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On generalized Berwald manifolds with semi-symmetric compatible linear connections

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Abstract. Generalized Berwald manifolds are special Finsler manifolds admitting compatible linear connections on the base manifold. Compatibility means that the parallel transports preserve the Finslerian length of tangent vectors. It is known [13] that any compatible linear connection is Riemann metrizable by the averaged Riemannian metric which is given as the integral of the Riemann-Finsler metric over the indicatrix hypersurfaces. The basic questions are the unicity of the compatible linear connection and its expression in terms of the canonical data of the Finsler manifold (intrinsic characterization). Here we discuss the case of Finsler manifolds admitting compatible linear connections with vanishing trace-less part in the torsion. Our main results are the intrinsic characterization and the proof of the uniqueness of such a linear connection (if exists).

1. Introduction

I. Let M be a differentiable manifold with local coordinates $u^1, \dots, u^n$ on $U \subset M$. The induced coordinate system on the tangent manifold consists of the functions

$$x^1 := u^1 \circ \pi, \dots, x^n = u^n \circ \pi \quad \text{and} \quad y^1 := du^1, \dots, y^n = du^n,$$

where $\pi : TM \to M$ is the canonical projection. A Finsler structure on a differentiable manifold M is a smoothly varying family $F : TM \to \mathbb{R}$ of Finsler–Minkowski functionals in the tangent spaces satisfying the following conditions:

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- each non-zero element $v \in TM$ has an open neighbourhood such that the restricted function is of class at least $\mathcal{C}^4$ in all of its variables $x^1, \ldots, x^n$ and $y^1, \ldots, y^n$,
- F is positively homogeneous of degree one, i.e. $F(rv) = rF(v)$ for any positive real number r and $F(v) = 0$ if and only if v is the zero element of the tangent space,
- (regularity condition) the Hessian matrix
$$g_{ij} := \frac{\partial^2 E}{\partial y^j \partial y^i}$$
of the energy function $E := (1/2)F^2$ with respect to the variables $y^1, \ldots, y^n$ is positive definite.

Using the regularity condition we can introduce the metric tensor $g = g_{ij} dy^i \otimes dy^j$ on the tangent spaces. The so-called Riemann–Finsler metric g is defined only on the punctured space $T_pM \setminus \{\mathbf{0}\}$ because the second order partial differentiability of the energy function at the origin does not follow automatically. Further canonical objects are
$$d\mu = \sqrt{\det g_{ij}}\ dy^1 \wedge \ldots \wedge dy^n,$$
the Liouville vector field $C := y^1 \partial/\partial y^1 + \ldots y^n \partial/\partial y^n$ and the induced volume form
$$\mu = \frac{1}{F} \iota_C d\mu = \sqrt{\det g_{ij}}\ \sum_{i=1}^{n} (-1)^{i-1} \frac{y^i}{F} dy^1 \wedge \ldots \wedge dy^{i-1} \wedge dy^{i+1} \ldots \wedge dy^n$$
on the indicatrix hypersurface $\partial K_p := F^{-1}(1) \cap T_pM$ belonging to the Finsler–Minkowski functional of the tangent space. The Riemann–Finsler metric (together with the associated canonical objects) can be also interpreted as a Riemannian metric (volume forms, vector field) on the vertical subbundle or the pull-back bundle by the canonical projection. For the foundations of Finsler geometry see [3], [8] and [10].

II. Let $f : TM \to \mathbb{R}$ be a zero homogeneous function and let us define the average-valued function[1]

$$A_f(p) := \int_{\partial K_p} f \, \mu_p.$$

Especially we can introduce the averaged Riemannian metric

$$\gamma_p(v, w) := \int_{\partial K_p} g(v, w) \, \mu. \tag{1}$$

The basic version (1) of the averaged Riemannian metric tensors was introduced in [13]. For further processes to construct Riemannian metric tensors by averaging we can refer to [4], see also [2]. Weighted versions by the measure of the indicatrix body can be also used. For more technical results including the expression of the Lévi–Civita connection in terms of Finslerian objects see [4] and [17].

Definition 1. A linear connection on the base manifold is compatible with the Finslerian structure if the parallel transports preserve the Finslerian norm of tangent vectors. Finsler manifolds admitting compatible linear connections are called generalized Berwald manifolds.

Theorem 1 ([13]). *If a linear connection on the base manifold is compatible with the Finslerian structure then it must be metrical with respect to the averaged Riemannian metric* γ.

It is well-known that metrical linear connections are uniquely determined by the torsion tensor. Consider the decomposition

$$T(X, Y) := T_1(X, Y) + T_2(X, Y),$$

where

$$T_1(X, Y) := T(X, Y) - \frac{1}{n-1}\big(\tilde{T}(X)Y - \tilde{T}(Y)X\big),$$

$$T_2(X, Y) := \frac{1}{n-1}\left(\tilde{T}(X)Y - \tilde{T}(Y)X\right)$$

and $\tilde{T}$ is the trace tensor of the torsion. The trace-less part T_1 is automatically

[1]Integration is taken with respect to the orientation induced by the coordinate vector fields $\partial/\partial y^1, \dots, \partial/\partial y^n$. It can be easily seen that the integral

$$\int_K f \, d\mu_p = \int_{y(K)} f \circ y^{-1} \sqrt{\det g_{ij}} \circ y^{-1} \, dy^1 \dots dy^n$$

is independent of the choice of the coordinate system (orientation). Actually, the orientation is convenient but not necessary to make integrals of functions sense [18].

zero in case of $n = 2$. In case of $n \geq 3$ the trace-less part can be divided into two further components A_1 and S_1 by separating the axial (or totally anti-symmetric) part A_1. This means that its lowered tensor with respect to the Riemannian metric is totally anti-symmetric:

$$T(X,Y) = A_1(X,Y) + S_1(X,Y) + T_2(X,Y).$$

Then we have eight classes of metrical linear connections depending on that the terms A_1, S_1 and T_2 are surviving or not [1]. At the same time we have eight classes of generalized Berwald manifolds. They correspond to the elements of $(\mathbb{Z}_2)^3$ in such a way that we use the term 1 if the corresponding component is not identically zero. Generalized Berwald manifolds of type

- $(0,0,0)$ are the classical Berwald manifolds admitting torsion-free compatible linear connections on the base manifold [12].
- $(0,0,1)$ are Finsler manifolds admitting compatible linear connections with vanishing trace-less part in the torsion (the only surviving term is T_2). In an equivalent terminology, such a metrical linear connection is called semi-symmetric.
- $(1,0,0)$ are Finsler manifolds admitting compatible linear connections with totally anti-symmetric torsion tensor (the only surviving term is A_1). It is a well-known [1] that metric connections with totally anti-symmetric torsion have the same geodesics as the Lévi–Civita connection, i.e. all of these connections have an associated spray in common (the spray of the Lévi–Civita connection of the averaged Riemannian metric).

2. Problems and solutions

The paper is devoted to the discussion of generalized Berwald manifolds of type $(0,0,1)$. This means that we have a Finsler manifold admitting a compatible semi-symmetric linear connection on the base manifold. There are some partial results [14] in case of torsion tensors of the form

$$T = \frac{1}{2}\left(1 \otimes d\alpha - d\alpha \otimes 1\right), \tag{2}$$

where $\alpha : M \to \mathbb{R}$ is a smooth function. Hashiguchi–Ichyjio's theorem [7] states that a Finsler manifold admits a compatible linear connection with torsion (2) if and only if it is conformal to a Berwald manifold: the conformal change $E_\alpha =$

$e^{\alpha\circ\pi}E$ results in a Berwald manifold. Conformally Berwald Finsler manifolds or Finsler manifolds admitting compatible semi-symmetric linear connections with exact 1-forms in the torsion are called exact Wagner manifolds. A more general case is to consider closed 1-forms in the torsion[2]: for any point $p \in M$ there exist an open neighbourhood around p and a smooth function $\alpha_p : U_p \to \mathbb{R}$ such that

$$d\alpha_p = \beta \quad \text{and} \quad T = \frac{1}{2}\left(1 \otimes \beta - \beta \otimes 1\right). \tag{3}$$

Let us introduce the notion of closed Wagner manifolds for Finsler spaces admitting compatible semi-symmetric linear connections with closed 1-forms in the torsion. The main question is how to generalize Hasiguchi–Ichijyo's theorem. It is clear from the global version of the theorem that for any point of a closed Wagner manifold has a neighbourhood over which it is conformal equivalent to a Berwald manifold. This means that closed Wagner manifolds are locally conformal to Berwald manifolds. What about the converse? The exterior derivatives of the local scale functions constitute a globally well-defined 1-form if and only if they coincide on the intersection of overlapping neighbourhoods. In other words the compatible linear connection on the intersection of overlapping neighbourhoods should be uniquely determined. This gives the question that how many essentially different ways there are for a Finsler manifold to be conformal to a Berwald manifold. Alternatively: are there non-homothetic and non-Riemannian conformally equivalent Berwald spaces? This is just the MATSUMOTO's problem [9]. It has been solved in [14] and [15], see also [16]. The basic result is that up to homothetic changes there is (at most) one way for any non-Riemannian Finsler manifold to be conformal to a Berwald manifold. Therefore we have the following generalization of the classical Hashiguchi–Ichyjio's theorem.

Theorem 2 ([14], [15])**.** *A Finsler manifold is a closed Wagner manifold if and only if it is a locally conformally Berwald manifold.*

Another important task is to express the compatible linear connection in terms of the canonical objects of the Finsler manifold. This problem has been also solved in [14] for closed Wagner manifolds. The main point of the present paper is to solve the problem of the intrinsic characterization without any additional condition for the 1-form β. As a direct consequence we also have the uniqueness of such a linear connection (if exists).

[2]Semi-symmetric metric linear connections with closed 1-forms in formula (3) for the torsion are very important in differential geometry: if β is closed then all the classical curvature properties are satisfied which is crucial for the classification of the holonomy groups and Simon's theory of holonomy systems [11].

3. The main result

Consider a generalized Berwald manifold of type $(0,0,1)$. This means that we have a Finsler manifold admitting a compatible linear connection $\bar{\nabla}$ with torsion

$$T(X,Y)=\lambda(Y)X-\lambda(X)Y, \tag{4}$$

where λ is a 1-form on the base manifold M; for the sake of simplicity all the constant proportional terms are involved in λ. In what follows we use the symbol ∇^* for the Lévi–Civita connection of the averaged Riemannian metric γ. In the sense of Theorem 1 the connection $\bar{\nabla}$ must be metrical with respect to γ and a routine calculation shows that

$$\bar{\nabla}_X Y=\nabla^*_X Y+\lambda(Y)X-\gamma(X,Y)U, \tag{5}$$

where the vector field $U:=\lambda^\sharp$ is defined by the formula $\gamma(U,X)=\lambda(X)$. We will use the language of associated horizontal distributions. Let c be an arbitrary curve in the base manifold and consider a parallel vector field X_t along c with respect to $\bar{\nabla}$. If F is the Finslerian fundamental function then the compatibility condition implies that F is constant along X_t. By differentiation

$$0=(F\circ X_t)'=c^{k\prime}\frac{\partial F}{\partial x^k}\circ X_t+X^{k\prime}\frac{\partial F}{\partial y^k}\circ X_t$$
$$=c^{k\prime}\frac{\partial F}{\partial x^k}\circ X_t-\bar{\Gamma}^k_{ij}\circ c\, c^{i\prime}X^j\frac{\partial F}{\partial y^k}\circ X_t=c^{i\prime}\left(\frac{\partial F}{\partial x^i}-\bar{\Gamma}^k_{ij}\circ\pi y^j\frac{\partial F}{\partial y^k}\right)\circ X_t$$

which means that $\bar{\nabla}$ is compatible with the Finslerian structure if and only if

$$X_i^{\bar{h}}F=0\quad(i=1,\dots,n), \tag{6}$$

where the vector fields of type

$$X_i^{\bar{h}}=\frac{\partial}{\partial x^i}-\bar{\Gamma}^k_{ij}\circ\pi y^j\frac{\partial}{\partial y^k}$$

span the associated horizontal distribution. The associated spray can be given by a simple contraction:

$$\bar{S}:=y^i\frac{\partial}{\partial x^i}-\bar{\Gamma}^k_{ij}\circ\pi y^i y^j\frac{\partial}{\partial y^k}.$$

Using equation (5) we have the relationships between the associated sprays and the horizontal distributions:

$$\bar{S}=S^*-\lambda_j\circ\pi y^j C+2E^*U^v \tag{7}$$

and

$$X_i^{\bar{h}} = X_i^{h^*} - \lambda_j \circ \pi y^j X_i^v + (X_i^v E^*) U^v, \tag{8}$$

where the vector fields

$$X_1^v := \frac{\partial}{\partial y^1}, \dots, X_n^v := \frac{\partial}{\partial y^n}$$

span the vertical subspaces (recall that the vertical and the horizontal subspaces are direct complement to each other),

$$C := y^1 X_1^v + \dots y^n X_n^v$$

is the Liouville vector field,

$$E^*(v) := \frac{1}{2}\gamma(v, v)$$

means the Riemannian energy and the vector field

$$U^v := U^1 \circ \pi X_1^v + \dots U^n \circ \pi X_n^v$$

denotes the vertical lift of U. From (7) we have immediately that

$$U^v E = \frac{\bar{S}E - S^*E}{2E^*} + \frac{E}{E^*}\lambda_j \circ \pi y^j$$

because the homogenity of the Finslerian energy implies that $CE = 2E$. Equation (8) says that

$$\frac{X_i^{\bar{h}} E}{E} = \frac{X_i^{h^*} E}{E} - \lambda_j \circ \pi y^j \frac{X_i^v E}{E} + (X_i^v E^*) \frac{U^v E}{E}. \tag{9}$$

It can be written into the form

$$\frac{X_i^{\bar{h}} E}{E} = \frac{X_i^{h^*} E}{E} + \lambda_j \circ \pi y^j \left(\frac{X_i^v E^*}{E^*} - \frac{X_i^v E}{E} \right) + \frac{1}{2} \frac{\bar{S}E - S^*E}{E} \frac{X_i^v E^*}{E^*}. \tag{10}$$

We put

$$\bar{\rho} := \frac{d_{\bar{h}} E}{E} - \frac{1}{2} \frac{\bar{S}E}{E} \frac{d_J E^*}{E^*}, \tag{11}$$

$$\rho^* := \frac{d_{h^*} E}{E} - \frac{1}{2} \frac{S^*E}{E} \frac{d_J E^*}{E^*} \quad \text{and} \quad f := \log \frac{E^*}{E}, \tag{12}$$

where J is the canonical vertical endomorphism (almost tangent structure) on the tangent manifold defined by the formulas

$$J\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k} \quad \text{and} \quad J\left(\frac{\partial}{\partial y^k}\right) = 0,$$

$\bar{h}$ and h^* are the horizontal endomorphisms [5], [6] associated with $\bar{\nabla}$ and ∇^*, respectively. In general

$$h\left(\frac{\partial}{\partial x^k}\right) = X_k^h \quad \text{and} \quad h\left(\frac{\partial}{\partial y^k}\right) = 0.$$

Recall that any 1-form (endomorphism) K on the tangent manifold can be directly composed with the exterior derivative of a function $\Phi : TM \to \mathbb{R}$ in the following way:

$$d_K \Phi(\Lambda) := K(\Lambda)\Phi,$$

where $\Lambda : TM \to TTM$ is a vector field on the tangent manifold. By the help of notations (11) and (12) we can write

$$\bar{\rho} = \rho^* + \lambda_j \circ \pi y^j d_J f \tag{13}$$

and, consequently,

$$d_J \bar{\rho} = d_J \rho^* + \lambda_j \circ \pi dx^j \wedge d_J f \tag{14}$$

because of ${d_J}^2 = 0$. Let us introduce the following gradient-type vector field

$$\Theta = E^* \frac{\partial f}{\partial y^i} \gamma^{ij} \circ \pi \frac{\partial}{\partial x^j}.$$

Substituting Θ into formula (14)

$$\iota_\Theta d_J \bar{\rho} = \iota_\Theta d_J \rho^* + E^* \lambda_j \circ \pi \gamma^{ij} \circ \pi \frac{\partial f}{\partial y^i} d_J f - \frac{1}{E^*} \|J\Theta\|^2 \lambda_j \circ \pi dx^j,$$

where the norm is taken with respect to the vertical lift of the Riemannian metric tensor γ. Explicitly

$$\|J\Theta\|^2 = E^{*2} \gamma^{ij} \circ \pi \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}.$$

On the other hand

$$E^* \lambda_j \circ \pi \gamma^{ij} \circ \pi \frac{\partial f}{\partial y^i} = E^* \gamma^v (J\Theta, U^v) = E^* U^v f.$$

Using equation (7)

$$E^* U^v f = \frac{\bar{S} f - S^* f}{2}$$

because the zero homogeneity of f implies that $Cf = 0$. On the other hand $S^* E^* = 0$ and, consequently,

$$S^* f = -S^* \log E = -\frac{S^* E}{E}.$$

In a similar way

$$\bar{S}f = -\frac{\bar{S}E}{E}.$$

Therefore

$$\iota_\Theta d_J\bar{\rho} = \iota_\Theta d_J\rho^* + \frac{S^*E - \bar{S}E}{2E} d_J f - \frac{1}{E^*}\|J\Theta\|^2\lambda_j \circ \pi dx^j$$

and we have that

$$\iota_\Theta d_J\bar{\rho} + \frac{1}{2}\frac{\bar{S}E}{E} d_J f = \iota_\Theta d_J\rho^* + \frac{1}{2}\frac{S^*E}{E} d_J f - \frac{1}{E^*}\|J\Theta\|^2\lambda_j \circ \pi dx^j. \tag{15}$$

Let us define[3] the 1-forms

$$\eta^*(X_p) := \int_{\partial K_p^*} \left(d_J\rho^*(\Theta, X^h) + \frac{1}{2}\frac{S^*E}{E} X_i^v f \right) \mu^*$$

and

$$\bar{\eta}(X_p) := \int_{\partial K_p^*} \left(d_J\bar{\rho}(\Theta, X^h) + \frac{1}{2}\frac{\bar{S}E}{E} X_i^v f \right) \mu^*.$$

Then by (15) we can write

$$\bar{\eta}(X_p) = \eta^*(X_p) - 2\sigma(p)\lambda(X_p), \tag{16}$$

where

$$\sigma(p) := \int_{\partial K_p^*} \frac{1}{2E^*}\|J\Theta\|^2\,\mu^*. \tag{17}$$

Before dividing by the function σ we discuss its possible values. It is clear from the definition that $\sigma(p) = 0$ if and only if ∂K_p and ∂K_p^* are homothetic, i.e. the Finslerian indicatrix reduces to a Riemannian one (a quadratic hypersurface with respect to the averaged Euclidean inner product at p). In the special case of generalized Berwald manifolds we can formulate the following result.

Lemma 1. *For any generalized Berwald manifold σ is a constant function. It is identically zero or strictly positive depending on that the manifold is Riemannian or not.*

[3]Note that the integrand is semibasic and we can use arbitrary horizontal lifting process $X \mapsto X^h$ in its second argument.

Proof. If we have a generalized Berwald manifold then we can compare the Finslerian indicatrices in different tangent spaces by the parallel transport with respect to the compatible linear connection. Such a linear connection is compatible with both the Finslerian and the averaged Riemannian structure. Therefore we have an invariant integral (17) under changing the base point and σ is constant. Its vanishing at a single point implies that all of the Finslerian indicatrices reduce to Riemannian ones. □

Theorem 3. *A non-Riemannian Finsler manifold is a generalized Berwald manifold admitting a semi-symmetric compatible linear connection if and only if $\sigma(p) > 0$ for any $p \in M$ and the linear connection*

$$\bar{\nabla}_X Y = \nabla^*_X Y + \frac{1}{2\sigma}\left(\eta^*(Y)X - \gamma(X,Y)\eta^{*\sharp}\right),$$

is compatible with the Finslerian structure.

Proof. From the compatibility condition $\bar{\rho} = 0$ and $\bar{\eta} = 0$. Now the result follows immediately from equation (16) because $\bar{\nabla}$ is the only metrical connection with

$$T(X,Y) = \frac{1}{2\sigma}\left(\eta^*(Y)X - \eta^*(X)Y\right), \tag{18}$$

i.e.

$$\lambda := \frac{\eta^*}{2\sigma}$$

in formula (4) for the torsion. □

4. Conformally invariant characterization

In what follows we are going to present Theorem 3 in terms of conformally invariant quantities (functions, differential forms).

Lemma 2. *The exterior derivative*

$$\theta := \frac{1}{\sigma}\left(d\eta^* - \frac{1}{\sigma} d\sigma \wedge \eta^*\right) \tag{19}$$

of η^*/σ and $d_{\bar{h}} \log E$ are conformally invariant.

Proof. Consider the conformal change

$$E_\alpha = e^{\alpha\circ\pi} E \;\Rightarrow\; g_\alpha = e^{\alpha\circ\pi} g$$

of the Riemann–Finsler metric, where $\alpha : M \to \mathbb{R}$ is a smooth function on the base manifold.

First step. The averaged Riemannian metrics are also conformally related with the same proportional term: $\gamma_\alpha = e^\alpha \gamma$ because

$$\partial K_\alpha = \frac{1}{\sqrt{e^{\alpha\circ\pi}}}\,\partial K,$$

i.e. the indicatrices are pointwise homothetic and

$$\det g_\alpha = (e^{\alpha\circ\pi})^n \det g \;\Rightarrow\; d\mu_\alpha = (e^{\alpha\circ\pi})^{n/2}\, d\mu \quad \text{and } \mu_\alpha = (e^{\alpha\circ\pi})^{\frac{n-1}{2}}\,\mu.$$

For any zero-homogeneous function f (especially for any component of the Riemann–Finsler metric g_α)

$$\int_{\partial K} f\,\mu = (e^{\alpha\circ\pi})^{\frac{n-1}{2}} \int_{\partial K_\alpha} f\,\mu = \int_{\partial K_\alpha} f\,\mu_\alpha. \tag{20}$$

Second step. As a routine calculation shows

$$\nabla^*_\alpha(X,Y) = \nabla^*_X Y + \frac{1}{2}\left(X\alpha Y + Y\alpha X - \gamma(X,Y)\ \mathrm{grad}^*\alpha\right),$$

where the gradient operator is taken with respect to the averaged Riemannian metric γ. Therefore

$$\nabla^*_\alpha(X,Y) = \tilde{\nabla}_X Y + \frac{1}{2} X\alpha Y \tag{21}$$

and

$$X_i^{h^*_\alpha} = X_i^{\tilde{h}} - \frac{1}{2}\frac{\partial\alpha}{\partial u^i}\circ\pi C, \tag{22}$$

where $\tilde{h}$ is the horizontal distribution associated to the metric linear connection $\tilde{\nabla}$ under the choice

$$\lambda := \frac{1}{2} d\alpha$$

in formula (4) for the torsion (cf. formula (2)). By a simple contraction

$$S^*_\alpha = \tilde{S} - \frac{1}{2}\alpha^c C, \tag{23}$$

where

$$\alpha^c := \frac{\partial\alpha}{\partial u^i}\circ\pi y^i$$

is the complete lift of the function α. Another type of lifting proccess for functions is the so-called vertical lift $\alpha^v := \alpha\circ\pi$. It can be easily seen that

$$X^h\alpha^v = (X\alpha)\circ\pi \quad \text{and} \quad S\alpha^v = \alpha^c \tag{24}$$

independently of the choice of the horizontal distribution and the associated spray. We put

$$\tilde{\rho} := \frac{d_{\tilde{h}}E}{E} - \frac{1}{2}\frac{\tilde{S}E}{E}\frac{d_J E^*}{E^*} \tag{25}$$

$$\rho^*_\alpha := \frac{d_{h^*_\alpha}E_\alpha}{E_\alpha} - \frac{1}{2}\frac{S^*_\alpha E_\alpha}{E_\alpha}\frac{d_J E^*_\alpha}{E^*_\alpha} \quad \text{and} \quad f_\alpha := \log\frac{E^*_\alpha}{E_\alpha} = \log\frac{E^*}{E} = f. \tag{26}$$

Using equation (24)

$$\frac{d_{h^*_\alpha}E_\alpha}{E_\alpha} = d\alpha^v + \frac{d_{h^*_\alpha}E}{E} \overset{(22)}{=} d\alpha^v + \frac{d_{\tilde{h}}E}{E} - d\alpha^v = \frac{d_{\tilde{h}}E}{E}$$

because of the homogeneity property $CE = 2E$. On the other hand

$$\frac{d_J E^*_\alpha}{E^*_\alpha} = \frac{d_J E^*}{E^*} \tag{27}$$

and

$$\frac{S^*_\alpha E_\alpha}{E_\alpha} \overset{(24)}{=} \alpha^c + \frac{S^*_\alpha E}{E} \overset{(23)}{=} \alpha^c + \frac{\tilde{S}E}{E} - \alpha^c = \frac{\tilde{S}E}{E}. \tag{28}$$

Therefore

$$\rho^*_\alpha = \tilde{\rho}. \tag{29}$$

Using that $f_\alpha = f$ and

$$\Theta_\alpha := E^*_\alpha \frac{\partial f_\alpha}{\partial y^i}\gamma^{ij}_\alpha \circ \pi \frac{\partial}{\partial x^j} = E^* \frac{\partial f}{\partial y^i}\gamma^{ij} \circ \pi \frac{\partial}{\partial x^j} = \Theta$$

we can follow directly the procedure in section 3 to conclude that

$$\eta^*_\alpha(X_p) = \eta^*(X_p) - 2\sigma(p)\lambda(X_p), \quad \text{where} \quad \lambda = \frac{1}{2}d\alpha \tag{30}$$

and

$$\sigma(p) := \int_{\partial K^*_p} \frac{1}{2E^*}\|J\Theta\|^2 \mu^*. \tag{31}$$

Equation (20) says that $\sigma_\alpha = \sigma$ and, consequently,

$$\frac{\eta^*_\alpha}{\sigma_\alpha} = \frac{\eta^*}{\sigma} - d\alpha. \tag{32}$$

This means that the exterior derivative (19) is conformally invariant.

Third step. Using that

$$\bar{\nabla}_X Y = \nabla^*_X Y + \frac{1}{2\sigma}\left(\eta^*(Y)X - \gamma(X,Y)\eta^{*\sharp}\right)$$

we have

$$\bar{\nabla}_\alpha(X,Y) = \nabla^*_\alpha(X,Y) + \frac{1}{2\sigma_\alpha}\left(\eta^*_\alpha(Y)X - \gamma_\alpha(X,Y)\eta^{*\sharp_\alpha}_\alpha\right). \tag{33}$$

Since the sharp operators of conformally equivalent Riemannian metrics are related by the simple formula

$$X^{\sharp_\alpha} = \frac{1}{e^\alpha}X^\sharp$$

it follows that

$$\begin{aligned}\frac{1}{2\sigma_\alpha}\eta^{*\sharp_\alpha}_\alpha &= \frac{1}{2}\left(\frac{\eta^*_\alpha}{\sigma_\alpha}\right)^{\sharp_\alpha} = \frac{1}{2e^\alpha}\left(\frac{\eta^*_\alpha}{\sigma_\alpha}\right)^{\sharp} \overset{(32)}{=} \frac{1}{2e^\alpha}\left(\frac{\eta^*}{\sigma} - d\alpha\right)^{\sharp}\\ &= \frac{1}{2e^\alpha}\left(\left(\frac{\eta^*}{\sigma}\right)^{\sharp} - \operatorname{grad}^*\alpha\right)\end{aligned}$$

and thus

$$\begin{aligned}\bar{\nabla}_\alpha(X,Y) &= \nabla^*_\alpha(X,Y) + \frac{1}{2\sigma}\left(\eta^*(Y)X - \gamma(X,Y)\eta^{*\sharp}\right) - \frac{1}{2}Y\alpha X + \frac{1}{2}\gamma(X,Y)\operatorname{grad}^*\alpha\\ &= \nabla^*_X Y + \frac{1}{2}X\alpha Y + \frac{1}{2\sigma}\left(\eta^*(Y)X - \gamma(X,Y)\eta^{*\sharp}\right) = \bar{\nabla}_X Y + \frac{1}{2}X\alpha Y.\end{aligned}$$

Therefore

$$X_i^{\bar{h}_\alpha} = X_i^{\bar{h}} - \frac{1}{2}\frac{\partial\alpha}{\partial u^i}\circ\pi C \tag{34}$$

and equation (34) shows that

$$d_{\bar{h}_\alpha}\log E_\alpha = d\alpha^v + d_{\bar{h}_\alpha}\log E = d\alpha^v + d_{\bar{h}}\log E - d\alpha^v = d_{\bar{h}}\log E$$

as was to be proved. □

Theorem 4. *A non-Riemannian Finsler manifold is a generalized Berwald manifold admitting a semi-symmetric compatible linear connection if and only if* $\sigma(p) > 0$ *for any* $p \in M$ *and* $d_{\bar{h}}\log E = 0$.

Corollary 1. *The class of generalized Berwald manifolds admitting semi-symmetric compatible linear connections is closed under the conformal change of the Riemann–Finsler metric.*

As a special case of compatible linear connections with closed 1-forms in the torsion we have the following result.

Theorem 5 ([14])**.** *A non-Riemannian Finsler manifold is a locally conformally Berwald manifold if and only if* $\sigma(p) > 0$ *for any* $p \in M$,

$$d_{\bar{h}} \log E = 0 \quad \text{and} \quad \theta := \frac{1}{\sigma}\left(d\eta^* - \frac{1}{\sigma} d\sigma \wedge \eta^*\right) = 0.$$

The local conformal change to a Berwald manifold around a point p in M is given by a function $\alpha_p : U \to \mathbb{R}$ *satisfying* $d\alpha_p = \eta^*/\sigma$.

Corollary 2. *The class of generalized Berwald manifolds admitting semi-symmetric compatible linear connections with closed 1-forms in the torsion is closed under the conformal change.*

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