Publ. Math. Debrecen
83/4 (2013), 697–705
DOI: 10.5486/PMD.2013.5703

Multidegrees of tame automorphisms with one prime number

By JIANTAO LI (Changchun) and XIANKUN DU (Changchun)

Abstract. Let $3 \leq d_1 \leq d_2 \leq d_3$ be integers. We show the following results: (1) If d_2 is a prime number and $\frac{d_1}{\gcd(d_1,d_3)} \neq 2$, then (d_1, d_2, d_3) is a multidegree of a tame automorphism if and only if $d_1 = d_2$ or $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$; (2) If d_3 is a prime number and $\gcd(d_1, d_2) = 1$, then (d_1, d_2, d_3) is a multidegree of a tame automorphism if and only if $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$. We also show that the condition $\frac{d_1}{\gcd(d_1,d_3)} \neq 2$ in (1) cannot be removed.

1. Introduction

Throughout this paper, let $F = (F_1, \ldots, F_n) : k^n \to k^n$ be a polynomial map, where k is a field of characteristic 0. Denote by $\operatorname{Aut} k^n$ the group of all polynomial automorphisms of k^n. Denote by $\operatorname{mdeg} F := (\deg F_1, \ldots, \deg F_n)$ the *multidegree* of F and by mdeg the mapping from the set of all polynomial maps into the set $\mathbb{N}^n$, where $\mathbb{N}$ denotes the set of all nonnegative integers.

A polynomial automorphism $F = (F_1, \ldots, F_n)$ of k^n is said to be *elementary* if

$$F = (x_1, \ldots, x_{i-1}, \alpha x_i + f, x_{i+1}, \ldots, x_n)$$

for some $1 \leq i \leq n$, $\alpha \in k^*$ and $f \in k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Denote by $\operatorname{Tame} k^n$ the subgroup of $\operatorname{Aut} k^n$ that is generated by all the elementary automorphisms. An element in $\operatorname{Tame} k^n$ is called a *tame automorphism*. The clas-

---

Mathematics Subject Classification: 14R10.
Key words and phrases: multidegree, tame automorphism, elementary reduction, Poisson bracket.
This research was supported by NSF of China (No. 11071097, No. 11101176) and "211 Project" and "985 Project" of Jilin University.

sical JUNG-VAN DER KULK theorem [4], [17] says that every polynomial automorphism of k^2 is tame. In 2004, SHESTAKOV and UMIRBAEV [14], [15] proved $\operatorname{Aut} k^3 \neq \operatorname{Tame} k^3$ by showing that the famous Nagata automorphism is not tame.

The multidegree plays an important role in the description of polynomial automorphisms. For example, the Jacobian conjecture is equivalent to the assertion that, if (F_1, F_2) is a polynomial map satisfying the Jacobian condition, then $\operatorname{mdeg} F = (\deg F_1, \deg F_2)$ is principal, that is, $\deg F_1 \mid \deg F_2$ or $\deg F_2 \mid \deg F_1$ [1]. But it is difficult to describe the multidegrees of polynomial maps in higher dimensional cases, even in the case of dimension three. Recently, some authors present papers concerning the multidegrees of tame automorphisms in dimension three, see [5], [6], [7], [8], [10], [16].

In [5], KARAŚ proposed the following conjecture.

Conjecture 1.1 ([5, Conjecture 4.1]). Let $3 \leq p_1 \leq d_2 \leq d_3$ be integers with p_1 a prime number. Then $(p_1, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame} k^3)$ if and only if $p_1 \mid d_2$ or $d_3 \in p_1\mathbb{N} + d_2\mathbb{N}$.

In [6], KARAŚ showed that if $\frac{d_3}{d_2} \neq \frac{3}{2}$ or $\frac{d_3}{d_2} = \frac{3}{2}$ and $d_2 > 2p_1 - 4$, then Conjecture 1.1 is valid. In [16], SUN and CHEN gave a stronger result that Conjecture 1.1 is true if $\frac{d_3}{d_2} \neq \frac{3}{2}$, or if $\frac{d_3}{d_2} = \frac{3}{2}$ and $d_2 > 2p_1 - 5$.

In this paper, we consider a variation of the conjecture of Karaś. Let $3 \leq d_1 \leq d_2 \leq d_3$ be integers. We show the following results: (1) If d_2 is a prime number and $\frac{d_1}{\gcd(d_1,d_3)} \neq 2$, then $(d_1, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame} k^3)$ if and only if $d_1 = d_2$ or $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$; (2) If d_3 is a prime number and $\gcd(d_1, d_2) = 1$, then $(d_1, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame} k^3)$ if and only if $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$. We also show that the condition $\frac{d_1}{\gcd(d_1,d_3)} \neq 2$ in (1) cannot be removed.

2. Preliminaries

Recall that, in [14], [15], a pair $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ is said to be $*$-*reduced* if

(1) f, g are algebraically independent;

(2) $\bar{f}, \bar{g}$ are algebraically dependent, where $\bar{f}$ denotes the highest homogeneous component of f;

(3) $\bar{f} \notin \langle \bar{g} \rangle$ and $\bar{g} \notin \langle \bar{f} \rangle$.

The following inequality plays an important role in the proof of the Nagata conjecture in [14], [15] and is also essential in our proofs.

Theorem 2.1 ([14, Theorem 3])**.** *Let $f, g \in k[x_1, \ldots, x_n]$ be a $*$-reduced pair, and $G(x, y) \in k[x, y]$ with $\deg_y G(x, y) = pq + r$, $0 \le r < p$, where $p = \frac{\deg f}{\gcd(\deg f, \deg g)}$. Then*

$$\deg G(f, g) \ge q(p \deg g - \deg f - \deg g + \deg[f, g]) + r \deg g.$$

Here, $[f, g]$ means the Poisson bracket of f and g defined by

$$[f, g] = \sum_{1 \le i < j \le n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right) [x_i, x_j].$$

By definition, $\deg[x_i, x_j] = 2$ for $i \ne j$, $\deg 0 = -\infty$, and

$$\deg[f, g] = \max_{1 \le i < j \le n} \deg \left\{ \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right) [x_i, x_j] \right\}.$$

Note that $[f, g] \ne 0$ if and only if f, g are algebraically independent over k. If this is the case, we have

$$\deg[f, g] = 2 + \max_{1 \le i < j \le n} \deg \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right) \ge 2.$$

Remark 2.2. The statement of Theorem 2.1 holds if only f and g do not belong to k due to Kuroda [11, Corollary 3.5] (see also [8], [18]).

Recall that a polynomial automorphism $F = (F_1, F_2, F_3)$ is said to admit an *elementary reduction* if there exists a permutation σ of the set $\{1, 2, 3\}$ and $g \in k[x, y]$ such that

$$\deg(F_{\sigma(1)} - g(F_{\sigma(2)}, F_{\sigma(3)})) < \deg F_{\sigma(1)}.$$

Theorem 2.3 ([15, Theorem 2])**.** *Let $F = (F_1, F_2, F_3)$ be a tame automorphism of k^3. If $\deg F_1 + \deg F_2 + \deg F_3 > 3$, then F admits either an elementary reduction or a reduction of types* I–IV.

We refer to [15, Definitions 1–4] for the definitions of reductions of types I–IV.

Remark 2.4. It is shown by Kuroda that there is no tame automorphism on $k[x, y, z]$ admitting reductions of type IV, see [13, Theorem 7.1].

In this paper, we consider when (d_1, d_2, d_3) is a multidegree of a tame automorphism on k^3. Note that, if (F_1, F_2, F_3) is a tame automorphism, then $(F_{\sigma(1)}, F_{\sigma(2)}, F_{\sigma(3)})$ is also a tame automorphism for any permutation σ of $\{1, 2, 3\}$. If $d_1 \le d_2 \le d_3$ and $d_1 < 3$, then $(d_1, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame} k^3)$ by [8, Example 3.1]. Thus, without loss of generality, we can assume that $3 \le d_1 \le d_2 \le d_3$.

3. Multidegree (d_1, p_2, d_3) with p_2 a prime number

In this section, let $3 \le d_1 \le p_2 \le d_3$ be integers with p_2 a prime number. We start with the following lemma.

Lemma 3.1. *If $(d_1, p_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame} k^3)$, then there exists a tame automorphism with multidegree (d_1, p_2, d_3) which admits an elementary reduction.*

Proof. Let F be a tame automorphism with $\operatorname{mdeg} F = (d_1, p_2, d_3)$. By Theorem 2.3 and Remark 2.4, F admits an elementary reduction or a reduction of types I–III.

If F admits a reduction of type III, then by [15, Definition 3] there exists $n \in \mathbb{N}$ such that

$$n < d_1 \le \frac{3}{2}n, \quad p_2 = 2n, \quad d_3 = 3n; \quad \text{or} \tag{3.1}$$

$$d_1 = \frac{3}{2}n, \quad p_2 = 2n, \quad \frac{5n}{2} < d_3 \le 3n. \tag{3.2}$$

Since p_2 is a prime number greater than 3, (3.1) and (3.2) cannot be satisfied. Thus, F admits no reduction of type III.

By the definitions of reductions of types I and II, if F admits a reduction of type I or II, then there exists a tame automorphism with the same multidegree that admits an elementary reduction (see [6, Proposition 20]). $\square$

To prove our main theorem, we also use the following well-known result (see e.g. [2]).

Lemma 3.2. *If a, b are positive integers with $\gcd(a, b) = 1$, then $l \in a\mathbb{N} + b\mathbb{N}$ for all integers $l \ge (a-1)(b-1)$.*

We are now in a position to show our main result in this section.

Theorem 3.3. *Let $3 \le d_1 \le p_2 \le d_3$ be integers with p_2 a prime number. If $\frac{d_1}{\gcd(d_1, d_3)} \ne 2$, then $(d_1, p_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame} k^3)$ if and only if $d_1 = p_2$ or $d_3 \in d_1\mathbb{N} + p_2\mathbb{N}$.*

Proof. Thanks to [8, Proposition 2.2], it suffices to prove the "only if" part. Suppose that $d_1 < p_2$ and $d_3 \notin d_1\mathbb{N} + p_2\mathbb{N}$. Then, we have $d_3 < (d_1 - 1)(p_2 - 1)$ by Lemma 3.2. We prove that no tame automorphism has multidegree (d_1, p_2, d_3) by contradiction. If the assertion is false, then there exists a tame automorphism $F = (F_1, F_2, F_3)$ with $\operatorname{mdeg} F = (d_1, p_2, d_3)$ admitting an elementary reduction by Lemma 3.1. There exist three cases to be considered as follows.

Case 1: If F admits an elementary reduction for some $g \in k[x,y]$ such that $\deg(F_3 - g(F_1,F_2)) < \deg F_3$, then we have $\deg F_3 = \deg g(F_1,F_2)$. Since F is a polynomial automorphism, F_i, F_j $(i,j = 1,2,3, i \neq j)$ are algebraically independent, and hence $\deg[F_i,F_j] \geq 2$. Moreover, $\bar{F_i} \notin \langle \bar{F_j} \rangle$ since otherwise we have $\deg F_i \mid \deg F_j$, which contradicts that $d_1 \nmid p_2$ and $d_3 \notin d_1\mathbb{N}+p_2\mathbb{N}$. Note that $\frac{\deg F_1}{\gcd(\deg F_1, \deg F_2)} = d_1$. Set $\deg_y g(x,y) = d_1 q + r,\ 0 \leq r < d_1$. By Theorem 2.1 and Remark 2.2,

$$\begin{aligned} d_3 = \deg F_3 = \deg g(F_1,F_2) &\geq q(d_1p_2 - d_1 - p_2 + \deg[F_1,F_2]) + rp_2 \\ &\geq q(d_1p_2 - d_1 - p_2 + 2) + rp_2. \end{aligned}$$

Since $d_3 < (d_1-1)(p_2-1)$, we have $q = 0$. Hence, we get $\deg_y g(x,y) = r$. Noting that $0 \leq r < d_1$, we can write $g(x,y) = g_0(x) + g_1(x)y + \cdots + g_{d_1-1}(x)y^{d_1-1}$. It follows from $\gcd(d_1,p_2) = 1$ that the sets $d_1\mathbb{N}, d_1\mathbb{N}+p_2, \ldots, d_1\mathbb{N}+(d_1-1)p_2$ are disjoint. Thus,

$$\begin{aligned} d_3 = \deg g(F_1,F_2) &= \deg(g_0(F_1) + g_1(F_1)F_2 + \cdots + g_{d_1-1}(F_1)F_2^{d_1-1}) \\ &= \max_{0 \leq i \leq d_1-1}\{\deg F_1 \deg g_i + i \deg F_2\} = \max_{0 \leq i \leq d_1-1}\{d_1 \deg g_i + ip_2\}, \end{aligned}$$

which contradicts $d_3 \notin d_1\mathbb{N} + p_2\mathbb{N}$.

Case 2: If F admits an elementary reduction for some $g \in k[x,y]$ such that $\deg(F_1 - g(F_2,F_3)) < \deg F_1$, then $\deg F_1 = \deg g(F_2,F_3)$. Note that $\frac{\deg F_2}{\gcd(\deg F_2, \deg F_3)} = p_2$. Set $\deg_y g(x,y) = p_2 q + r,\ 0 \leq r < p_2$. Then

$$\begin{aligned} d_1 = \deg F_1 = \deg g(F_2,F_3) &\geq q(p_2d_3 - p_2 - d_3 + \deg[F_2,F_3]) + rd_3 \\ &\geq q(3d_3 - p_2 - d_3 + 2) + rd_3 \geq q((d_3 - p_2) + d_3 + 2) + rd_3. \end{aligned}$$

Since $d_1 < (d_3 - p_2) + d_3 + 2$ and $d_1 < d_3$, it follows that $q = r = 0$. Hence, $g(x,y)$ belongs to $k[x]$. Thus, $d_1 = \deg F_1 = \deg g_1(F_2)$ belongs to $p_2\mathbb{N}$. This contradicts $d_1 < p_2$.

Case 3: If F admits an elementary reduction for some $g \in k[x,y]$ such that $\deg(F_2 - g(F_1,F_3)) < \deg F_2$, then $\deg F_2 = \deg g(F_1,F_3)$. It follows from $d_3 \notin d_1\mathbb{N} + p_2\mathbb{N}$ that $\gcd(d_1,d_3) \neq d_1$, whence $p = \frac{d_1}{\gcd(d_1,d_3)} \geq 2$. Moreover, since $\frac{d_1}{\gcd(d_1,d_3)} \neq 2$ by assumption, we have $p \geq 3$. Let $\deg_y g(x,y) = pq + r,\ 0 \leq r < p$. Then

$$\begin{aligned} p_2 = \deg F_2 = \deg g(F_1,F_3) &\geq q(pd_3 - d_1 - d_3 + \deg[F_1,F_3]) + rd_3 \\ &\geq q(3d_3 - d_1 - d_3 + 2) + rd_3 = q((d_3 - d_1) + d_3 + 2) + rd_3. \end{aligned}$$

Since $p_2 < d_3$, we get $q = r = 0$. Hence, $g(x,y)$ belongs to $k[x]$. Thus, $p_2 = \deg F_2 = \deg g_1(F_1)$ belongs to $d_1\mathbb{N}$. This contradicts that p_2 is a prime number with $p_2 > d_1$.

Therefore, F cannot admit any elementary reduction, the contradiction implies that there exists no tame automorphism with multidegree (d_1, p_2, d_3) if $d_1 < p_2$ and $d_3 \notin d_1\mathbb{N} + p_2\mathbb{N}$. □

We claim that the condition $\frac{d_1}{\gcd(d_1,d_3)} \neq 2$ in Theorem 3.3 cannot be removed. Indeed, for each positive integers p and q, KURODA [12] constructed a tame automorphism with multidegree $(2m, 2pm+p+1, (2p+1)m)$ with $m = pq+p+q$ which admits a reduction of type I. In the special cases where $(p, q) = (2, 1), (2, 3)$, the multidegrees are equal to $(10, 23, 25)$ and $(22, 47, 55)$, for which we have $\frac{d_1}{gcd(d_1,d_3)} = 2$. We note that, if $(p, q) = (2, 1)$, this tame automorphism is described as

$$\begin{cases} f_1 = x + y^2 - g^2, \\ f_2 = \dfrac{256}{25} f_1^5 + g + h^2, \\ f_3 = f_2 + h, \end{cases}$$

where $g = z + 3x^2y + 3xy^3 + y^5$ and $h = y - 6(x + y^2)^2 g + 8(x + y^2)g^3 - \frac{16}{5} g^5$.

From the proof of Theorem 3.3, we see that a more precise lower bound of $\deg[F_1, F_3]$ gives a better description of $\operatorname{mdeg}(\operatorname{Tame} k^3)$. We mention that SUN–CHEN [16] and KARAŚ [9] gave notable results on the multidegrees of tame automorphisms by improving the lower bound of the degrees of Poisson brackets.

4. Multidegree (d_1, d_2, p_3) with p_3 a prime number

In this section, let $3 \le d_1 \le d_2 \le p_3$ be integers with $\gcd(d_1, d_2) = 1$ and p_3 a prime number.

Lemma 4.1. *If $(d_1, d_2, p_3) \in \operatorname{mdeg}(\operatorname{Tame} k^3)$ with $\gcd(d_1, d_2) = 1$ and p_3 a prime number, then there exists a tame automorphism with multidegree (d_1, d_2, p_3) which admits an elementary reduction.*

PROOF. Let F be a tame automorphism with $\operatorname{mdeg} F = (d_1, d_2, p_3)$. By Theorem 2.3 and Remark 2.4, F admits an elementary reduction or a reduction of types I–III.

If F admits a reduction of type III, then by [15, Definition 3] there exists $n \in \mathbb{N}$ such that

$$n < d_1 \le \frac{3}{2}n, \quad d_2 = 2n, \quad p_3 = 3n; \quad \text{or} \tag{4.1}$$

$$d_1 = \frac{3}{2}n, \quad d_2 = 2n, \quad \frac{5n}{2} < p_3 \leq 3n. \tag{4.2}$$

Since p_3 is a prime number greater that 3, (4.1) cannot be satisfied. If (d_1, d_2, p_3) satisfies (4.2), it follows from $\gcd(d_1, d_2) = 1$ that $n = 2$. Hence $5 < p_3 \leq 6$, and so $p_3 = 6$. This contradicts that p_3 is a prime number. Thus, F admits no reduction of type III.

By [6, Proposition 20], if F admits a reduction of type I or II, then there exists a tame automorphism with the same multidegree that admits an elementary reduction. □

We can now formulate our main result in this section.

Theorem 4.2. *Let* $3 \leq d_1 \leq d_2 \leq p_3$ *be integers with* $\gcd(d_1, d_2) = 1$ *and* p_3 *a prime number. Then* $(d_1, d_2, p_3) \in \operatorname{mdeg}(\operatorname{Tame} k^3)$ *if and only if* $p_3 \in d_1\mathbb{N} + d_2\mathbb{N}$.

Proof. Thanks to [8, Proposition 2.2], it suffices to prove the "only if" part. Suppose that $p_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$, whence $p_3 < (d_1 - 1)(d_2 - 1)$ by Lemma 3.2. We prove that no tame automorphism has multidegree (d_1, p_2, d_3) by contradiction. If the assertion is false, then there exists a tame automorphism $F = (F_1, F_2, F_3)$ with $\operatorname{mdeg} F = (d_1, d_2, p_3)$ admitting an elementary reduction by Lemma 4.1. There exist three cases to be considered as follows.

Case 1: If F admits an elementary reduction for some $g \in k[x, y]$ such that $\deg(F_1 - g(F_2, F_3)) < \deg F_1$, then $\deg F_1 = \deg g(F_2, F_3)$. Note that $\frac{\deg F_2}{\gcd(\deg F_2, \deg F_3)} = d_2$. Set $\deg_y g(x, y) = d_2 q + r,\ 0 \leq r < d_2$. Then

$$\begin{aligned} d_1 = \deg F_1 = \deg g(F_2, F_3) &\geq q(d_2 p_3 - d_2 - p_3 + \deg[F_2, F_3]) + r p_3 \\ &\geq q(3p_3 - d_2 - p_3 + 2) + r p_3 \geq q((p_3 - d_2) + p_3 + 2) + r p_3. \end{aligned}$$

Thus, $q = r = 0$. Hence, $g(x, y)$ belongs to $k[x]$. Thus, $d_1 = \deg F_1 = \deg g_1(F_2)$ belongs to $d_2\mathbb{N}$. This contradicts $d_1 < d_2$.

Case 2: If F admits an elementary reduction for some $g \in k[x, y]$ such that $\deg(F_2 - g(F_1, F_3)) < \deg F_2$, then $\deg F_2 = \deg g(F_1, F_3)$. Note that $\frac{\deg F_1}{\gcd(\deg F_1, \deg F_3)} = d_1$. Set $\deg_y g(x, y) = d_1 q + r,\ 0 \leq r < d_1$. Then

$$\begin{aligned} d_2 = \deg F_2 = \deg g(F_1, F_3) &\geq q(d_1 p_3 - d_1 - p_3 + \deg[F_1, F_3]) + r p_3 \\ &\geq q(3p_3 - d_1 - p_3 + 2) + r p_3 = q((p_3 - d_1) + p_3 + 2) + r p_3. \end{aligned}$$

Thus, $q = r = 0$. Hence, $g(x, y)$ belongs to $k[x]$. Thus, $d_2 = \deg F_2 = \deg g_1(F_1)$ belongs to $d_1\mathbb{N}$. This contradicts $\gcd(d_1, d_2) = 1$.

Case 3: If F admits an elementary reduction for some $g \in k[x,y]$ such that $\deg(F_3 - g(F_1,F_2)) < \deg F_3$, then $\deg F_3 = \deg g(F_1,F_2)$. It follows from $\gcd(d_1,d_2) = 1$ that $\frac{\deg F_1}{\gcd(\deg F_1, \deg F_2)} = d_1$. Set $\deg_y g(x,y) = d_1 q + r$, $0 \le r < d_1$. Then

$$\begin{aligned} p_3 = \deg F_3 = \deg g(F_1,F_2) &\ge q(d_1 d_2 - d_1 - d_2 + \deg[F_1,F_2]) + r d_2 \\ &\ge q(d_1 d_2 - d_1 - d_2 + 2) + r d_2. \end{aligned}$$

Since $p_3 < (d_1-1)(d_2-1)$, we have $q = 0$. Hence, we get $\deg_y g(x,y) = r$. Noting that $0 \le r < d_1$, we can write $g(x,y) = g_0(x) + g_1(x)y + \cdots + g_{d_1-1}(x)y^{d_1-1}$. It follows from $\gcd(d_1,d_2) = 1$ that the sets $d_1\mathbb{N}, d_1\mathbb{N} + d_2, \ldots, d_1\mathbb{N} + (d_1-1)d_2$ are disjoint. Thus,

$$\begin{aligned} p_3 &= \deg g(F_1,F_2) = \deg(g_0(F_1) + g_1(F_1)F_2 + \cdots + g_{d_1-1}(F_1)F_2^{d_1-1}) \\ &= \max_{0 \le i \le d_1-1} \{\deg F_1 \deg g_i + i \deg F_2\} = \max_{0 \le i \le d_1-1} \{d_1 \deg g_i + i d_2\}, \end{aligned}$$

which contradicts $p_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$.

Thus, F admits no elementary reduction, the contradiction implies that there exists no tame automorphism with multidegree (d_1, d_2, p_3) if $p_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$. □

Acknowledgment. The authors would like to thank the editor and referees for their careful reading and constructive comments.

References

[1] S. S. Abhyankar, Some thoughts on the Jacobian conjecture, I, *J. Algebra* **319**(2) (2008), 493–548.

[2] A. Brauer, On a problem of partitions, *Amer. J. Math.* **64** (1942), 299–312.

[3] V. Drensky and J.-T. Yu, Degree estimate for commutators, *J. Algebra* **322**(7) (2009), 2321–2334.

[4] H. W. E. Jung, Über ganze birationale Transformationen der Ebene, *J. Reine Angew. Math.* **184** (1942), 161–174.

[5] M. Karaś, Tame automorphisms of $\mathbb{C}^3$ with multidegree of the form $(3, d_2, d_3)$, *J. Pure Appl. Algebra* **214**(12) (2010), 2144–2147.

[6] M. Karaś, Multidegrees of tame automorphisms of $\mathbb{C}^n$, *Dissertationes Math. (Rozprawy Mat.)* **477** (2011), 55pp.

[7] M. Karaś, Tame automorphisms of $\mathbb{C}^3$ with multidegree of the form (p_1, p_2, d_3), *Bull. Pol. Acad. Sci. Math.* **59**(1) (2011), 27–32.

[8] M. Karaś, There is no tame automorphism of $\mathbb{C}^3$ with multidegree $(3,4,5)$, *Proc. Amer. Math. Soc.* **139**(3) (2011), 769–775.

[9] M. Karaś, There is no tame automorphism of $\mathbb{C}^3$ with multidegree $(4,5,6)$, arXiv:math.AG/1104.1061v1.

[10] M. Karaś and J. Zygadło, On multidegrees of tame and wild automorphisms of $\mathbb{C}^3$, *J. Pure Appl. Algebra* **215**(12) (2011), 2843–2846.

[11] S. Kuroda, A generalization of the Shestakov–Umirbaev inequality, *J. Math. Soc. Japan* **60** (2008), 495–510.

[12] S. Kuroda, Automorphisms of a polynomial ring which admit reductions of type I, *Publ. Res. Inst. Math. Sci.* **45**(3) (2009), 907–917.

[13] S. Kuroda, Shestakov–Umirbaev reductions and Nagata's conjecture on a polynomial automorphism, *Tohoku Math. J.* (2) **62**(1) (2010), 75–115.

[14] I. P. Shestakov and U. U. Umirbaev, Poisson brackets and two-generated subalgebras of rings of polynomials, *J. Amer. Math. Soc.* **17**(1) (2004), 181–196.

[15] I. P. Shestakov and U. U. Umirbaev, The tame and the wild automorphisms of polynomial rings in three variables, *J. Amer. Math. Soc.* **17**(1) (2004), 197–227.

[16] X. Sun and Y. Chen, Multidegrees of tame automorphisms in dimension three, *Pub. Res. Inst. Math. Sci* **48**(1) (2012), 129–137.

[17] W. van der Kulk, On polynomial rings in two variables, *Nieuw Arch. Wiskunde* (3) **1** (1953), 33–41.

[18] S. Vénéreau, A parachute for the degree of a polynomial in algebraically independent ones, *Math. Ann.* **349** (2011), 589–597.

JIANTAO LI
SCHOOL OF MATHEMATICS
JILIN UNIVERSITY
130012 CHANGCHUN
CHINA

E-mail: jtlimath@gmail.com

XIANKUN DU
SCHOOL OF MATHEMATICS
JILIN UNIVERSITY
130012 CHANGCHUN
CHINA

E-mail: duxk@jlu.edu.cn

(Received November 15, 2012; revised March 6, 2013)