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A characterization of exponential polynomials

By LÁSZLÓ SZÉKELYHIDI (Debrecen)

Abstract. Using spectral synthesis on finite dimensional varieties and some additional ring-theoretical tools we give a new proof for the fact that exponential polynomials on Abelian groups can be characterized by the property that they span a finite dimensional translation invariant linear space. In particular, we characterize exponential monomials, too.

1. Introduction

Throughout this paper $\mathbb{C}$ denotes the set of complex numbers and G is an Abelian group equipped with the discrete topology, further $\mathbb{C}^G$ denotes the locally convex topological vector space of all complex valued functions defined on G, equipped with the pointwise operations and the product topology. For each function f in $\mathbb{C}^G$ we define $\check{f}(x) = f(-x)$, whenever x is in G. By a *ring* we always mean a commutative ring with unit.

The dual of $\mathbb{C}^G$ can be identified with $\mathcal{M}_c(G)$, the space of all finitely supported complex measures on G. This space is also identified with the set of all finitely supported complex valued functions on G in the following obvious way. If the point mass concentrated at the element x is denoted by δ_x, then each measure μ in $\mathcal{M}_c(G)$ has a unique representation in the form

$$\mu = \sum_{x \in G} \mu(x)\delta_x$$

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with some finitely supported function $\mu : G \to \mathbb{C}$. "Identification" means that we use the same letter μ for both the measure and the representing function. In this sense δ_x is the characteristic function of the singleton $\{x\}$. The pairing between $\mathbb{C}^G$ and $\mathcal{M}_c(G)$ is given by the formula

$$\langle \mu, f\rangle = \int f\, d\mu = \sum_{x\in G} f(x)\,\mu(x).$$

Convolution on $\mathcal{M}_c(G)$ will be defined by

$$\mu * \nu(x) = \int \mu(x-y)\, d\nu(y) = \sum_{y\in G} \mu(x-y)\nu(y)$$

for any μ, ν in $\mathcal{M}_c(G)$ and x in G. Convolution converts the linear space $\mathcal{M}_c(G)$ into a commutative algebra with unit δ_0, 0 being the identity in G. One realizes immediately, that the algebra $\mathcal{M}_c(G)$ is identical with the finite *group algebra* of G. Hence we can use the alternative notation $\mathbb{C}G$ for $\mathcal{M}_c(G)$, which may be more familiar for algebraists.

We also define convolution of measures in $\mathbb{C}G$ with arbitrary functions on G by the same formula

$$\mu * f(x) = \int f(x-y)\, d\mu(y) = \sum_{y\in G} f(x-y)\mu(y)$$

for each μ in $\mathbb{C}G$, f in $\mathbb{C}^G$ and x in G. The linear operators $f \mapsto \mu * f$ on $\mathbb{C}^G$ are called *convolution operators*.

From the above definition it is clear that convolution operators arise from translation operators. *Translation with the element y in G* is the linear operator mapping the function f in $\mathbb{C}^G$ onto its *translate* $\tau_y f$ defined by $\tau_y f(x) = f(x+y)$ for each x in G. Clearly, τ_y is a convolution operator, namely, it is the convolution with the measure δ_{-y}. A subset of $\mathbb{C}^G$ is called *translation invariant*, if it contains all translates of its elements. A closed linear subspace of $\mathbb{C}^G$ is called a *variety* on G, if it is translation invariant. For each function f the smallest variety containing f is called the *variety generated by f* and is denoted by $\tau(f)$.

Spectral analysis and spectral synthesis deal with the description of varieties on Abelian groups. The fundamental question of spectral analysis for a given variety is if there is a nonzero finite dimensional subvariety in the variety. If so, then we say that *spectral analysis* holds for the variety. The problem of spectral synthesis, however, is if there are sufficiently many finite dimensional subvarieties in a given variety in the sense that the linear span of them is dense in the variety. In the affirmative case we say that *spectral synthesis* holds for the variety. If spectral analysis, resp. spectral synthesis holds for each nonzero variety on the group, then we say that *spectral analysis*, resp. *spectral synthesis holds on the group.* For more about spectral analysis, resp. spectral synthesis the reader should refer to [12], [10], [16], [8], [9].

It turns out that the problems of spectral analysis and spectral synthesis on varieties is closely related to some basic classes of functions, which have the property that they generate finite dimensional varieties. These functions are the so-called *exponential polynomials.* These can be considered the basic building bricks of varieties. In this paper we study exponential polynomials on Abelian groups. Our purpose is to give new characterizations of these functions using classical tools from ring theory. Utilizing some classical results from the theory of Noether rings we obtain new characterization theorems for exponential monomials and for polynomial functions on Abelian groups, see 12, 13, 17. In particular, we give a new proof for spectral synthesis on finite dimensional varieties, in Corollary 19.

2. Polynomial functions

Using translation one introduces *difference operators* $\Delta_y = \tau_y - \tau_0$ and higher order difference operators $\Delta_{y_1,y_2,\dots,y_n} = \Pi_{i=1}^n \Delta_{y_i}$ for each $y_1, y_2 \dots, y_n$ in G. Obviously, $\Delta_{y_1,y_2,\dots,y_n}$ is a convolution operator, namely

$$\Delta_{y_1,y_2,\dots,y_n} f = \Pi_{i=1}^n (\delta_{-y_i} - \delta_0) * f\,,$$

where Π denotes convolution product.

Difference operators, in particular, higher order difference operators are related to an important function class. A function $f : G \to \mathbb{C}$ is called a *generalized polynomial*, if there is a nonnegative integer n such that

$$\Delta_{y_1,y_2,\dots,y_{n+1}} f = 0 \tag{1}$$

holds for each $y_1, y_2, \dots, y_{n+1}$ in G. In this case we say that f is *of degree at most* n and the *degree* of f is the smallest n for which f is of degree at most n.

A homomorphism of G in the additive group of complex numbers is called an *additive function*. Clearly, every nonzero additive function is a generalized polynomial of degree 1.

A special class of generalized polynomials is formed by functions of the form

$$p(x) = P\big(a_1(x), a_2(x), \dots, a_k(x)\big), \tag{2}$$

where $P : \mathbb{C}^k \to \mathbb{C}$ is an ordinary polynomial in k variables and $a_i : G \to \mathbb{C}$ is additive for $i = 1, 2, \dots, k$. These functions are called simply *polynomials*. In particular, additive functions are polynomials. The fact, that polynomials are generalized polynomials is proved in the following theorem, together with their characterization.

Theorem 1. *Let G be an Abelian group. Then every polynomial on G is a generalized polynomial. Moreover, the following statements are equivalent for any $f : G \to \mathbb{C}$:*

(i) *f is a polynomial;*

(ii) *all functions $\Delta_y f$ for y in G lie in a finite dimensional linear space of polynomials;*

(iii) *f is a generalized polynomial and $\tau(f)$ is finite dimensional.*

PROOF. Suppose first, that $f : G \to \mathbb{C}$ has the form (2). We assume that P has degree $n > 0$. By the Taylor Formula we have for each x, y in G

$$\begin{aligned} f(x+y) &= P\big(a_1(x)+a_1(y), a_2(x)+a_2(y), \dots, a_k(x)+a_k(y)\big) \\ &= \sum \frac{1}{\alpha_1! \dots \alpha_k!} \partial_1^{\alpha_1} \dots \partial_k^{\alpha_k} P\big(a_1(x), \dots, a_k(x)\big) a_1(y)^{\alpha_1}, \dots, a_k(y)^{\alpha_k}, \end{aligned} \tag{3}$$

where the summation extends to all multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ in $\mathbb{N}^k$ with

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_k \le n.$$

It follows

$$\Delta_y f(x) = \sum_{1 \le |\alpha| \le n} \frac{1}{\alpha_1! \dots \alpha_k!} \partial_1^{\alpha_1} \dots \partial_k^{\alpha_k} P\big(a_1(x), \dots, a_k(x)\big) a_1(y)^{\alpha_1}, \dots, a_k(y)^{\alpha_k},$$

that is, $\Delta_y f(x) = Q_y\big(a_1(x), a_2(x), \dots, a_k(x)\big)$ holds for each x in G with some polynomial $Q_y : \mathbb{C}^k \to \mathbb{C}$ of degree at most $n-1$. Repeating this argument we get, that

$$\Delta_{y_1, y_2, \dots, y_n} f(x) = Q_{y_1, y_2, \dots, y_n}\big(a_1(x), a_2(x), \dots, a_k(x)\big)$$

for all $x, y_1, y_2, \dots, y_n$ in G, and $Q_{y_1, y_2, \dots, y_n}$ is a constant, hence

$$\Delta_{y_1, y_2, \dots, y_n, y_{n+1}} f(x) = 0$$

for all $x, y_1, y_2, \dots, y_n, y_{n+1}$ in G, showing, that f is a generalized polynomial, as it was to be proved.

Observe, that if f is a polynomial, then, by (3), $\tau(f)$ is generated by the finite set of functions $\{\partial_1^{\alpha_1}\dots\partial_k^{\alpha_k}P\big(a_1(x),\dots,a_k(x)\big):|\alpha|\leq n\}$, hence it is finite dimensional. This means, that the conditions given in (ii)–(iii) are necessary for f to be a polynomial.

Now suppose, that $f:G\to\mathbb{C}$ has the property that all the differences $\Delta_y f$ for y in G lie in a finite dimensional linear space X of polynomials. We show, that f is a polynomial. Let $a_1,a_2,\dots,a_k$ be linearly independent additive functions on G such that all elements of X are ordinary polynomials of these functions. We write $a=(a_1,a_2,\dots,a_k)$, then we have, that

$$\Delta_y f(x)=\sum_{|\alpha|\leq n}c_\alpha(y)a(x)^\alpha, \tag{4}$$

where n is an upper bound for the degrees of the polynomials $\Delta_y f$ with y in G and $c_\alpha:G\to\mathbb{C}$ are functions for $|\alpha|\leq n$. Here

$$a(x)^\alpha=a_1(x)^{\alpha_1}a_2(x)^{\alpha_2}\dots a_k(x)^{\alpha_k}$$

for each x in G and multi-index $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_k)$. Then f satisfies

$$\Delta_{y_1,y_2,\dots,y_{n+2}}f(x)=0$$

for each $x,y_1,y_2,\dots,y_{n+2}$ in G. We remark, that if $n=0$, then obviously f is additive plus constant, hence it is a polynomial. In general, it follows, that f is a generalized polynomial of degree at most $n+1$, hence, by the results of [3], it has a unique representation in the form

$$f(x)=\sum_{j=0}^{n+1}A_j^{(j)}(x) \tag{5}$$

for each x in G, where $A_j:G^j\to\mathbb{C}$ is a j-additive symmetric function for $j=1,2,\dots,n+1$, $A_0^{(0)}$ is a constant, and

$$A_j^{(j)}(x)=A_j(x_1,x_2,\dots,x_j)$$

with $x_1=x_2=\dots=x_j=x$ for $j=1,2,\dots,n+1$. Here $A_j^{(j)}$ is called the *homogeneous term* of degree j of f for $j=0,1,\dots,n+1$. More generally, we write for $1\leq i\leq j-1$

$$A_j^{(i)}(x,y)=A_j(x_1,x_2,\dots,x_j)$$

with $x_1 = x_2 = \cdots = x_i = x$ and $x_{i+1} = x_{i+2} = \cdots = x_j = y$. From (4) and (5) we have

$$\sum_{j=0}^{n+1} \big[A_j^{(j)}(x+y) - A_j^{(j)}(x)\big] = \sum_{|\alpha|\le n} c_\alpha(y) a(x)^\alpha \tag{6}$$

for each x, y in G. Using the uniqueness of the representations of the form (5) and comparing the homogeneous terms of the same degree on both sides, by the obvious binomial identity

$$A_j^{(j)}(x+y) = A_j^{(j)}(x) + \sum_{i=1}^{j-1} \binom{j}{i} A_j^{(i)}(x,y) + A_j^{(j)}(y)$$

we have, that the homogeneous terms of highest degree on the left and on the right hand side in (6) are equal:

$$(n+1)\, A_{n+1}^{(n)}(x,y) = \sum_{|\alpha|=n} c_\alpha(y) a(x)^\alpha \tag{7}$$

for each x, y in G. As the functions a^α are linearly independent for $|\alpha| = n$ (see [15], Lemma 2.7, p. 29.), there are elements x_β, $|\beta| = n$ such that the matrix $\big(a(x_\beta)^\alpha_{|\alpha|,|\beta|=n}\big)$ is regular. Substituting $x = x_\beta$ in (7) we get a system of linear equations for the unknowns $c_\alpha(y)$ ($|\alpha| = n$) with regular fundamental matrix, from which it follows, that the functions c_α – as linear combinations of the additive functions $y \mapsto A_{n+1}^{(n)}(x_\beta, y)$ – are additive. Then putting $y = x$ in (7) we obtain, that A_{n+1} is a polynomial. Now we let $f_n = f - A_{n+1}$ and we infer, that the differences of f_n lie in a finite dimensional space of polynomials of degree at most $n-1$. Applying induction we arrive at our statement. This means, that (i) and (ii) are equivalent.

Now suppose, that f is a generalized polynomial and $\tau(f)$ is finite dimensional. The statement is obviously true for generalized polynomials of degree at most 1 – they are actually additive plus constant, hence they are polynomials. We continue by induction: for each y in G the degree of the generalized polynomial $\Delta_y f$ is one less than $\deg f$. As $\tau(\Delta_y f)$ is a subset of $\tau(f)$, and all polynomials of degree at most $\deg f - 1$ in a finite dimensional linear space form a finite dimensional linear space of polynomials, hence, by (ii), f is a polynomial and the theorem is proved. $\square$

3. Exponential polynomials

Another basic function class is formed by the joint eigenfunctions of all translation operators, that is, by those nonzero functions $\varphi : G \to \mathbb{C}$ satisfying

$$\tau_y \varphi = m(y) \cdot \varphi \tag{8}$$

with some $m : G \to \mathbb{C}$, that is

$$\varphi(x+y) = m(y)\,\varphi(x) \tag{9}$$

for all x, y in G. It follows

$$\varphi(y) = \varphi(0) \cdot m(y),$$

which implies, that $\varphi(0) \neq 0$ and, by (9),

$$m(x+y) = m(x)\,m(y) \tag{10}$$

for all x, y in G. Nonzero functions $m : G \to \mathbb{C}$ satisfying (10) for each x, y in G are called *exponentials.* Clearly, every exponential generates a one dimensional variety, and, conversely, every one dimensional variety is generated by an exponential.

We shall also use *modified difference operators* defined as follows: given an exponential m, a function f and an element y in G, then we let

$$\Delta_{m;y} f(x) = f(x+y) - m(y)\,f(x)$$

for each x in G. The iterates are also defined for any positive integer n and for each $y_1, y_2, \dots, y_n$ in G by

$$\Delta_{m;y_1,y_2,\dots,y_n} = \Pi_{i=1}^{n} \Delta_{m;y_i}.$$

Obviously, these operators are also convolution operators, namely

$$\Delta_{m;y_1,y_2,\dots,y_n} f = \Pi_{i=1}^{n} (\delta_{-y_i} - m(y_i)\,\delta_0) * f$$

holds. On the right hand side Π is meant as a convolution. In particular, for $m = 1$ we have $\Delta_{1;y_1,y_2,\dots,y_n} = \Delta_{y_1,y_2,\dots,y_n}$. The following formula can be proved easily by induction:

$$\Delta_{m;y_1,y_2,\dots,y_n} f(x) = m(x + y_1 + y_2 + \dots + y_n)\,\Delta_{y_1,y_2,\dots,y_n}\big(f \cdot \check{m}\big)(x) \tag{11}$$

for every positive integer n, exponential m, function f and arbitrary elements $x, y_1, y_2, \dots, y_n$ in G.

Modified difference operators are related to another basic function class. A complex valued function on G is called an *exponential monomial*, if it is the product of a polynomial and an exponential. Linear combinations, or what is the same, sums of exponential monomials are called *exponential polynomials*. Further, a complex valued function f on G is called a *generalized exponential monomial*, if it is the product of a generalized polynomial and an exponential, and linear combinations of generalized monomials are called *generalized exponential polynomials*.

Similarly to generalized polynomials, generalized exponential monomials can be characterized by a functional equation, namely, the following theorem holds.

Theorem 2. *Given an Abelian group G the function $f : G \to \mathbb{C}$ is a generalized exponential monomial if and only if there is a positive integer n and an exponential m such that*

$$\Delta_{m;y_1,y_2,\dots,y_n} f = 0 \tag{12}$$

holds for each $y_1, y_2, \dots, y_n$ in G. In this case $f = p\,m$ with some generalized polynomial p.

Proof. This statement is obvious using formula (11) and the fact, that exponentials never vanish. $\square$

The following characterization of exponential monomials is also straightforward.

Theorem 3. *On an Abelian group a generalized exponential monomial is an exponential monomial if and only if it generates a finite dimensional variety.*

Proof. Indeed, for any exponential m the map $f \longleftrightarrow f \cdot m$ is an isomorphism between the varieties $\tau(f)$ and $\tau(f \cdot m)$, hence the statement follows from Theorem 1, (iii). $\square$

4. Ideals and annihilators

For any subset H in G the *annihilator of* H is the set $H^\perp$ of all measures in $\mathbb{C}G$, which vanish on H. Clearly, this is an ideal, which is proper if and only if H is nonzero. For a function f the annihilator of the set $\{f\}$, or, what is the same, of the variety $\tau(f)$ is called simply the annihilator of f. Similarly, for any subset K in $\mathbb{C}G$ the *annihilator of* K is the set $K^\perp$ of all functions in $\mathbb{C}^G$, which are annihilated by all measures in K. Clearly, this is a variety on G, which is nonzero if and only if K is nonzero. As above, for a measure μ the annihilator of the set $\{\mu\}$ is called simply the variety of μ. Clearly, this is the solution space of the convolution equation $\mu * f = 0$. By the Hahn–Banach Theorem it is clear, that $V^{\perp\perp} = V$ for each variety on G and it is obvious, that $I^{\perp\perp} \supseteq I$ for any ideal I in $\mathbb{C}G$. As it is shown in [9], p. 105, actually equality holds here, too.

The basic question of spectral analysis on a variety V in $\mathbb{C}^G$ can be reformulated as follows: does V contain a nonzero exponential monomial? In [8] M. Laczkovich and G. Székelyhidi presented a complete characterization of Abelian groups having spectral analysis: spectral analysis holds on G if and only if the torsion free rank of G is less than the continuum.

The basic problem of spectral synthesis on a variety V can be reformulated in terms of exponential monomials, too: given a variety V in $\mathbb{C}^G$, do the exponential monomials in this variety span a dense subspace? This is the case, for instance, if G is a finitely generated free Abelian group, by a result of M. Lefranc [10]. In [4] R. J. Elliot presented a theorem stating, that spectral synthesis holds for any Abelian group, however, his proof was defective, and, as it was shown in [16], actually Elliot's theorem is false. In [9] the authors proved that spectral synthesis holds on an Abelian group if and only if the torsion free rank of the group is finite.

Even if spectral analysis or spectral synthesis does not hold on the group G it may hold on some special varieties. Concerning spectral analysis, we have the following simple result.

Theorem 4 (Spectral analysis for finite dimensional varieties)**.** *Given an Abelian group G spectral analysis holds for all nonzero finite dimensional varieties in $\mathbb{C}^G$.*

Proof. We have to show that every nonzero finite dimensional variety in $\mathbb{C}^G$ contains an exponential. If V is a nonzero finite dimensional variety, then it is a common invariant subspace of all translation operators τ_y for y in G, which commute, hence they have a common eigenfunction in V, which generates a one dimensional variety, and it is, as we have seen above, generated by an exponential. $\square$

In the subsequent paragraphs we shall obtain the corresponding result for spectral synthesis in Corollary 19, which is one of our main results: spectral synthesis holds for every finite dimensional variety. Also, in Theorem 18, we get a new proof for the fact, that exponential polynomials are characterized by the property spanning a finite dimensional variety. Similar results has been obtained previously by several authors using different methods (see e.g. [1], [5], [7], [11], [13], [14]). Our approach depends on the following fundamental theorem, the

Noether–Lasker decomposition theorem (see [18]). We recall, that a ring is called a *Noether ring*, if it satisfies the *ascending chain condition* for the ideals: every ascending chain of ideals terminates. We shall also use another related concept: a ring is called an *Artin ring*, if it satisfies the *descending chain condition* for the ideals: every descending chain of ideals terminates. It is known, that every Artin ring is Noether, however, the converse is not true in general. For basic knowledge on Noether and Artin rings see [6], Vol. II.

Theorem 5 (Noether–Lasker)**.** *In any Noether ring every proper ideal is a finite intersection of primary ideals.*

This theorem is really the key of our work. Namely, it is obvious, that if the ideal I in $\mathbb{C}G$ is the intersection of the ideals $I_1, I_2, \dots, I_n$, then

$$I^\perp = I_1^\perp + I_2^\perp + \cdots + I_n^\perp.$$

Hence, to prove spectral synthesis for a given variety it is enough to show, that the annihilators of primary ideals in $\mathbb{C}G$ including the variety consist of exponential monomials. When proving this, as a by-product, we shall obtain a characterization of exponential monomials. We shall do this in the sequel.

Theorem 6. *Given an Abelian group G and a finite dimensional variety V in $\mathbb{C}^G$, then the ring $\mathbb{C}G/V^\perp$ is Artin.*

Proof. A descending chain of ideals in $\mathbb{C}G/V^\perp$ generates a descending chain of ideals including $V^\perp$ in $\mathbb{C}G$, which generates an ascending chain of varieties in V. As V is finite dimensional, this chain of varieties, hence also the original chain of ideals must be stationary, that is, $\mathbb{C}G/V^\perp$ is Artin. □

Theorem 7. *Given an Abelian group G and a finite dimensional variety V in $\mathbb{C}^G$, then the annihilator $V^\perp$ is the intersection of a finite number of primary ideals.*

Proof. Indeed, as the ring $\mathbb{C}G/V^\perp$, by the previous theorem, is Artin, hence it is Noether, and, by Theorem 5, its zero ideal (0) is the intersection of a finite number of primary ideals, which means, that $V^\perp$ is the intersection of a finite number of primary ideals including $V^\perp$ in $\mathbb{C}G$. □

Theorem 8. *Given an Abelian group G and a proper variety V in $\mathbb{C}^G$, then it is sufficient for an ideal I containing $V^\perp$ to be maximal is that its annihilator in V is a one dimensional variety, generated by an exponential. If spectral analysis holds for V, then it is also necessary.*

Proof. A variety is one dimensional if and only if it consists of all scalar multiples of an exponential. For each exponential m let M_m denote the annihilator of m. Let $M \supseteq M_m$ be a maximal ideal, then $M^{\perp}$, as a nonzero subvariety of the variety generated by m, is necessarily equal to the variety of m, hence $M = M^{\perp\perp} = M_m$. This means that M_m is maximal. For the converse we suppose, that spectral analysis holds for V and M is a maximal ideal containing $V^{\perp}$. Then its annihilator $M^{\perp}$ is a nonzero subvariety in V, hence spectral analysis holds for it, therefore it contains an exponential m. Thus $M^{\perp} \supseteq \tau(m)$, that is, $M = M_m$ and $M^{\perp} = M_m^{\perp}$, which is the one dimensional variety generated by m. □

Theorem 9. *Let G be an Abelian group and m an exponential. A proper ideal in $\mathbb{C}G$ contains the measures $\delta_{-x} - m(x)\,\delta_0$ for all x in G if and only if it is maximal and equals to M_m.*

Proof. Let I be a proper ideal in $\mathbb{C}G$ containing all measures $\delta_{-x} - m(x)\,\delta_0$ and let φ be in $I^{\perp}$. Then $\tau_y\varphi$ is in $I^{\perp}$ for each y in G, hence we have

$$0 = (\delta_{-x} - m(x)\,\delta_0)(\tau_y\varphi) = \varphi(x+y) - m(x)\,\varphi(y)$$

for all x, y in G. This implies that $\varphi = \varphi(0)\,m$, that is φ is in $\tau(m)$. We have $I^{\perp} \subseteq \tau(m)$, which implies $I = \left(I^{\perp}\right)^{\perp} \supseteq M_m$, but the latter is maximal, hence $I = M_m$. Conversely, it is obvious, that M_m contains all measures of the form $\delta_{-x} - m(x)\,\delta_0$ with x in G. □

We have the two simple corollaries.

Corollary 10. *Given an Abelian group and a proper variety V in $\mathbb{C}^G$, for which spectral analysis holds, then it is necessary and sufficient for an ideal I containing $V^{\perp}$ to be maximal, that it is of the form M_m with some exponential m in V.*

Corollary 11. *Given an Abelian group and a proper variety V in $\mathbb{C}^G$, for which spectral analysis holds, then it is necessary and sufficient for an ideal I containing $V^{\perp}$ to be maximal, that it contains all measures of the form $\delta_{-x} - m(x)\,\delta_0$ with some exponential m in V.*

5. Characterization theorems

Using the above results we obtain new characterization theorems for exponential monomials.

Theorem 12. *Given an Abelian group G and a function $f : G \to \mathbb{C}$ such that spectral analysis holds for $\tau(f)$. Then the following statements are equivalent:*

(i) *f is an exponential monomial;*

(ii) *$\mathbb{C}G/\tau(f)^{\perp}$ is a local Artin ring.*

Proof. As spectral analysis holds for $\tau(f)$, it follows, by Corollary 10, that every maximal ideal containing $\tau(f)^{\perp}$ is of the form M_m with some exponential m. Suppose, that $f = p\,m$ is an exponential monomial with $p : G \to \mathbb{C}$ polynomial and $m : G \to \mathbb{C}$ exponential. Then, clearly, the unique maximal ideal containing $\tau(f)^{\perp}$ is M_m, hence the ring $\mathbb{C}G/\tau(f)^{\perp}$ is local. On the other hand, $\tau(f)$ is finite dimensional by Theorem 3, hence, by Theorem 6, the ring $\mathbb{C}G/\tau(f)^{\perp}$ is Artin.

Now suppose that $\mathbb{C}G/\tau(f)^{\perp}$ is a local Artin ring and let M_m denote its maximal ideal, where m is an exponential in $\tau(f)$. By Krull's Intersection Theorem (see [6], Vol. II, pp. 442–443.)

$$\cap_{n=1}^{\infty} M_m^n = \tau(f)^{\perp}.$$

On the other hand, as the ring $\mathbb{C}G/\tau(f)^{\perp}$ is Artin, the descending chain of ideals $M_m \supseteq M_m^2 \supseteq \dots$ terminates, that is, $M_m^n = M_m^{n+1} = \dots$ for some positive integer n. It follows $\tau(f)^{\perp} = M_m^n$. For each y in G the measure $\delta_{-y} - m(y)\,\delta_0$ belongs to M_m, hence $\Pi_{i=1}^n \big(\delta_{-y_i} - m(y_i)\,\delta_0\big)$ belongs to M_m^n, which implies that we have

$$\Delta_{m;y_1,y_2,\dots,y_n} f = 0$$

for each $y_1, y_2, \dots, y_n$ in G. This means, that f is a generalized exponential monomial having the form $f = p\,m$ with some exponential m and generalized polynomial p of degree at most $n-1$. Suppose, that p is not a polynomial. It is clear, that the measure μ in $\mathbb{C}G$ annihilates f if and only if the measure $\check{m} \cdot \mu$ annihilates p, hence the rings $\mathbb{C}G/\tau(f)^{\perp}$ and $\mathbb{C}G/\tau(p)^{\perp}$ are isomorphic. It follows, that $\mathbb{C}G/\tau(p)^{\perp}$ is a local Artin ring with maximal ideal M_1. Based on the assumption, that p is not a polynomial we construct a strictly ascending chain of subvarieties in $\tau(p)$, which generates a strictly descending chain of ideals in $\mathbb{C}G/\tau(p)^{\perp}$, contradicting the fact, that it is an Artin ring. By assumption $n \geq 3$ and $\tau(p)$ is infinite dimensional.

Let $V_0 = \{0\}$ and for each $i = 0, 1, \dots$ we define

$$V_{i+1} = \{\varphi \mid \varphi \in \tau(p) \text{ and } \Delta_y \varphi \in V_i \text{ for each } y \in G\}.$$

Obviously we have $V_i \subseteq V_{i+1}$ for $i = 0, 1, \dots$ and $V_n = \tau(p)$. If V_{n-1} is finite dimensional, then, by part (iii) in Theorem 1, it follows, that p is a polynomial, which is not the case. On the other hand, V_1 is finite dimensional. Hence there exists a k with $1 \leq k < n-1$ such that V_k is finite dimensional and V_{k+1} is infinite dimensional. It follows again from (iii) in Theorem 1, that the functions in V_{k+1} are polynomials of degree at most n. Let $\varphi_1, \varphi_2, \dots, \varphi_d$ be a basis of V_k and we choose a sequence $\psi_1, \psi_2, \dots$ in V_{k+1} such, that ψ_{i+1} is not in the variety W_i generated by the set of functions

$$\{\varphi_1, \varphi_2, \dots, \varphi_d\} \cup \{\psi_1, \psi_2, \dots, \psi_i\}$$

for each $i = 1, 2, \dots$. This is possible, because this set consists of a finite number of polynomials, hence it generates a finite dimensional variety, however V_{k+1} is infinite dimensional. We have, that the chain of varieties W_i $(i = 1, 2, \dots)$ is strictly ascending, which is a contradiction and our theorem is proved. $\square$

Theorem 13. *Given an Abelian group a generalized exponential polynomial is an exponential monomial if and only if the factor ring with respect to its annihilator is a local Noether ring.*

Proof. If f is an exponential monomial, then obviously spectral analysis holds for $\tau(f)$ and, by the previous theorem, $\mathbb{C}G/\tau(f)^{\perp}$ is local Artin, hence it is local Noether.

Conversely, suppose, that f is a generalized exponential monomial, that is, $f = p\,m$, where p is a generalized polynomial of degree n and m is an exponential, further $\mathbb{C}G/\tau(f)^{\perp}$ is a local Noether ring. Clearly, spectral analysis holds for $\tau(f)$ and $\tau(f)^{\perp}$ is included in a unique maximal ideal M_m. On the other hand, obviously $M_m^{n+1} \subseteq \tau(f)^{\perp}$, hence the maximal ideal of the local Noether ring $\mathbb{C}G/\tau(f)^{\perp}$ is nilpotent. By Theorem 7.15 in [6], Vol. II. on pp. 426–427. it follows, that $\mathbb{C}G/\tau(f)^{\perp}$ is Artin, hence, by Theorem 12, f is an exponential monomial. $\square$

Theorem 14. *Given an Abelian group G and a variety V in $\mathbb{C}^G$, for which $\mathbb{C}G/V^{\perp}$ is Artin. Then every prime ideal containing $V^{\perp}$ is maximal.*

Proof. It is enough to show that every prime ideal in $\mathbb{C}G/V^{\perp}$ is maximal. Let P be a prime ideal in $\mathbb{C}G/V^{\perp}$. Then $(\mathbb{C}G/V^{\perp})/P$ is a domain, hence, as $\mathbb{C}G/V^{\perp}$ is Artin, it is an Artin domain, which is a field. It follows, that P is maximal. $\square$

Corollary 15. *Given an Abelian group G and a variety V in $\mathbb{C}^G$, for which $\mathbb{C}G/V^{\perp}$ is Artin. Then for every primary ideal I containing $V^{\perp}$ the ring $\mathbb{C}G/I$ is local.*

Proof. It is enough to show, that if $I \supseteq V^\perp$ is primary, then there is a unique maximal ideal containing I. Let M be the radical of I, then, as I is primary, by Krull's Theorem (see [6], Vol. 2, Theorem 7.1, p. 392.), M is prime, hence, by Theorem 14, M is maximal. Suppose, that M_1 is another maximal ideal containing I. As the radical of I is the intersection of all prime ideals containing I, and M_1 is prime, it follows, that M_1 includes M, but this implies $M_1 = M$. □

Corollary 16. *Let G be an Abelian group and V a variety in $\mathbb{C}^G$, for which spectral analysis holds. If $V^\perp$ is primary and $\mathbb{C}G/V^\perp$ is Artin, then there is an exponential m in V such that every element in V is an exponential monomial of the form $p \cdot m$ with some polynomial p.*

The following theorem is a summary of our previous results.

Theorem 17. *Let G be an Abelian group and $f : G \to \mathbb{C}$ a function. Then the following statements are equivalent:*

(i) *f is an exponential monomial;*

(ii) *spectral analysis holds for $\tau(f)$, $\tau(f)^\perp$ is primary and $\mathbb{C}G/\tau(f)^\perp$ is a Noether ring;*

(iii) *spectral analysis holds for $\tau(f)$, $\tau(f)^\perp$ is primary and $\mathbb{C}G/\tau(f)^\perp$ is an Artin ring.*

Corollary 18. *A complex valued function on an Abelian group is an exponential polynomial if and only if it is contained in a finite dimensional variety.*

Proof. This is a consequence of the Noether–Lasker theorem and the remark following it. □

Corollary 19 (Spectral synthesis for finite dimensional varieties)**.** *Given an Abelian group G spectral synthesis holds for all nonzero finite dimensional varieties in $\mathbb{C}^G$.*

Proof. By Theorem 7, for the finite dimensional variety V its annihilator $V^\perp$ is the intersection of a finite number of primary ideals, hence we have

$$V^\perp = \bigcap_{j=1}^{k} V_j^\perp, \tag{13}$$

where V_j is a subvariety of V, further, by Corollary 16, V_j consists of exponential monomials. Obviously, we have

$$V = \sum_{j=1}^{k} V_j,$$

which proves our statement. □

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LÁSZLÓ SZÉKELYHIDI
INSTITUTE OF MATHEMATICS
DEBRECEN UNIVERSITY
4032 DEBRECEN
HUNGARY

E-mail: lszekelyhidi@gmail.com

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