) corresponds to Metric Differential Geometry and Relativistic Mechanics and (b) to Geometrical Optics and Non-relativistic Mechanics. So, in the sequel, we shall call M_n, for (a) a generalized *metric* space ([3], [4], [5], [15]), (b) a generalized *Lagrange* space ([7]) and (c) a generalized *Finsler* space ([1], [2], [6], [12], [13], [14]).

The geometry of a generalized metric space M_n is closely related to that of $F_n^*(g)$. However, its geometry is in contrast with that of (ordinary) Finsler space $F_n := (M_T, F(x,y))$. That is, there exist two characteristic tensors C_{ij} and $P^i{}_j$. For a given metric tensor g_{ij} in M_n, the metric tensor $g^*{}_{ij}$ of its associated Finsler space $F_n^*(g)$ is related as

$$(0.1) \qquad g^*{}_{ij} = g_{ij} + C_{ij}, \quad C_{ij} := y^h \dot{\partial}_j g_{ih} \qquad ([3], (2.8)(b)),$$

where the tensor C_{ij} satisfies $C_{ij} = C_i{}^0{}_j$ and $C_{ij} = C_{ji}$ ([3],(2.9)). Vanishing of the tensor C_{ij} means that the M_n itself reduces to a Finsler space.

To determine the non-linear connection N, we assume that geodesics in M_n are coincident with those in $F_n^*(g)$, that is,

$$(\mathbf{A0}) \qquad 2G^i = N^i_j y^j.$$

Therefore another characteristic tensor $P^i{}_k$ satisfies the following relations:

$$(0.2) \quad N^i_k = G^i_k - P^i{}_k, \quad P^i{}_0 = 0, \quad C_{ij/0} = 2g^*{}_{ih}P^h{}_j, \quad ([3], (2.16)(f)),$$

where G^i_j is a unique non-linear connection of $F_n^*(g)$ and N^i_k is an arbitrary non-linear connection in M_n. (0.2) shows that the arbitrary tensor $P^i{}_k$ has disappeared in Finsler geometry. The fact that some differential equation does not contain the tensor $P^i{}_j$ explicitly, implies that the geometrical property described by this equation is free from any choice of the non-linear connection.

However, examples of a generalized metric space are very few. Let us consider the following metric in an M_n:

$$(0.3) \quad g_{ij}(x,y) = a_{ij}(x) - \alpha(x,y)h_{ij}(x,y), \qquad C_{ij} = \alpha h_{ij} \quad \text{(cf. [5])},$$

where the tensor $a_{ij}(x)$ is a Riemannian metric. This metric defines a generalized metric space M_n which is not a Finsler space and its associated Finsler space is a Riemannian space (cf. §3).

It is well known that in a Finsler space $F_n^*(g)$ we can define three types of connection: $[C^*]$, $[R^*]$ and $[B^*]$ (cf. §2) in a natural way. On the other hand, in a space M_n([3]) we defined three types of connection: $[C]$, $[R]$ and $[B]$ (cf. §1). However, the connection $[B]$ in M_n and the connection $[B^*]$ in $F_n^*(g)$ are coincident. In a same underlying space M_T, we can consider five connections: $[C]$, $[R]$, $[B]$, $[C^*]$ and $[R^*]$ originating from only one *structure*: the metric tensor $g_{ij}(x,y)$.

One of the purposes of the present paper is to find the relations between $[C]$ in a space M_n and $[C^*]$ in a space $F_n^*(g)$. In virtue of these equations, the properties of M_n are investigated by means of well-known theorems in a Finsler space $F_n^*(g)$, which suggest some properties in M_n. As we see, the tensor C_{ij} holds a key to investigate the geometry of spaces M_n. Especially, the most important fact is that the connection parameters $F_j{}^i{}_k$ of $[C]$ and ${}^*\Gamma_j{}^i{}_k$ of $[C^*]$ are coincident if and only if $C_{ij/k} = 0$ (Theorem 2.4).

Roughly speaking, if a generalized metric space M_n itself is a Finsler-, a Riemannian- or a g-Minkowski space, then its associated Finsler space $F_n^*(g)$ preserves this property. Our interest is in the inverse problem.

§1 is the summary of results obtained in M_n. §2 is devoted to deriving the relations between $[C]$ and $[C^*]$ in terms of the tensors in M_n. In §§3, 4, we investigate a generalized metric space whose associated Finsler space is a Riemannian or a Minkowski space. We shall show that

[**A**] If an RM_n space satisfies the condition $C_{ij/k} = 0$, then the space M_n is a g-Berwald space (Theorem 3.7).

[**B**] A necessary and sufficient condition for a space M_n to be a g-Minkowski space is that the curvature tensors $K_h{}^i{}_{jk}$ and $F_h{}^i{}_{jk}$ vanish (Theorem 4.1).

[**C**] A necessary and sufficient condition for a space M_n to be an MM_n space is that the curvature tensors $H_h{}^i{}_{jk}$ and $G_h{}^i{}_{jk}$ vanish (Theorem 4.2).

[**D**] If an MM_n space satisfies the condition $C_{ij/k} = 0$, then the space is a g-Minkowski space (Theorem 4.4).

We raise or lower the indices by means of g_{ij} only without comment.

§1. Preliminaries in M_n

The purpose of this section is to summarize the connections in M_n.

1.1. Assumptions on the metric tensor $g_{ij}(x, y)$.

Let M be an n-dimensional manifold of class C^∞ with local coordinates (x^i) and $T(M)$ its tangent vector bundle with local coordinates (x^i, y^i). Let us denote by M_T a manifold of non-vanishing tangent vectors: $M_T := T(M) - \{0\}$. A generalized metric space is a pair $M_n = (M_T, g_{ij}(x, y))$, where the metric tensor g_{ij} satisfies the following conditions:

(**A1**) $g_{ij}(x, y)$ is positively homogeneous of degree 0 in y,
(**A2**) $g_{ij}X^iX^j$ is positive definite,
(**A3**) $g^*{}_{ij} := \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2$ is non-degenerate, where $F(x,y) = \sqrt{g_{ij}y^iy^j}$ and $\dot{\partial}_j := \partial/\partial y^j$.

From conditions (**A2**) and (**A3**) a pair $F_n^*(g) = (M_T, F(x, y))$ is a Finsler space (called the *associated Finsler space* of M_n). In [3], we introduced the following three types of connection:

$[C]$ the metrical connection $C\Gamma(N) : \omega_j^i = F_j{}^i{}_k dx^k + C_j{}^i{}_k \delta y^k;\ \delta y^k = dy^k + N_h^k dx^h$ such that $\delta g_{ij} = dg_{ij} - \omega_i^h g_{hj} - \omega_j^h g_{ih} = g_{ij/k}dx^k + g_{ij/(k)}\delta y^k = 0$, where

$$g_{ij/k} := d_k g_{ij} - F_i{}^h{}_k g_{hj} - F_j{}^h{}_k g_{ih} = 0, \qquad d_k := \partial_k - N_k^r \dot{\partial}_r,$$
$$g_{ij/(k)} := g_{ij(k)} - C_i{}^h{}_k g_{hj} - C_j{}^h{}_k g_{ih} = 0, \qquad g_{ij(k)} := \dot{\partial}_k g_{ij},$$

and satisfies the following conditions:

$$(\mathbf{A4}) \quad (a) \quad N^i_k = F_j{}^i{}_k y^j, \quad (b) \quad F_j{}^i{}_k = F_k{}^i{}_j, \quad (c) \quad C_j{}^i{}_k = C_k{}^i{}_j.$$

$[R]$ the h-metrical connection $R\Gamma(N) : \omega^i_j = F_j{}^i{}_k dx^k$ so that $g_{ij/k} = 0$.
$[B]$ the non-metrical connection $B\Gamma(G) : \omega^i_j = G_j{}^i{}_k dx^k; \quad G_j{}^i{}_k := \dot\partial_k G^i_j$, where

$$G^i_j := \dot\partial_j G^i, \quad 4G^i := g^{*ih}(y^j \partial_j \dot\partial_h F^2 - \partial_h F^2),$$
$$\partial_h = \partial/\partial x^h, \quad g^{*ih} g^*{}_{hj} = \delta^i_j.$$

It is evident that $[B]$ in M_n is coincident with $[B^*]$ in $F^*_n(g)$. However, the general non-linear connection N^i_j of $[C]$ satisfies $(\mathbf{A0})$ $N^i_j y^j = 2G^i$ implicitly. So differentiating this equation, we have

$$(1.1) \quad N^i_j = G^i_j - P^i{}_j, \quad P^i{}_j := \frac{1}{2}(y^h \dot\partial_j N^i_h - N^i_j), \quad P^i{}_0 := P^i{}_j y^j = 0,$$

where the index 0 means the transvection with y.

The conditions $(\mathbf{A1})$ and $(\mathbf{A4})(c)$ give

$$(1.2) \quad \begin{aligned} &(a) \quad g^*{}_{ij} = g_{ij} + C_{ij}, \quad C_{ij} := y^h \dot\partial_j g_{ih} = C_{ji} \quad ([3],(2.8)),\\ &(b) \quad C_0{}^i{}_k = C_j{}^i{}_0 = 0,\\ &(c) \quad C_0{}^0{}_k = \frac{1}{2} g_{hj(k)} y^h y^j = 0 \qquad ([3],(2.3),(2.6)). \end{aligned}$$

The connection parameters for $C\Gamma(N)$ are given by

$$(1.3) \quad \begin{aligned} F_j{}^i{}_k &= \frac{1}{2} g^{ih}(d_k g_{hj} + d_j g_{hk} - d_h g_{jk}),\\ C_j{}^i{}_k &= \frac{1}{2} g^{ih}(g_{hj(k)} + g_{hk(j)} - g_{jk(h)}), \qquad C_i{}^0{}_j = C_{ij}. \end{aligned}$$

Then we have

$$(1.4) \quad \begin{aligned} &(a) \quad y_j = g_{ij} y^i = g^*{}_{ij} y^i, \quad y^i = g^{*ih} y_h, \quad y^i{}_{(j)} = y^i{}_{/(j)} = \delta^i_j,\\ &(b) \quad y_{i(j)} = g^*{}_{ij}, \qquad y_{i/(j)} = g_{ij}, \qquad y_{i/j} = 0. \end{aligned}$$

Remark. The homogeneous condition $(\mathbf{A1})$ implies that if there exists a coordinate system such that the metric g_{ij} is expressed by $g_{ij} = e^{2\sigma(x,y)} a_{ij}(x)$ ([6],[14]), then the metric itself is Riemannian. In fact, because the scalar $\sigma(x, y)$ must be p–homogeneous of degree 0 in y, the relation $C_{ij} = C_{ji}$ gives $y_i \sigma_{(j)} = y_j \sigma_{(i)}$. This means $\sigma_{(i)} = 0$.

1.2. The curvature and torsion tensors.

For curvature and torsion forms, we defined in [3] as follows:

$$(1.5)\qquad \begin{aligned} &(a)\quad \Omega^i_j := [d\omega^i_j] + [\omega^i_h \omega^h_j],\\ &(b)\quad \Omega^{(i)} := [\delta\delta y^i] = [d\delta y^i] + [\omega^i_h \delta y^h] = \Omega^i_0,\\ &(c)\quad \Omega^i := [\delta dx^i] = [ddx^i] + [\omega^i_h dx^h]. \end{aligned}$$

We shall denote

$$\begin{aligned} &[C]\quad C\Gamma(N):\quad \Omega^i_j = -\frac{1}{2}R_j{}^i{}_{kl}[k,l] - P_j{}^i{}_{kl}[k,(l)] - \frac{1}{2}S_j{}^i{}_{kl}[(k),(l)],\\ &\qquad\qquad\qquad\quad \Omega^{(i)} = -\frac{1}{2}R^i{}_{kl}[k,l] - P^i{}_{kl}[k,(l)],\quad \Omega^i = -C_j{}^i{}_k[j,(k)];\\ &[R]\quad R\Gamma(N):\quad \Omega^i_j = -\frac{1}{2}K_j{}^i{}_{kl}[k,l] - F_j{}^i{}_{kl}[k,(l)],\\ &\qquad\qquad\qquad\quad \Omega^{(i)} = -\frac{1}{2}R^i{}_{kl}[k,l] - P^i{}_{kl}[k,(l)],\quad \Omega^i = 0;\\ &[B]\quad B\Gamma(G):\quad \Omega^i_j = -\frac{1}{2}H_j{}^i{}_{kl}[k,l] - G_j{}^i{}_{kl}[k,(l)^*],\\ &\qquad\qquad\qquad\quad \Omega^{(i)} = -\frac{1}{2}H^i{}_{kl}[k,l],\quad \Omega^i = 0, \end{aligned}$$

where $[k,l] := [dx^k, dx^l]$, $[k,(l)] := [dx^k, \delta y^l]$, $[(k),(l)] := [\delta y^k, \delta y^l]$ and $[k,(l)^*] := [dx^k, \delta^* y^l] = [dx^k, \delta y^l + P^l{}_h dx^h] = [k,(l)] + P^l{}_h[k,h]$.

The covariant derivatives for a vector $v^i(x,y)$ with respect to x^k and y^k are defined as follows:

$$\begin{aligned} &v^i{}_{/k} := d_k v^i + F_j{}^i{}_k v^j, \qquad v^i{}_{/(k)} := v^i{}_{(k)} + C_j{}^i{}_k v^j \quad \text{for [C], [R]},\\ &v^i{}_{/\!\!/k} := \bar{d}_k v^i + G_j{}^i{}_k v^j, \qquad v^i{}_{(k)} := \dot{\partial}_k v^i \qquad\qquad \text{for [B]}, \end{aligned}$$

where $\bar{d}_k := \partial_k - G^h_k \dot{\partial}_h = d_k - P^h{}_k \dot{\partial}_h$.

We shall list the identities for curvature and torsion tensors in M_n:

$$(1.6)\qquad \begin{aligned} &(a)\quad C_{0j} = C_{i0} = 0,\quad P^i{}_0 = P^0{}_k = 0,\quad (b)\quad g_{ij(k)} = C_{ijk} + C_{jik},\\ &(c)\quad P^i{}_{0k} = 2P^i{}_k \qquad\qquad \text{([3], Proposition 2.6)}, \end{aligned}$$

$$
(1.7)\quad
\begin{aligned}
&(a)\quad R_h{}^i{}_{jk} = K_h{}^i{}_{jk} + C_h{}^i{}_r R^r{}_{jk}, \quad F_h{}^i{}_{jk} := \dot{\partial}_k F_h{}^i{}_j,\\
&\qquad P_h{}^i{}_{jk} = F_h{}^i{}_{jk} - C_h{}^i{}_{k/j} + C_h{}^i{}_m P^m{}_{jk}, \quad P^i{}_{jk} = N^i_{j(k)} - F_k{}^i{}_j,\\
&(b)\quad R_0{}^i{}_{jk} = K_0{}^i{}_{jk} = R^i{}_{jk}, \quad H_0{}^i{}_{jk} = H^i{}_{jk},\\
&\qquad P_0{}^i{}_{jk} = F_0{}^i{}_{jk} = F_j{}^i{}_{0k} = P^i{}_{jk}, \quad S_0{}^i{}_{jk} = 0,\\
&(c)\quad R^0{}_{jk} = 0, \quad P^0{}_{jk} = 0, \quad P^i{}_{j0} = 0, \quad H^0{}_{jk} = 0,\\
&(d)\quad R_h{}^0{}_{jk} = -g_{hr} R^r{}_{jk}, \quad K_h{}^0{}_{jk} = -g^*{}_{hr} R^r{}_{jk},\\
&\qquad H_h{}^0{}_{jk} = -g^*{}_{hr} H^r{}_{jk}, \quad F_h{}^0{}_{jk} = C_{hk/j} - g^*{}_{hr} P^r{}_{jk},\\
&\qquad S_h{}^0{}_{jk} = C_{hj(k)} + C_{hjk} - j|k = 0,
\end{aligned}
$$

$$
(1.8)\quad
\begin{aligned}
&(a)\quad R_{hijk} + R_{ihjk} = 0, \quad P_{hijk} + P_{ihjk} = 0, \quad S_{hijk} + S_{ihjk} = 0,\\
&(b)\quad K_{hijk} + K_{ihjk} = -g_{hi(r)} R^r{}_{jk},\\
&(c)\quad F_{hijk} + F_{ihjk} = g_{hi(k)/j} - g_{hi(r)} P^r{}_{jk},
\end{aligned}
$$

$$
(1.9)\quad
\begin{aligned}
&(a)\quad C_{hj/k} - C_{hk/j} = g^*{}_{jr} P^r{}_{kh} - g^*{}_{kr} P^r{}_{jh},\\
&(b)\quad g_{hi(k)/0} = g_{ir} P^r{}_{hk} + g_{hr} P^r{}_{ik} + 2 g_{hi(r)} P^r{}_k,\\
&(c)\quad C_{jk/0} = 2 g^*{}_{jr} P^r{}_k = g^*{}_{jr} P^r{}_k + g^*{}_{kr} P^r{}_j,
\end{aligned}
$$

$$
(1.10)\quad
\begin{aligned}
&(a)\quad H^i{}_{jk(h)} = H_h{}^i{}_{jk},\\
&(b)\quad H^i{}_{k(j)} - j|k = 3 H^i{}_{jk}, \quad H^i{}_k := H^i{}_{0k},\\
&(c)\quad H_{hj} := H_h{}^i{}_{ji} = H_{j(h)}, \quad H_j := H^i{}_{ji},
\end{aligned}
$$

where $j|k$ means the interchange of the indices j, k in the foregoing terms.

1.3. Relations between $[C]$ and $[B]$; Difference tensor $D_j{}^i{}_k$.

It is easily seen that for a vector v^i we find

(1.11)

$$v^i{}_{/\!/k} = \bar{d}_k v^i + G_h{}^i{}_k v^h = v^i{}_{/k} + D_h{}^i{}_k v^h - P^h{}_k v^i{}_{(h)}, \quad D_h{}^i{}_k := G_h{}^i{}_k - F_h{}^i{}_k.$$

Hence we have for the metric tensor g_{ij}

$$
(1.12)\quad
\begin{aligned}
&(a)\quad g_{ij/\!/k} = -D_i{}^h{}_k g_{hj} - D_j{}^h{}_k g_{ih} - P^h{}_k g_{ij(h)},\\
&(b)\quad -2 D_j{}^i{}_k = g^{ih}(g_{hj/\!/k} + g_{hk/\!/j} - g_{jk/\!/h} + g_{hj(r)} P^r{}_k\\
&\qquad\qquad + g_{hk(r)} P^r{}_j - g_{jk(r)} P^r{}_h),\\
&(c)\quad g_{ij/\!/0} = -g_{ih} P^h{}_j - g_{jh} P^h{}_i.
\end{aligned}
$$

Proposition 1.1 ([3], Proposition 3.1). *The difference tensor* ${D_j}^i{}_k$ *is expressed by*

$$(1.13)\qquad {D_j}^i{}_k = P^i{}_{jk} + P^i{}_{j(k)} = {D_k}^i{}_j,$$

and satisfies the following relations:

$$(1.14)\qquad \begin{aligned} &(a)\quad {D_0}^i{}_k = P^i{}_k, \qquad (b)\quad {D_j}^0{}_k = -g^*{}_{jh}P^h{}_k,\\ &(c)\quad {D_j}^i{}_{k(l)} = {G_j}^i{}_{kl} - {F_j}^i{}_{kl}, \quad (d)\quad {D_j}^i{}_{k(l)}y^j = -P^i{}_{kl}. \end{aligned}$$

The following relations are known:

$$(1.15)\qquad y^i{}_{/\!\!/ k} = 0, \qquad y_{j/\!\!/ k} = 0,$$

$$(1.16)\qquad \begin{aligned} &(a)\quad H_{hijk} + H_{ihjk} = -g_{hi/\!\!/ j/\!\!/ k} + g_{hi/\!\!/ k/\!\!/ j} - g_{hi(r)}H^r{}_{jk},\\ &(b)\quad {G_h}^0{}_{jk} = g^*{}_{hj/\!\!/ k} = g_{hj/\!\!/ k} + C_{hj/\!\!/ k},\\ &(c)\quad G_{hijk} + G_{ihjk} = -g_{hi/\!\!/ j(k)} + g_{hi(k)/\!\!/ j}, \end{aligned}$$

$$(1.17)\qquad \begin{aligned} &(a)\quad {H_h}^i{}_{jk} = {K_h}^i{}_{jk} + {E_h}^i{}_{jk},\\ &\qquad {E_h}^i{}_{jk} := {D_h}^i{}_{j/k} + {D_h}^r{}_j{D_r}^i{}_k - {G_h}^i{}_{jr}P^r{}_k - j|k,\\ &(b)\quad E^i{}_{jk} := {E_0}^i{}_{jk} = H^i{}_{jk} - R^i{}_{jk} = P^i{}_{j/k} + P^r{}_j{D_r}^i{}_k - j|k. \end{aligned}$$

1.4. Projection to the indicatrix.

Let us denote by $\mathrm{p}\cdot T$ the *projection* of a tensor T to the indicatrix, e.g., for a tensor $T^i{}_j$, we shall define $\mathrm{p}\cdot T^i{}_j := h^i_a T^a{}_b h^b_j$. If $\mathrm{p}\cdot T = T$ holds, then the tensor T is called an *indicatric* tensor. For example, as the torsion vector $C_j := {C_j}^k{}_k$ is p-homogeneous of degree -1, we find

$$(1.18)\qquad F\mathrm{p}\cdot C_{j/(k)} = Fh^a_j h^b_k C_{a/(b)} = FC_{j/(k)} + l_jC_k + l_kC_j.$$

Proposition 1.2 (cf. [10], (3.18)). *Let* $K(x, y)$ *be a scalar*, p*-homogeneous of degree* 0 *in* y, *and put* $K_j := FK_{(j)}$, $K_{jk} = K_{kj} := F\mathrm{p}\cdot K_{j(k)}$ *and* $K_{hjk} := F\mathrm{p}\cdot K_{jk(h)}$. *Then we have*

$$(1.19)\qquad K_{hjk} + K_h h^*{}_{jk} - h|j = 0, \quad h^*{}_{jk} = h_{jk} + C_{jk}.$$

Therefore the scalar K *is independent of* y *if* $K_j = 0$ *or* $K_{jk} = 0$ *holds.*

§2. The associated Finsler space $F_n^*(g)$ of M_n

In this section, we shall find the relations in which the connections and curvature and torsion tensors of $F_n^*(g)$ are expressed in terms of M_n.

2.1. Connection parameters of $[C^*]$ and $[C]$.

As usual, we can define the connections in $F_n^*(g)$.
$[C^*]$ the metrical connection $CF^*(G)$: $\omega^{*i}_j := {}^*\Gamma_j{}^i{}_k dx^k + C^*{}_j{}^i{}_k \delta^* y^k$,
$\delta^* y^k := \delta y^k + P^k{}_h dx^h$ such that $\delta^* g^*{}_{ij} = 0$,
${}^*\Gamma_j{}^i{}_k = {}^*\Gamma_k{}^i{}_j$, $C^*{}_j{}^i{}_k = \frac{1}{2} g^{*ih} g^*{}_{hj(k)}$.
$[R^*]$ the h-metrical connection $RF^*(G)$: $\omega^{*i}_j := {}^*\Gamma_j{}^i{}_k dx^k$, $g^*{}_{ij\overset{*}{/}k} = 0$.
Let us put

$$\omega^{*i}_j = \omega^i_j + t^i_j, \qquad t^i_j := A_j{}^i{}_k dx^k + B_j{}^i{}_k \delta y^k.$$

Accordingly we have

$$(2.1)\quad (a)\quad {}^*\Gamma_j{}^i{}_k = F_j{}^i{}_k + A_j{}^i{}_k - C^*{}_j{}^i{}_h P^h{}_k, \quad (b)\quad C^*{}_j{}^i{}_k = C_j{}^i{}_k + B_j{}^i{}_k,$$

and using the symmetric property of ${}^*\Gamma_j{}^i{}_k$, $F_j{}^i{}_k$, $C^*{}_j{}^i{}_k$ and $C_j{}^i{}_k$, we see

$$(2.2)\qquad \begin{aligned} &A_j{}^i{}_k + A_k{}^i{}_j = 2({}^*\Gamma_j{}^i{}_k - F_j{}^i{}_k) + C^*{}_j{}^i{}_h P^h{}_k + C^*{}_k{}^i{}_h P^h{}_j, \\ &A_j{}^i{}_k - A_k{}^i{}_j = C^*{}_j{}^i{}_h P^h{}_k - C^*{}_k{}^i{}_h P^h{}_j, \quad B_j{}^i{}_k = B_k{}^i{}_j. \end{aligned}$$

To determine the tensors $A_j{}^i{}_k$ and $B_j{}^i{}_k$, we give

Lemma 2.1. *The form t^i_j satisfies the following relation*:

$$(2.3)\qquad \delta C_{ij} = t^h_i g^*{}_{hj} + t^h_j g^*{}_{hi}.$$

Proof. Because both connections are metrical, we see

$$\begin{aligned} 0 = \delta^* g^*{}_{ij} &= dg^*{}_{ij} - \omega^{*h}_i g^*{}_{hj} - \omega^{*h}_j g^*{}_{hi} \\ &= dg_{ij} + dC_{ij} - (\omega^h_i + t^h_i)(g_{hj} + C_{hj}) - (\omega^h_j + t^h_j)(g_{hi} + C_{hi}) \\ &= \delta g_{ij} + \delta C_{ij} - t^h_i g^*{}_{hj} - t^h_j g^*{}_{hi}. \end{aligned}$$

Hence the condition $\delta g_{ij} = 0$ gives (2.3). □

From (2.3) we see

$$(2.4)\quad C_{ij/k} = A_i{}^h{}_k g^*{}_{hj} + A_j{}^h{}_k g^*{}_{hi}, \quad C_{ij/(k)} = B_i{}^h{}_k g^*{}_{hj} + B_j{}^h{}_k g^*{}_{hi}.$$

Now, applying the Christoffel process to (2.4) and using (2.2), we obtain

Proposition 2.2. *Two tensors ${A_j}^i{}_k$ and ${B_j}^i{}_k$ are given by*

(2.5)

$$(a)\quad {A_j}^i{}_k = \frac{1}{2} g^{*ih}(C_{hj/k} + C_{hk/j} - C_{jk/h}) - {C^*}_k{}^i{}_r P^r{}_j + g^{*ih} {C^*}_{jkr} P^r{}_h,$$

$$(b)\quad {B_j}^i{}_k = \frac{1}{2} g^{*ih}(C_{hj/(k)} + C_{hk/(j)} - C_{jk/(h)}),$$

and satisfy the following relations:

$$(2.6)\quad \begin{aligned} &(a)\quad {A_0}^i{}_k = {A_k}^i{}_0 = P^i{}_k, \quad {A_j}^0{}_k = -\frac{1}{2} C_{jk/0} = -{g^*}_{jh} P^h{}_k,\\ &(b)\quad {B_0}^i{}_k = {B_k}^i{}_0 = 0, \qquad {B_j}^0{}_k = -C_{jk},\\ &(c)\quad t^i_0 = P^i{}_k dx^k. \end{aligned}$$

We shall prove

Proposition 2.3. *In a generalized metric space, we have that*

(*a*) ${A_j}^i{}_k = 0$ *is equivalent to* $C_{ij/k} = 0$,
(*b*) ${B_j}^i{}_k = 0$ *is equivalent to* $C_{ij/(k)} = 0$,
(*c*) $C_{ij/(k)} = 0$ *is equivalent to* $C_{ij} = 0$.

Proof. If ${A_j}^i{}_k = 0$ or ${B_j}^i{}_k = 0$, we have from (2.4) $C_{ij/k} = 0$ or $C_{ij/(k)} = 0$, respectively. The inverse of (*a*) is obvious from (1.9)(*c*) and (2.5)(*a*). (*b*) and (*c*) are evident. □

By means of $C_{jk/0} = 2{g^*}_{jr} P^r{}_k$ and (2.5)(*a*), the relation (2.1)(*a*) shows the following

Theorem 2.4. *A necessary and sufficient condition for the connection parameters ${F_j}^i{}_k$ of $[C]$ and ${}^*{\Gamma_j}^i{}_k$ of $[C^*]$ to be coincident is that the condition $C_{ij/k} = 0$ holds.*

2.2. Curvature forms of $[C^*]$ and $[C]$.

Lemma 2.5. *The curvature forms ${\Omega^*}^i_j$ of $CF^*(G)$ and Ω^i_j of $C\Gamma(N)$ are related as follows*:

$$(2.7)\qquad {\Omega^*}^i_j = \Omega^i_j + [\delta t^i_j] + [t^i_h t^h_j].$$

Proof. From the definition and the relation $\omega^* = \omega + t$ (without indices), we see

$$\begin{aligned} \Omega^* &= [d\omega^*] + [\omega^*\omega^*] = [d\omega] + [dt] + [(\omega + t)(\omega + t)]\\ &= [d\omega] + [\omega\omega] + [dt] + [\omega t] + [t\omega] + [tt] = \Omega + [\delta t] + [tt], \end{aligned}$$

where we used the matrix product rule. □

We remark that

$$[t\omega] = [t^i_h \omega^h_j] = -[\omega^h_j t^i_h] = -[\omega t] \quad (\text{for the 1-form } t^i_j),$$
$$[\delta t^i_j] := [dt^i_j] + [\omega^i_h t^h_j] - [\omega^h_j t^i_h] \qquad (\text{definition}).$$

As usual in a Finsler space $F^*_n(g)$, we put

$$\Omega^{*i}_j = -\frac{1}{2} R^*{}_j{}^i{}_{kl}[k,l] - P^*{}_j{}^i{}_{kl}[k,(l)^*] - \frac{1}{2} S^*{}_j{}^i{}_{kl}[(k)^*,(l)^*],$$

where $[(k)^*,(l)^*] := [(k),(l)] + P^k{}_r[r,(l)] + P^l{}_r[(k),r] + P^k{}_r P^l{}_s[r,s]$. Hence we get

$$\begin{aligned}\Omega^{*i}_j = &-\frac{1}{2}(R^*{}_j{}^i{}_{kl} + P^*{}_j{}^i{}_{kr}P^r{}_l - P^*{}_j{}^i{}_{lr}P^r{}_k + S^*{}_j{}^i{}_{rs}P^r{}_k P^s{}_l)[k,l] \\ &- (P^*{}_j{}^i{}_{kl} + S^*{}_j{}^i{}_{rl}P^r{}_k)[k,(l)] - \frac{1}{2} S^*{}_j{}^i{}_{kl}[(k),(l)].\end{aligned} \tag{2.8}$$

Let us now carry out the following calculations:

$$\begin{aligned}(a)\quad [\delta t^i_j] &= [\delta(A_j{}^i{}_k dx^k + B_j{}^i{}_k \delta y^k)] \\ &= [\delta A_j{}^i{}_k, dx^k] + [\delta B_j{}^i{}_k, \delta y^k] + A_j{}^i{}_h[\delta dx^h] + B_j{}^i{}_h[\delta\delta y^h] \\ &= -\frac{1}{2}(A_j{}^i{}_{k/l} - A_j{}^i{}_{l/k} + B_j{}^i{}_h R^h{}_{kl})[k,l] \\ &\quad - (A_j{}^i{}_{k/(l)} - B_j{}^i{}_{l/k} + A_j{}^i{}_h C_k{}^h{}_l + B_j{}^i{}_h P^h{}_{kl})[k,(l)] \\ &\quad - B_j{}^i{}_{k/(l)}[(k),(l)], \\ (b)\quad [t^i_h t^h_j] &= -A_j{}^h{}_k A_h{}^i{}_l[k,l] - (A_j{}^h{}_k B_h{}^i{}_l - B_j{}^h{}_l A_h{}^i{}_k)[k,(l)] \\ &\quad - B_j{}^h{}_k B_h{}^i{}_l[(k),(l)],\end{aligned} \tag{2.9}$$

where we used (1.5)(c) and (b). By means of (2.8) and (2.9), the relation (2.7) gives us the following

Proposition 2.6. *In a space M_n, the curvature tensors of $CF^*(G)$ and $C\Gamma(N)$ are connected by the following relations*:

$$\begin{aligned}(a)\quad & R^*{}_j{}^i{}_{kl} + P^*{}_j{}^i{}_{kr}P^r{}_l - P^*{}_j{}^i{}_{lr}P^r{}_k + S^*{}_j{}^i{}_{rs}P^r{}_k P^s{}_l \\ &= R_j{}^i{}_{kl} + B_j{}^i{}_h R^h{}_{kl} + (A_j{}^i{}_{k/l} + A_j{}^h{}_k A_h{}^i{}_l - k|l), \\ (b)\quad & P^*{}_j{}^i{}_{kl} + S^*{}_j{}^i{}_{rl}P^r{}_k \\ &= P_j{}^i{}_{kl} + A_j{}^i{}_{k/(l)} - B_j{}^i{}_{l/k} + A_j{}^i{}_h C_k{}^h{}_l + B_j{}^i{}_h P^h{}_{kl} \\ &\qquad + A_j{}^h{}_k B_h{}^i{}_l - B_j{}^h{}_l A_h{}^i{}_k, \\ (c)\quad & S^*{}_j{}^i{}_{kl} = S_j{}^i{}_{kl} + (B_j{}^i{}_{k/(l)} + B_j{}^h{}_k B_h{}^i{}_l - k|l).\end{aligned} \tag{2.10}$$

2.3. Torsion forms of $[C^*]$ and $[C]$.

Lemma 2.7. *The torsions* Ω^{*i}, $\Omega^{*(i)}$ *of* $CF^*(G)$ *and* Ω^i, $\Omega^{(i)}$ *of* $C\Gamma(N)$ *are related as follows*:

$$(2.11)\qquad \begin{aligned} &(a)\quad \Omega^{*i} = \Omega^i + [t^i_j dx^j],\\ &(b)\quad \Omega^{*(i)} = \Omega^{(i)} + [t^i_j \delta y^j] + [\delta t^i_0] + [t^i_h t^h_0], \end{aligned}$$

where $\Omega^{*i} := [\delta^* dx^i]$ *and* $\Omega^{*(i)} := [\delta^*\delta^* y^i] = \Omega^{*i}_{\ 0}$.

Proof. For (a), we see

$$\Omega^{*i} = [\delta^* dx^i] = [\delta dx^i] + [t^i_j dx^j] = \Omega^i + [t^i_j dx^j].$$

For (b), we see

$$\begin{aligned} \Omega^{*(i)} &= [\delta^*\delta^* y^i] = [\delta\delta^* y^i] + [t^i_h \delta^* y^h] = [\delta(\delta y^i + t^i_0)] + [t^i_h(\delta y^h + t^h_0)]\\ &= \Omega^{(i)} + [\delta t^i_0] + [t^i_h \delta y^h] + [t^i_h t^h_0]. \end{aligned}$$

□

Let us carry out the following calculations:

$$\begin{aligned} &\Omega^{*i} = -{C^*}_j{}^i{}_k[j,(k)^*] = -{C^*}_j{}^i{}_h P^h{}_k[j,k] - {C^*}_j{}^i{}_k[j,(k)],\\ &\Omega^i + [t^i_j dx^j] = -C_j{}^i{}_k[j,(k)] - A_j{}^i{}_k[j,k] - B_j{}^i{}_k[j,(k)],\\ &\Omega^{*(i)} = -\frac{1}{2}H^{*i}{}_{jk}[j,k] - P^{*i}{}_{jh}P^h{}_k[j,k] - P^{*i}{}_{jk}[j,(k)],\\ &[\delta t^i_0] = [\delta P^i{}_k, dx^k] + P^i{}_h[\delta dx^h]\\ &\qquad\ = -P^i{}_{j/k}[j,k] - P^i{}_{j/(k)}[j,(k)] - P^i{}_h C_j{}^h{}_k[j,(k)],\\ &[t^i_j \delta y^j] = A_k{}^i{}_j[j,(k)], \qquad (B_j{}^i{}_k = B_k{}^i{}_j),\\ &[t^i_h t^h_0] = -P^h{}_j A_h{}^i{}_k[j,k] - P^h{}_j B_h{}^i{}_k[j,(k)]. \end{aligned}$$

Using the above and (2.2), we see from (2.11)

$$(2.12)\qquad \begin{aligned} (a)\quad & H^{*i}{}_{jk} + (P^{*i}{}_{jh}P^h{}_k - j|k)\\ &\qquad = R^i{}_{jk} + (P^i{}_{j/k} + P^h{}_j A_h{}^i{}_k - j|k),\\ (b)\quad & P^{*i}{}_{jk} = P^i{}_{jk} + P^i{}_{j/(k)} - A_k{}^i{}_j + P^i{}_h C_j{}^h{}_k + P^h{}_j B_h{}^i{}_k\\ &\qquad = P^i{}_{jk} + P^i{}_{j(k)} - A_k{}^i{}_j + C_h{}^i{}_k P^h{}_j + P^h{}_j B_h{}^i{}_k\\ &\qquad = D_j{}^i{}_k - A_k{}^i{}_j + {C^*}_h{}^i{}_k P^h{}_j\\ &\qquad = D_j{}^i{}_k - A_j{}^i{}_k + {C^*}_h{}^i{}_j P^h{}_k. \end{aligned}$$

If we substitute $P^{*i}{}_{jh}$ in (2.12)(b) into (a), then we have

$$\begin{aligned} H^{*i}{}_{jk} - R^i{}_{jk} &= P^i{}_{j/k} + P^h{}_j A_h{}^i{}_k - (D_j{}^i{}_h - A_h{}^i{}_j + C^*{}_r{}^i{}_h P^r{}_j)P^h{}_k - j|k \\ &= P^i{}_{j/k} + P^h{}_j D_h{}^i{}_k - j|k = E^i{}_{jk}, \quad ((1.17)\ (b)). \end{aligned}$$

Hence we have

Proposition 2.8. *In a space M_n, the torsion tensors of $CF^*(G)$ and $C\Gamma(N)$ are related by the following equations*:

$$\begin{aligned} &(a) \quad P^{*i}{}_{jk} = D_j{}^i{}_k - A_j{}^i{}_k + C^*{}_j{}^i{}_r P^r{}_k, \\ (2.13) \qquad &\qquad {}^*\Gamma_j{}^i{}_k - F_j{}^i{}_k = D_j{}^i{}_k - P^{*i}{}_{jk} = A_j{}^i{}_k - C^*{}_j{}^i{}_r P^r{}_k, \\ &(b) \quad H^{*i}{}_{jk} = R^i{}_{jk} + E^i{}_{jk} = H^i{}_{jk}. \end{aligned}$$

2.4. Curvature tensors of $[R^*]$ and $[R]$.

After the similar calculations of the metrical case, we have for the h-metrical case

Proposition 2.9. *In a space M_n, the curvature tensors of $RF^*(G)$ and $R\Gamma(N)$ are related by the following equations*:

$$\begin{aligned} &(a) \quad K^*{}_j{}^i{}_{kl} + {}^*\Gamma_j{}^i{}_{kh} P^h{}_l - {}^*\Gamma_j{}^i{}_{lh} P^h{}_k \\ (2.14) \qquad &\qquad = K_j{}^i{}_{kl} + \{A_j{}^i{}_{k/l} - C^*{}_j{}^i{}_{h/l} P^h{}_k - C^*{}_j{}^i{}_h P^h{}_{k/l} \\ &\qquad\quad + (A_j{}^h{}_k - C^*{}_j{}^h{}_r P^r{}_k)(A_h{}^i{}_l - C^*{}_h{}^i{}_r P^r{}_l) - k|l\}, \\ &(b) \quad {}^*\Gamma_j{}^i{}_{kl} = F_j{}^i{}_{kl} + A_j{}^i{}_{k(l)} - C^*{}_j{}^i{}_{h(l)} P^h{}_k - C^*{}_j{}^i{}_h P^h{}_{k(l)}. \end{aligned}$$

2.5. The space M_n with $C_{ij/k} = 0$ or $C_{ij/0} = 0$.

Using Proposition 2.3 and Theorem 2.4, we have from (2.10), (2.12) and (2.14)

Proposition 2.10. *In a space M_n with $C_{ij/0} = 0$ we have*

$$\begin{aligned} &(a) \quad P^i{}_k = 0, \quad A_j{}^i{}_k = \frac{1}{2} g^{*ih}(C_{hj/k} + C_{hk/j} - C_{jk/h}), \\ &(b) \quad R^*{}_j{}^i{}_{kl} = R_j{}^i{}_{kl} + B_j{}^i{}_h R^h{}_{kl} + (A_j{}^i{}_{k/l} + A_j{}^h{}_k A_h{}^i{}_l - k|l), \\ (2.15) \qquad &(c) \quad K^*{}_j{}^i{}_{kl} = K_j{}^i{}_{kl} + (A_j{}^i{}_{k/l} + A_j{}^h{}_k A_h{}^i{}_l - k|l), \\ &(d) \quad H^i{}_{jk} = R^i{}_{jk}, \quad P^{*i}{}_{jk} = P^i{}_{jk} - A_j{}^i{}_k, \quad E^i{}_{jk} = 0, \\ &(e) \quad {}^*\Gamma_j{}^i{}_{kl} = F_j{}^i{}_{kl} + A_j{}^i{}_{k(l)}. \end{aligned}$$

Proposition 2.11. *In a space M_n with $C_{ij/k}=0$ we have*

$$
\begin{aligned}
&(a)\quad R^*{}_j{}^i{}_{kl}=R_j{}^i{}_{kl}+B_j{}^i{}_h R^h{}_{kl},\\
&(b)\quad P^*{}_j{}^i{}_{kl}=P_j{}^i{}_{kl}-B_j{}^i{}_{l/k}+B_j{}^i{}_h P^h{}_{kl},\quad P^{*i}{}_{jk}=P^i{}_{jk},\\
&(c)\quad K^*{}_j{}^i{}_{kl}=K_j{}^i{}_{kl},\quad {}^*\Gamma_j{}^i{}_{kl}=F_j{}^i{}_{kl}.
\end{aligned}
\tag{2.16}
$$

§3. A generalized metric space whose associated Finsler space is a Riemannian space

If the metric g_{ij} is independent of y: $C_j{}^i{}_k=0$, then the space M_n itself is a Riemannian space and then its associated Finsler space is also a Riemannian space from the definition.

Definition. A generalized metric space M_n whose associated Finsler space $F^*_n(g)$ is a Riemannian space ($C^*{}_j{}^i{}_k=0$) is called an RM_n space (abbreviation). If the Riemannian space is of constant curvature, then the space M_n is called an $RccM_n$ space.

By means of (2.1)(b) and Proposition 2.3, we see

Theorem 3.1. *If an RM_n space satisfies the condition $C_{ij/(k)}=0$, then the space is a Riemannian space.*

From (2.1)(b) and (2.5)(b) we see

$$3C^*{}_{ijk}=C_{ijk}+C_{jki}+C_{kij}+\frac{1}{2}(C_{ij(k)}+C_{jk(i)}+C_{ki(j)}). \tag{3.1}$$

Hence we have the following

Theorem 3.2. *A space M_n reduces to an RM_n space if the following condition holds*:

$$C_{ijk}+C_{jki}+C_{kij}+\frac{1}{2}(C_{ij(k)}+C_{jk(i)}+C_{ki(j)})=0.$$

S. Numata proved the following theorem ([8],Theorem 2): *A Landsberg space* (in the sense of Finsler geometry) *of scalar curvature K is a Riemannian space of constant curvature provided* $K\neq 0$. Hence we have

Theorem 3.3. *An LM_n space* (cf. §5) *of scalar curvature K is an $RccM_n$ space.*

C. Shibata proved the following theorem ([11], Theorem 4): *If a Finsler space of scalar curvature satisfies the condition* $P^i{}_{hj/k}-j|k=0$ (in the notation of ordinary Finsler geometry), *then the space is a Riemannian space of constant curvature.* Hence we have

Theorem 3.4. *If the Finsler space $F_n^*(g)$ of scalar curvature K satisfies the condition ${P^*}^i{}_{hj}{}^*_{/k} - j|k = 0$, then the space is an $RccM_n$ space.*

From the theory of Finsler spaces, we see that in an RM_n space we have the following relations:

$$
\begin{aligned}
&(a)\quad {}^*\Gamma_j{}^i{}_k = G^*{}_j{}^i{}_k = G_j{}^i{}_k = \{{}_j{}^i{}_k\},\\
(3.2)\quad &(b)\quad P^{*i}{}_{jk} = 0, \quad P^*{}_j{}^i{}_{kl} = {}^*\Gamma_j{}^i{}_{kl} = G^*{}_j{}^i{}_{kl} = G_j{}^i{}_{kl} = S^*{}_j{}^i{}_{kl} = 0,\\
&(c)\quad R^*{}_j{}^i{}_{kl} = K^*{}_j{}^i{}_{kl} = H^*{}_j{}^i{}_{kl} = H_j{}^i{}_{kl}(x),
\end{aligned}
$$

where $\{{}_j{}^i{}_k\}$ is the Christoffel symbol with respect to $g^*{}_{ij}(x)$.
Using (2.10), (2.12), (2.13), (2.14) and (3.2), we have

Proposition 3.5. *In an RM_n space, we have*

$$
\begin{aligned}
&(a)\quad A_j{}^i{}_k = D_j{}^i{}_k = \frac{1}{2}g^{*ih}(C_{hj/k} + C_{hk/j} - C_{jk/h}),\\
&\qquad F_j{}^i{}_k = \{{}_j{}^i{}_k\} - A_j{}^i{}_k, \quad C_j{}^i{}_k = -B_j{}^i{}_k,\\
&\qquad P^i{}_{kl} = A_k{}^i{}_l - P^i{}_{k(l)},\\
(3.3)\quad &(b)\quad H_j{}^i{}_{kl}(x) = K_j{}^i{}_{kl} + E_j{}^i{}_{kl}, \quad H^i{}_{jk} = R^i{}_{jk} + E^i{}_{jk},\\
&\qquad E_j{}^i{}_{kl} = A_j{}^i{}_{k/l} + A_j{}^h{}_k A_h{}^i{}_l - k|l,\\
&\qquad E^i{}_{jk} = P^i{}_{j/k} + P^h{}_j A_h{}^i{}_k - j|k,\\
&(c)\quad P_j{}^i{}_{kl} = -A_j{}^i{}_{k(l)} - C_j{}^i{}_{l/k} + C_j{}^i{}_h P^h{}_{kl},\\
&\qquad F_j{}^i{}_{kl} = -A_j{}^i{}_{k(l)}, \quad G_j{}^i{}_{kl} = 0.
\end{aligned}
$$

Because of $g^{*ih}{}_{(k)} = 0$, Proposition 2.3 and (3.2)(a), we can easily prove

Lemma 3.6. *In an RM_n space, the following four conditions are equivalent:*

$$(a)\quad A_j{}^i{}_k = 0, \quad (b)\quad C_{hj/k} = 0, \quad (c)\quad A_j{}^i{}_{k(l)} = 0, \quad (d)\quad C_{hj/k(l)} = 0.$$

Theorem 3.7. *If an RM_n space satisfies the condition $C_{hj/k} = 0$, then the space M_n is a g-Berwald space ($F_j{}^i{}_{kl} = 0$, cf. §5).*

§4. A generalized metric space whose associated Finsler space is a Minkowski space

Definition. If there exists a coordinate system such that the metric tensor g_{ij} is independent of x: $g_{ij}(y)$ and $P^i{}_k = 0$, then the space M_n

is called a *g-Minkowski* space. If $C_{ij} = 0$, then the g-Minkowski space is called a *Minkowski* space.

Definition. A generalized metric space M_n whose associated Finsler space $F^*_n(g)$ is a Minkowski space is called an MM_n space (abbreviation).

Remark. From the definition $g^*{}_{ij}(y) = \dot\partial_i \dot\partial_j (g_{hk}(y) y^h y^k)/2$, a g-Minkowski space is an MM_n space. However, from the relation: $g^*{}_{ij}(y) = g_{ij}(x,y) + C_{ij}(x,y)$, being an MM_n space ($\partial_k g^*{}_{ij} = 0$) does not mean that the space M_n is a g-Minkowski space ($\partial_k g_{ij} = 0$).

Theorem 4.1 (cf. [6],[12]). *A necessary and sufficient condition for a generalized metric space M_n to be a g-Minkowski space is that the curvature tensors $K_j{}^i{}_{kl}$ and $F_j{}^i{}_{kl}$ vanish ($\Omega^i_j = 0$ for $R\Gamma(G)$).*

Proof. Let us assume that the generalized metric space M_n is a g-Minkowski space. Then we have $F^2(x,y) = \bar F^2 := \bar g_{ab}(\bar y)\bar y^a \bar y^b$ in some suitable coordinate system, hence $\partial_c \bar F^2 = \partial \bar F^2/\partial \bar x^c = 0$. From the definition in §1, we find

$$4\bar G^a = \bar g^{*ab}(\bar y^c \dot\partial_b \partial_c \bar F^2 - \partial_b \bar F^2) = 0, \quad \dot\partial_b = \partial/\partial \bar y^b,$$

$$\bar N^a_b = \bar G^a_b = 0, \quad \partial_c \bar g_{ab} = 0, \quad \bar F_b{}^a{}_c = 0, \quad \bar F_b{}^a{}_{cd} = 0, \quad \bar K_b{}^a{}_{cd} = 0.$$

Conversely, $F_j{}^i{}_{kl} = F_j{}^i{}_{k(l)} = 0$ means that the connection parameters $F_j{}^i{}_k$ are functions of x^i only. Therefore the curvature tensor $K_j{}^i{}_{kl}$ is also a function of x^i only. When $K_j{}^i{}_{kl}(x) = 0$, we know as in a Riemannian space that there exists a coordinate system $(\bar x^a)$ for which the connection parameters $\bar F_b{}^a{}_c$ vanish, that is,

$$\bar g_{ad}\bar F_b{}^d{}_c = \frac{1}{2}(\partial_b \bar g_{ac} + \partial_c \bar g_{ab} - \partial_a \bar g_{bc}) = 0, \quad \bar N^a_c = \bar F_b{}^a{}_c \bar y^b = 0. \tag{4.1}$$

Making $+a|c$ in (4.1), we get $\partial_a \bar g_{bc} = 0$ which means that $\bar g_{bc}$ does not contain $\bar x^a$. Moreover we get $\bar P^a{}_b = 0$ from (1.1). □

Remark. From (1.7)(a), (b) and Theorem 5.14(cf. §5), we see that the conditions in Theorem 4.2 are equivalent to the conditions $R_j{}^i{}_{kl} = 0$ and $C_j{}^i{}_{k/l} = 0$ for $C\Gamma(N)$.

By virtue of a well known theorem on Finsler spaces, we have

Theorem 4.2. *A necessary and sufficient condition for a generalized metric space M_n to be an MM_n space is that the curvature tensors $H_j{}^i{}_{kl}$ and $G_j{}^i{}_{kl}$ vanish ($\Omega^i_j = 0$ for $B\Gamma(G)$).*

From the theory of Finsler spaces, in an MM_n space, we have

$$
(4.2)\qquad
\begin{aligned}
&(a)\quad R^*{}_j{}^i{}_{kl} = K^*{}_j{}^i{}_{kl} = H^*{}_j{}^i{}_{kl} = H_j{}^i{}_{kl} = 0,\\
&\qquad\ R^{*i}{}_{jk} = H^{*i}{}_{jk} = H^i{}_{jk} = 0,\\
&(b)\quad C^*{}_j{}^i{}_{k\overset{*}{/}l} = {}^*\Gamma_j{}^i{}_{kl} = G^*{}_j{}^i{}_{kl} = G_j{}^i{}_{kl} = 0,\\
&\qquad\ P^{*i}{}_{jk} = 0, \quad P^*{}_h{}^i{}_{jk} = 0.
\end{aligned}
$$

Using the relations in §2 and (4.2), we obtain

Proposition 4.3. *In an MM_n space, we have*

$$
\begin{aligned}
&(a)\quad D_j{}^i{}_k = A_j{}^i{}_k - C^*{}_j{}^i{}_h P^h{}_k, \quad R^i{}_{jk} = -E^i{}_{jk},\\
&(b)\quad R_j{}^i{}_{kl} - S^*{}_j{}^i{}_{rs} P^r{}_k P^s{}_l - B_j{}^i{}_h E^h{}_{kl} = -A_j{}^i{}_{k/l} - A_j{}^h{}_k A_h{}^i{}_l - k|l,\\
&(c)\quad F_j{}^i{}_{kl} = -A_j{}^i{}_{k(l)} + C^*{}_j{}^i{}_{h(l)} P^h{}_k + C^*{}_j{}^i{}_h P^h{}_{k(l)},\\
&(d)\quad P_j{}^i{}_{kl} = S^*{}_j{}^i{}_{hl} P^h{}_k - A_j{}^i{}_{k/(l)} + B_j{}^i{}_{l/k} - A_j{}^i{}_h C_k{}^h{}_l - B_j{}^i{}_h P^h{}_{kl}\\
&\qquad\qquad - A_j{}^h{}_k B_h{}^i{}_l + B_j{}^h{}_l A_h{}^i{}_k.
\end{aligned}
$$

In virtue of Proposition 2.3 and $C_{jk/0} = 2g^*{}_{jh} P^h{}_k$, we have that if an MM_n space satisfies the condition $C_{ij/k} = 0$, then the following relations hold:

$$
\begin{aligned}
&(a)\quad D_j{}^i{}_k = 0, \quad (b)\quad R^i{}_{jk} = -E^i{}_{jk} = 0, \quad P^i{}_{jk} = 0,\\
&(c)\quad R_j{}^i{}_{kl} = K_j{}^i{}_{kl} = 0, \quad (d)\quad F_j{}^i{}_{kl} = C_j{}^i{}_{k/l} = 0.
\end{aligned}
$$

Hence we have

Theorem 4.4. *If an MM_n space satisfies the condition $C_{ij/k} = 0$, then the space is a g-Minkowski space.*

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