Publ. Math. Debrecen
83/4 (2013), 707–714
DOI: 10.5486/PMD.2013.5712

A characterization of type A real hypersurfaces in complex projective space

By JUAN DE DIOS PÉREZ (Granada) and YOUNG JIN SUH (Taegu)

Abstract. We classify real hypersurfaces in complex projective space whose shape operator is of Codazzi type with respect to a generalized Tanaka–Webster connection with a condition on the principal curvature of the structure vector field. As a consequence we classify real hypersurfaces in complex projective space whose shape operator is generalized Tanaka–Webster parallel with the same condition.

1. Introduction

Let $\mathbb{C}P^m$, $m \geq 2$, be a *complex projective space* endowed with the metric g of constant holomorphic sectional curvature 4. Let M be a *connected real hypersurface* of $\mathbb{C}P^m$ without boundary, ∇ the Levi–Civita connection on M and J the complex structure of $\mathbb{C}P^m$. Take a locally defined unit normal vector field N on M and denote by $\xi = -JN$. This is a tangent vector field to M called the structure vector field on M. On M there exists an almost contact metric structure (ϕ, ξ, η, g) induced by the Kaehlerian structure of $\mathbb{C}P^m$, where ϕ is the tangent component of J and η is an one-form given by $\eta(X) = g(X, \xi)$ for any X tangent to M. Let us denote by A the shape operator on M associated to N. We will say that M is Hopf if the structure vector field is principal, that is, $A\xi = \alpha\xi$ for

Mathematics Subject Classification: 53C15, 53B25.
Key words and phrases: g-Tanaka–Webster connection; complex projective space, real hypersurface; shape operator, Codazzi type.
This work was supported by grant Proj. No. NRF-2011-220-1-C00002 from National Research Foundation of Korea. First author is partially supported by MEC Project MTM 2010-18099. Second author is partially supported by Project No. NRF-2012-R1A2A2A-01043023.

a certain function α on M. We will also denote by $\mathfrak{D}$ the maximal holomorphic distribution on M, given by all vector fields orthogonal to ξ.

The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [8], [9], [10] and consists in six different types of real hypersurfaces. Kimura, [4], also proved that they are the unique Hopf real hypersurfaces with constant principal curvatures.

Type A_1 are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$. They have 2 distinct constant principal curvatures, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$ and $\cot(r)$ with eigenspace $\mathfrak{D}$.

Type A_2 are tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m-1$. They have 3 distinct constant principal curvatures, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot(r)$ and $-\tan(r)$. The corresponding eigenspaces of $\cot(r)$ and $-\tan(r)$ are complementary and ϕ-invariant distributions in $\mathfrak{D}$.

From now on, we will call type A real hypersurfaces to both of either type A_1 or type A_2.

Type B are tubes of radius r, $0 < r < \frac{\pi}{4}$, over the complex quadric. They have 3 distinct constant principal curvatures, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot\left(r - \frac{\pi}{4}\right)$ and $-\tan\left(r - \frac{\pi}{4}\right)$ whose corresponding eigenspaces are complementary and equal dimensional distributions in $\mathfrak{D}$ such that $\phi T_{\cot(r-\frac{\pi}{4})} = T_{-\tan(r-\frac{\pi}{4})}$.

Type C are tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^n$, where $2n+1 = m$ and $m \geq 5$. They have 5 distinct constant principal curvatures, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot\left(r - \frac{\pi}{4}\right)$ with multiplicity 2, $\cot\left(r - \frac{\pi}{2}\right) = -\tan(r)$ with multiplicity $m-3$, $\cot\left(r - \frac{3\pi}{4}\right)$, with multiplicity 2 and $\cot(r-\pi) = \cot(r)$ with multiplicity $m-3$. Moreover $\phi T_{\cot(r-\frac{\pi}{4})} = T_{\cot(r-\frac{3\pi}{4})}$ and $T_{-\tan(r)}$ and $T_{\cot(r)}$ are ϕ-invariant.

Type D are tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Plucker embedding of the complex Grassmannian manifold $G(2,5)$ in $\mathbb{C}P^9$. They have the same principal curvatures as type C real hypersurfaces, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, and the other 4 principal curvatures have the same multiplicity 4 and their eigenspaces have the same behaviour with respect to ϕ as in type C.

Type E are tubes of radius r, $0 < r < \frac{\pi}{4}$, over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^{15}$.They also have the same principal curvatures as type C real hypersurfaces, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot\left(r - \frac{\pi}{4}\right)$ and $\cot\left(r - \frac{3\pi}{4}\right)$ have multiplicities equal to 6 and $-\tan(r)$ and $\cot(r)$ have multiplicities equal to 8. Their corresponding eigenspaces have the same behaviour with respect to ϕ as in type C.

The Tanaka–Webster connection, [11], [13], is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR-manifold. As a generalization of this connection, Tanno, [12], defined the generalized Tanaka–Webster connection for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y. \tag{1.1}$$

Using the naturally extended affine connection of Tanno's generalized Tanaka–Webster connection, a g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ for a real hypersurface M in $\mathrm{C}P^m$ is given, see [2], [3], by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.2}$$

for any X, Y tangent to M where k is a non-zero real number. Then $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)}g = 0$, $\hat{\nabla}^{(k)}\phi = 0$. In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the g-Tanaka–Webster connection coincides with the Tanaka–Webster connection.

From the Codazzi equation (see paragraph 2) it is very easy to see that there do not exist real hypersurfaces in $\mathrm{C}P^m$ satisfying $(\nabla_X A)Y = (\nabla_Y A)X$, for any X, Y tangent to M.

Our purpose is to study a similar condition for a g-Tanaka–Webster connection. We will say that the shape operator of M is of Codazzi type with respect to a g-Tanaka–Webster connection if it satisfies

$$(\hat{\nabla}^{(k)}_X A)Y = (\hat{\nabla}^{(k)}_Y A)X \tag{1.3}$$

for any X, Y tangent to M. Thus we will prove the following theorems

Theorem 1.1. *Let M be a real hypersurface of $\mathrm{C}P^m$, $m \geq 3$, whose shape operator is of Codazzi type with respect to a g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$. Then M must be a Hopf hypersurface.*

Theorem 1.2. *Let M be a Hopf hypersurface of $\mathrm{C}P^m$, $m \geq 2$ and let $\hat{\nabla}^{(k)}$ be a g-Tanaka–Webster connection. Then M is of Codazzi type with respect to $\hat{\nabla}^{(k)}$ and $\alpha \neq 2k$ if and only if M is locally congruent to a real hypersurface of type A.*

As a consequence we obtain the

Corollary 1.3. *Let M be a real hypersurface in $\mathrm{C}P^m$, $m \geq 3$, with $\alpha \neq 2k$. Then its shape operator is g-Tanaka–Webster parallel if and only if M is locally congruent to a real hypersurface of type A.*

2. Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi–Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M, see [1]. That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any tangent vectors X, Y to M. From (2.1) we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \phi AX \tag{2.4}$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \tag{2.6}$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M. We will call the maximal holomorphic distribution $\mathfrak{D}$ on M to the following one: at any $p \in M$, $\mathfrak{D}(p) = \{X \in T_pM \text{ such that } g(X, \xi) = 0\}$.

In the sequel we need the following results:

Theorem 2.1 ([7])**.** *Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geq 2$. Then the following are equivalent:*

1. *M is locally congruent to a real hypersurface of type A.*
2. *$\phi A = A\phi$.*

Theorem 2.2 ([5])**.** *Let M be a Hopf real hypersurface of $\mathbb{C}P^m$, $m \geq 2$, and let $X \in \mathfrak{D}$ such that $AX = \lambda X$. Then $\alpha = g(A\xi, \xi)$ is constant and $A\phi X = \frac{\lambda\alpha + 2}{2\lambda - \alpha}\phi X$.*

3. Proof of the Theorem

If we suppose that the shape operator A is of Codazzi type with respect to a g-Tanaka–Webster connection we obtain

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= g(\phi AY, AX)\xi - g(\phi AX, AY)\xi - \eta(AX)\phi AY \\ &+ \eta(AY)\phi AX - k\eta(Y)\phi AX + k\eta(X)\phi AY - g(\phi AY, X)A\xi + g(\phi AX, Y)A\xi \\ &+ \eta(X)A\phi AY - \eta(Y)A\phi AX + k\eta(Y)A\phi X - k\eta(X)A\phi Y. \qquad (3.1)\end{aligned}$$

From the Codazzi equation (3.1) becomes

$$\begin{aligned}\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi &= -2g(A\phi AX, Y)\xi - \eta(AX)\phi AY \\ &+ \eta(AY)\phi AX - k\eta(Y)\phi AX + k\eta(X)\phi AY + g((\phi A + A\phi)X, Y)A\xi \\ &+ \eta(X)A\phi AY - \eta(Y)A\phi AX + k\eta(Y)A\phi X - k\eta(X)A\phi Y. \qquad (3.2)\end{aligned}$$

for any X, Y tangent to M:

First we suppose that M is not Hopf. So we can write $A\xi = \alpha\xi + \beta U$ locally, where U is a unit vector field in $\mathfrak{D}$ and $\beta \neq 0$ is a function on M.

Taking $X = \xi$ in (3.2) and its scalar product with ξ we have

$$-3\beta A\phi U + \beta(\alpha + k)\phi U = 0. \qquad (3.3)$$

As $\beta \neq 0$ it follows

$$A\phi U = \frac{\alpha + k}{3}\phi U. \qquad (3.4)$$

Taking $X = \xi$, $Y = \phi U$ in (3.2), from (3.4) we get $-U = -2\beta\left(\frac{\alpha+k}{3}\right)\xi + \alpha\left(\frac{\alpha+k}{3}\right)U - k\left(\frac{\alpha+k}{3}\right)U + \alpha\beta\xi + \beta^2 U - \frac{\alpha+k}{3}AU + kAU$. That is,

$$(\alpha - 2k)AU = \beta(\alpha - 2k)\xi + (3 + 3\beta^2 + \alpha^2 - k^2)U. \qquad (3.5)$$

If $\alpha = 2k$, from (3.5) $3 + 3\beta^2 + 3k^2 = 0$, which is impossible. Thus

$$\alpha - 2k \neq 0 \qquad (3.6)$$

and

$$AU = \beta\xi + \frac{3 + 3\beta^2 + \alpha^2 - k^2}{\alpha - 2k}U. \qquad (3.7)$$

From (3.4) and (3.7) we have that $\mathfrak{D}_U = \{X \in \mathfrak{D} / g(X, U) = g(X, \phi U) = 0\}$ is a holomorphic (that is, ϕ-invariant) and A-invariant distribution. Let now X be a unit vector field in $\mathfrak{D}_U$ such that $AX = \lambda X$ for a certain function λ on M.

Introducing such an X in (3.2) and taking its scalar product with ξ we have $-2g(\phi X, Y) = -2\lambda g(A\phi X, Y) + \alpha g(A\phi X, Y) + \alpha\lambda g(\phi X, Y)$, for any Y tangent to M. This yields

$$(2\lambda - \alpha)A\phi X = (2 + \alpha\lambda)\phi X. \tag{3.8}$$

If $\alpha = 2\lambda$, from (3.8) $2 + 2\lambda^2 = 0$, which is impossible. Thus

$$A\phi X = \mu\phi X \tag{3.9}$$

where $\mu = \frac{2+\alpha\lambda}{2\lambda-\alpha}$. If we take such an $X \in \mathfrak{D}_U$ and $Y = \xi$ in (3.2) we have

$$\lambda(\alpha - k) - \lambda\mu + k\mu = -1. \tag{3.10}$$

Taking $Y = \xi$ and ϕX instead of X in (3.2) we get

$$\mu(k - \alpha) + \mu\lambda - k\lambda = 1. \tag{3.11}$$

From (3.10) and (3.11) we have $(\lambda - \mu)(\alpha - 2k) = 0$. As from (3.6) $\alpha \neq 2k$, $\lambda = \mu$. From (3.9) this yields

$$\lambda^2 - \alpha\lambda - 1 = 0. \tag{3.12}$$

From the Codazzi equation $(\nabla_X A)\phi X - (\nabla_{\phi X} A)X = -2\xi$. This yields $X(\lambda)\phi X + \lambda\nabla_X\phi X - A\nabla_X\phi X - (\phi X)(\lambda)X - \lambda\nabla_{\phi X}X + A\nabla_{\phi X}X = -2\xi$. Taking its scalar product with ξ we have $-2\lambda^2 + 2\alpha\lambda + \beta g([\phi X, X], U) = -2$. Then from (3.12) $\beta g([\phi X, X], U) = 0$. That is

$$g([\phi X, X], U) = 0. \tag{3.13}$$

But taking the scalar product of the above equation with U we get $(\gamma - \lambda)g([\phi X, X], U) + 2\beta\lambda = 0$, where $\gamma = \frac{3+3\beta^2+\alpha^2-k^2}{\alpha-2k}$. From (3.13) this yields $\beta\lambda = 0$. As $\beta \neq 0$, λ should vanish, which is impossible from (3.12). We have just proved that M must be Hopf. This gives a complete proof of our Theorem 1.1 in the introduction.

Now let us give the proof of Theorem 1.2. Since M is Hopf by Theorem 1.1, we may write $A\xi = \alpha\xi$ and take $X \in \mathfrak{D}$ such that $AX = \lambda X$. From Theorem 2.1, $A\phi X = \mu\phi X$ where $\mu = \frac{\lambda\alpha+2}{2\lambda-\alpha}$.

Taking $Y = \xi$ in (3.2) we obtain $-\phi X = \alpha\lambda\phi X - k\lambda\phi X - \lambda A\phi X + kA\phi X$. That is

$$(\lambda - k)A\phi X = (1 + \lambda(\alpha - k))\phi X \tag{3.14}$$

and the same equation as (3.10). If we take $Y = \xi$ and ϕX instead of X in (3.2) we obtain the same equation as (3.11).Thus $(\mu - \lambda)(2k - \alpha) = 0$. As we suppose

$\alpha \neq 2k$, $\mu = \lambda$, thus we have $A\phi = \phi A$ and from Theorem 2.1 M must be locally congruent to a real hypersurface of type A.

If M is a geodesic hypersphere, see [6], $\alpha = 2\cot(2r)$ and for any $X \in \mathfrak{D}$ $AX = \cot(r)X$. It is very easy to see that these real hypersurfaces satisfy (3.2).

If M is a tube of radius r over a $\mathrm{C}P^n$, $\alpha = 2\cot(2r)$ and for any $X \in \mathfrak{D}$, either $AX = \cot(r)X$ and $A\phi X = \cot(r)\phi X$ or $AX = -\tan(r)X$ and $A\phi X = -\tan(r)\phi X$. It is straightforward to check that they satisfy (3.2) and this finishes the proof of our Theorem 1.2.

Finally, let us mention the proof of our Corollary in the introduction as follows: If M satisfies $(\hat{\nabla}^{(k)}_X A)Y = 0$ for any X, Y tangent to M, its shape operator is of Codazzi type with respect to a g-Tanaka–Webster connection. Thus by Theorem 1.2, M should be of type A. It is very easy to check that the shape operator of these real hypersurfaces is parallel with respect to a g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$. So the Corollary is proved.

Remark. Let us suppose that $\alpha = 2k$. From (3.14) M must be a Hopf real hypersurface such that the principal curvatures in $\mathfrak{D}$ are not equal to k. As examples of such situations, we could give the principal curvatures of the six types in Takagi's list in the introduction. By virtue of the principal curvatures, we can easily check that any of the six types naturally satisfies (3.2). On the other hand, Kimura [4] proved that any Hopf hypersurfaces in $\mathbb{C}P^m$ with constant principal curvatures can be divided into 6 type of hypersurfaces in Takagi's list. So we conclude that any Hopf real hypersurface with constant principal curvatures for $\alpha = 2k$ has the shape operator of Codazzi type with respect to a g-Tanaka–Webster connection.

Remark. If you compare our results with the ones obtained by Cho in [2], where he asserts that type B real hypersurfaces have g-Tanaka–Webster parallel shape operator. Though he mentioned that real hypersurfaces of type B satisfy parallelism of A with respect to a g-Tanaka–Webster connection if we look at his formula (4.18) in [2], as for a type B real hypersurface $\lambda^2 - \alpha\lambda - \frac{c}{4} \neq 0$, for an $X \in V_\lambda$ $(\hat{\nabla}^{(k)}_\xi A)X = 0$ if and only if $\alpha = 2k$. This confirm our results.

Acknowledgements. The present authors would like to express their gratitude to the referees for their careful reading of our article and useful comments and valuable suggestions which helped us to develop the first version of our manuscript.

References

[1] D. E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Mathematics **203**, *Birkhauser Boston Inc. Boston*, 2002.

[2] J. T. Cho, CR-structures on real hypersurfaces of a complex space form, *Publ. Math. Debrecen* **54** (1999), 473–487.

[3] J. T. Cho, Pseudo-Einstein CR-structures on real hypersurfaces in a complex space form, *Hokkaido Math. J.* **37** (2008), 1–17.

[4] M. Kimura, Real hypersurfaces and complex submanifolds in complex projective space, *Trans. Amer. Math. Soc.* **296** (1986), 137–149.

[5] Y. Maeda, On real hypersurfaces of a complex projective space, *J. Math. Soc. Japan* **28** (1976), 529–540.

[6] R. Niebergall and P. J. Ryan, Real hypersurfaces in complex space forms, in Tight and Taut Submanifolds, *Math. Sci. Res. Inst. Publ.* **32** (1997), 233–305.

[7] M. Okumura, On some real hypersurfaces of a complex projective space, *Trans. Amer. Math. Soc.* **212** (1975), 355–364.

[8] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, *Osaka J. Math.* **10** (1973), 495–506.

[9] R. Takagi, Real hypersurfaces in complex projective space with constant principal curvatures, *J. Math. Soc. Japan* **27** (1975), 43–53.

[10] R. Takagi, Real hypersurfaces in complex projective space with constant principal curvatures II, *J. Math. Soc. Japan* **27** (1975), 507–516.

[11] N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, *Japan. J. Math. (N.S.)* **2** (1976), 131–190.

[12] S. Tanno, Variational problems on contact Riemennian manifolds, *Trans. Amer. Math. Soc.* **314** (1989), 349–379.

[13] S. M. Webster, Pseudohermitian structures on a real hypersurface, *J. Differential Geom.* **13** (1978), 25–41.

JUAN DE DIOS PÉREZ
DEPARTAMENTO DE GEOMETRÍA
Y TOPOLOGÍA
FACULTAD DE CIENCIAS
UNIVERSIDAD DE GRANADA
GRANADA 18071
SPAIN

E-mail: jdperez@ugr.es

YOUNG JIN SUH
KYUNGPOOK NATIONAL UNIVERSITY
DEPARTMENT OF MATHEMATICS
TAEGU 702-701
KOREA

E-mail: yjsuh@knu.ac.kr

(Received November 28, 2012; revised May 15, 2013)