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# The Gauss–Bonnet–Chern formula for Finslerian orbifolds

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*Dedicated to Professor Lajos Tamássy on the occasion of the 90th birthday*

**Abstract.** In this article, we prove a Gauss–Bonnet–Chern formula for Finsler metrics on a smooth orbifold. The main idea is that the intrinsic proof of the Gauss–Bonnet formula for Riemmanina manifolds of S. S. Chern is applicable to the very broad class of Finsler metrics on the very generalized orbifolds.

## 1. Introduction

In this paper we shall present a Gauss–Bonnet–Chern formula for Finsler metrics on an orbifold. The Gauss–Bonnet–Chern formula is one of the most important results in differential geometry. It discloses the intrinsic relation between the curvature, which is a geometric quantity, and the Euler characteristic number, which is a topological invariants. The intrinsic proof of Chern ([5], [6]) is of great significance in that it can be applied to a much larger class of metrics or spaces. For example, in 1949, Lichnerowich [11] proved a Gauss–Bonnet formula for Finsler metrics on a smooth manifold, using the Cartan connection on the Cartan–Berwald spaces. This formula was generalized to the more general class of Landsberg spaces by Bao–Chern in [3]. The essence of Bao–Chern's

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proof in [3] is exactly Chern's idea in [5], [6]. See also [15] and [10] for other forms of the Gauss–Bonnet–Chern formulas for Finsler metrics.

The notion of an orbifold is a generalization of manifolds. It was first introduced by I. Satake in [16] and [17], where it is called a $V$-manifolds. The study of $V$-manifolds was revalidated by W. Thurston in [18] where the terminology "orbifold" first appeared. The Gauss–Bonnet–Chern formula was generalized to Riemannian metrics on orbifolds by Satake [17] and for Riemannian metrics on orbifolds with boundary in [13] or [14]. It is interesting that Chern's method in [5] is also applicable in such generalized cases.

In this paper we will prove that the Gauss–Bonnet–Chern formula holds for some types of Finsler metrics on orbifolds. The main idea of the proof is adopted from the intrinsic proof of Chern for Riemannian metrics on closed manifolds ([5], [6]) and Bao–Chern's treatment for Landsberg metrics ([3]).

In Section 2, we recall the definitions and fundamental properties on orbifolds and Finsler manifolds. Section 3 deals with Finsler metrics on orbifolds. In Section 4, we prove the main results of this paper. Finally, in Section 5, we discuss orbifolds with boundary and show that certain forms of the Gauss–Bonnet–Chern formula holds for orbifolds with boundary.

**Notation Conventions.** Throughout the article, we assume the Einstein summation convention. Lower case Latin indices run from 1 to $m$ and lower case Greek indices run from 1 to $m-1$.

## 2. Preliminareis

In this section, we recall some definitions and facts of smooth orbifold structures and Finsler manifolds; see [1], [2] and [16], [17] for more details on orbifold structures, and [3], [4] for Finsler metrics on manifolds.

Let $M$ be a Hausdorff space. We say that $\{\widetilde{U}, G, \varphi\}$ is a $C^\infty$ orbifold chart or local uniformizing system over a uniformized open subset $U \subset M$ if it satisfies the following properties:

(i) $\widetilde{U}$ is a connected open subset of $\mathbb{R}^m$,

(ii) $G$ is a finite group of $C^\infty$ transformation of $\widetilde{U}$, whose set of fixed points has dimension $\leq m-2$,

(iii) $\varphi : \widetilde{U} \to U$ is a continuous map such that $\varphi \circ \sigma = \varphi$ for all $\sigma \in G$, and the induced map between $\widetilde{U}/G \longrightarrow U$ is a homeomorphism.

Given a chart $\{\widetilde{U}, G, \varphi\}$, if there exists an element $g \in G$, $g \neq 1$, such that $gx = x$, $\forall x \in \widetilde{U}$, we say that $G$ acts ineffectively on $\widetilde{U}$; otherwise, $G$ acts effectively, then the orbifold is called to be effective. In this paper, we always assume that each orbifold chart of $M$ is effective.

Let $\{\widetilde{U}, G, \varphi\}$ and $\{\widetilde{U}', G', \varphi'\}$ be two orbifold charts for $U$ and $U'$, respectively, and suppose $U \subset U'$. An injection $\lambda : \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$ is a $C^\infty$ isomorphism $\lambda$ from $\widetilde{U}$ onto an open subset of $\widetilde{U}'$ such that for any $\sigma \in G$ there exists $\sigma' \in G'$ such that $\lambda \circ \sigma = \sigma' \circ \lambda$ and $\varphi = \varphi' \circ \lambda$.

*Definition 2.1* (orbifold). A $C^\infty$ orbifold consists of a connected Hausdorff topological space $M$ and a family $\mathcal{F}$ of $C^\infty$ orbifold charts for open subsets in $M$ satisfying the following conditions:

(i) Each point $x \in M$ is contained in an open subset $U \subset M$ for which there exists an orbifold chart $\{\widetilde{U}, G, \varphi\} \in \mathcal{F}$ such that $\varphi(\widetilde{U}) = U$. If $x \in U_1 \cap U_2$, then there is a uniformized open subset $U_3 \subset U_1 \cap U_2$ such that $x \in U_3$.

(ii) If $\{\widetilde{U}, G, \varphi\}$, $\{\widetilde{U}', G', \varphi'\} \in \mathcal{F}$ and $\varphi(\widetilde{U}) = U \subset U' = \varphi'(\widetilde{U}')$, then there exists a $C^\infty$ injection $\lambda : \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$.

Moreover, an orbifold is called orientable if all the injections in (ii) are orientation preserving, i.e., det $\left(\frac{\partial \widetilde{x}'_j}{\partial \widetilde{x}^j}\right) > 0$ for any injection $\lambda : \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$, where $(\widetilde{x}^i)$ and $(\widetilde{x}'^i)$ are coordinates of $\widetilde{U}$ and $\widetilde{U}'$, respectively.

The notion of an orbifold $N$ with boundary can be defined similarly. We just need define the orbifold chart with boundary, and require that the boundary be preserved under the action of the finite group. The boundary $\partial N$ is also an orbifold with the structure inherited from $N$.

A map $f : M_1 \to M_2$ between orbifolds is called $C^\infty$, if for any local orbifold chart $\widetilde{U}_1$ the map $f|_{\widetilde{U}_1} \circ \varphi_1$ between $\widetilde{U}_1(\{\widetilde{U}_1, G_1, \varphi_1\} \in \mathcal{F}_1)$ and $\widetilde{U}_2(\{\widetilde{U}_2, G_2, \varphi_2\} \in \mathcal{F}_2)$ is $C^\infty$. Since $\mathbb{R}$ can be considered an orbifold with $\mathcal{F} = \{\mathbb{R}, \{1\}, \mathrm{id}\}$, one can define a $C^\infty$ function on an orbifold $M$ to be a $C^\infty$ map $f : M \to \mathbb{R}$.

For each point $x \in U \subset M$, there is an orbifold chart $\{\widetilde{U}, G, \varphi\} \in \mathcal{F}$ such that $x \in \varphi(\widetilde{U}) = U$. Fix a point $\widetilde{x} \in \widetilde{U}$ such that $\varphi(\widetilde{x}) = x$ and denote by $G_{\widetilde{x}}$ the isotropy subgroup of $\widetilde{x}$. Then $G_{\widetilde{x}}$ depends only on $x$. So we call $G_{\widetilde{x}}$ the isotropy group of $x$ and denote as $G_x$. If $G_x \neq 1$, then $x$ is called a singular point of $M$; otherwise, it is a nonsingular point.

Orbifold vector bundles are orbifolds which are locally of the form $\widetilde{U} \times \mathbb{R}^k$ where the group $G$ acts as bundle isomorphisms. In general, the fiber of an orbifold vector bundle is not always a vector space. A section of an orbifold vector bundle is a collection of locally $G$-equivariant sections on orbifold charts.

In particular, the orbifold tangent bundle $\pi : TM \to M$ of an orbifold $M$ is defined locally as the quotient $T\widetilde{U}/G$, with the action of $G$ given by $g(\widetilde{x}, \widetilde{y}) = (g\widetilde{x}, dg_{\widetilde{x}}(\widetilde{y}))$ for each $(\widetilde{x}, \widetilde{y}) \in T\widetilde{U}$.

For $x \in M$, denote by $T_xM$ the maximal vector space of $\pi^{-1}(x)$. The elements of $T_xM$ are called the tangent vectors at $x$.

The orbifold cotangent $T^*M$ can be defined in the same manner (see [17], p. 474, for details). For an orientable orbifolds $M$, one can define the integral $\int_M \omega$ of an $m$-form $\omega$ in the following way.

If the support of $\omega$ is contained in a uniformized open set
$U = \varphi(\widetilde{U})(\{\widetilde{U}, G, \varphi\} \in \mathcal{F})$, then we define

$$\int_U \omega = \frac{1}{|G|}\int_{\widetilde{U}} \omega_{\widetilde{U}}$$

If the orbifold $M$ is paracompact, then it admits a $C^\infty$ partition of unity $\{\psi_i\}$ on $M$ subordinate to a cover consisting of uniformized open subsets (see [7] or [8]). Then one can define the integral of the $m$-form $\omega$ by

$$\int_M \omega = \sum_i \int_M \psi_i \omega.$$

It can be checked that the above integral does not depend on the specific partition of unity. From now on we always assume that the orbifold $M$ is paracompact.

Let $X$ be a vector field on the orbifold $M$ with an isolated zero $x$. Let $\{\widetilde{U}, G, \varphi\} \in \mathcal{F}$, $\widetilde{x} \in \widetilde{U}$ be an orbifold chart such that $x \in \varphi(\widetilde{U})$ and $\varphi(\widetilde{x}) = x$. Let $\widetilde{X}$ be a corresponding vector field on $\widetilde{U}$. Then $\widetilde{X}$ has a zero at $\widetilde{x}$. The index $I_x(X)$ of $X$ at $x$ is given by

$$I_x(X) = \frac{1}{|G_{\widetilde{x}}|} I_{\widetilde{x}}(\widetilde{X}),$$

where $|G_{\widetilde{x}}|$ is the order of the isotropy of $G_{\widetilde{x}}$. The index $I_x(X)$ is uniquely determined by the vector field $X$ and $x$ and need not be an integer (see [17]).

Now we recall the notion of Euler characteristic number $\chi_V(M)$ of an orbifold $M$. Let $M$ be an $m$-dimensional orbifold and $X$ be a vector field on $M$ with isolated zero points $x_1, x_2, \ldots, x_s$ (the existence of such a vector field can be proved easily, see [17]). Then we define

$$\chi_V(M) = (-1)^m \sum_k I_{x_k}(X).$$

The number $\chi_V(M)$ is called the Euler characteristic number of $M$ as an orbifold. This notion was introduced by Satake in [17], where it is called the Euler characteristic number of $M$ as a $V$-mainfold. Note that $\chi_V(M)$ need not be an integer.

Next, we give some definitions and results on Finsler metrics on a manifold. Let $(\widetilde{U}, F)$ be a Finsler manifold. Given $\widetilde{x} \in \widetilde{U}$, let $\widetilde{y}^i$ be the global coordinates with the canonical coordinate base $\left(\frac{\partial}{\partial \widetilde{x}^i}\right)$ on the tangent space $T_{\widetilde{x}}\widetilde{U}$. By removing the origin $\widetilde{y} = 0$, the punctured linear space $T_{\widetilde{x}}\widetilde{U}\backslash 0$ becomes a Riemannian manifold when equipped with the following metric: $ds^2_{\widetilde{U}} = g_{ij}(\widetilde{x}, \widetilde{y}) d\widetilde{y}^i \otimes d\widetilde{y}^j$, where $g_{ij} = \frac{1}{2}(F)^2_{\widetilde{y}_i \widetilde{y}_j}$.

The volume element of $T_{\widetilde{x}}\widetilde{U}\backslash 0$ is $\sqrt{g} d\widetilde{y}^1 \wedge d\widetilde{y}^2 \wedge \cdots \wedge d\widetilde{y}^m$, where our convention is $d\widetilde{y} \wedge d\widetilde{z} = d\widetilde{y} \otimes d\widetilde{z} - d\widetilde{z} \otimes \widetilde{y}$. Let $S_{\widetilde{x}}\widetilde{U} := \{\widetilde{y} \in T_{\widetilde{x}}\widetilde{U} \mid F(\widetilde{x}, \widetilde{y}) = 1\}$. The indicatrix $S_{\widetilde{x}}\widetilde{U}$ is an $(m-1)$-dimensional submanifold of the punctured manifold $T_{\widetilde{x}}\widetilde{U}\backslash 0$ and hence inherits a Riemannian structure $h_{\widetilde{x}}$ from the metric $ds^2_{\widetilde{U}}$.

Since $\vec{n}_{out} = \widetilde{y}^i \frac{\partial}{\partial \widetilde{x}^i}$ is the outward-pointing unit normal field of $S_{\widetilde{x}}\widetilde{U}$, the volume element of $h_{\widetilde{x}}$ is

$$dV = \sqrt{g} \sum_{j=1}^{m} (-1)^{j-1} \widetilde{y}^j d\widetilde{y}^1 \wedge \cdots \wedge d\widetilde{y}^{j-1} \wedge d\widetilde{y}^{j+1} \wedge \cdots \wedge d\widetilde{y}^m$$

where the $\widetilde{y}^i$'$s$ are the elements satisfying $F(\widetilde{x}, \widetilde{y}) = 1$.

We consider the volume function

$$\mathrm{Vol}(\widetilde{x}) := \mathrm{Vol}(S_{\widetilde{x}}\widetilde{U}, h_{\widetilde{x}}) = \int_{S_{\widetilde{x}}\widetilde{U}} dV. \tag{1}$$

Now we define the covariant differential of the Cartan tensor $A = A_{ijk} d\widetilde{x}^i \otimes d\widetilde{x}^j \otimes d\widetilde{x}^k$ by

$$\nabla A = (dA_{ijk} - A_{sjk}\omega^s_i - A_{isk}\omega^s_j - A_{ijs}\omega^s_k) d\widetilde{x}^i \otimes d\widetilde{x}^j \otimes d\widetilde{x}^k := (\nabla A)_{ijk} d\widetilde{x}^i \otimes d\widetilde{x}^j \otimes d\widetilde{x}^k,$$

where $A_{ijk} = \frac{F}{4}[F^2]_{\widetilde{y}^i \widetilde{y}^j \widetilde{y}^k}(\widetilde{x}, \widetilde{y})$.

Denote $(\nabla A)_{ijk} = A_{ijk}|_s d\widetilde{x}^s + A_{ijk;s} \frac{\delta \widetilde{y}^s}{F}$ and define

$$\dot{A} = \dot{A}_{ijk} d\widetilde{x}^i \otimes d\widetilde{x}^j \otimes d\widetilde{x}^k,$$

where $\dot{A}_{ijk} := A_{ijk}|_s \frac{\widetilde{y}^s}{F}$.

*Definition 2.2.* If $\dot{A} = 0$, then $(\widetilde{U}, F)$ is called a Landsberg manifold.

**Proposition 2.3** ([3]). *If $(\widetilde{U}, F)$ is a Landsberg space, then the volume function* $\mathrm{Vol}(\widetilde{x})$ *is a constant.*

The second formal Christoffel symbols on $\widetilde{U}$ is defined by

$$\gamma^i_{jk} := \frac{1}{2} g^{is} \left( \frac{\partial g_{sj}}{\partial \widetilde{x}^k} - \frac{\partial g_{jk}}{\partial \widetilde{x}^s} + \frac{\partial g_{ks}}{\partial \widetilde{x}^j} \right).$$

We also define the quantities $N^i_j$ by

$$N^i_j = \gamma^i_{jk}\widetilde{y}^k - \frac{1}{F}A^i_{jk}\gamma^s_{kr}\widetilde{y}^r\widetilde{y}^s.$$

Set

$$\frac{\delta}{\delta\widetilde{x}^i} := \frac{\partial}{\partial\widetilde{x}^i} - N^i_j\frac{\partial}{\partial\widetilde{y}^i},$$

and

$$\delta\widetilde{y}^i := d\widetilde{y}^i + N^i_j d\widetilde{x}^j.$$

Then $\left\{\frac{\delta}{\delta\widetilde{x}^i}, F\frac{\partial}{\partial\widetilde{y}^i}\right\}$ and $\left\{d\widetilde{x}^i, \frac{\delta\widetilde{y}^i}{F}\right\}$ are natural local bases for the tangent bundle $TS\widetilde{U}$ and cotangent bundle $T^*(S\widetilde{U})$ of $S\widetilde{U}$, respectively.

Recall that $S\widetilde{U}$ has a natural Sasaki type metric (which is a Riemannian metric) defined by

$$g_{ij}d\widetilde{x}^i\otimes d\widetilde{x}^j + g_{ij}\frac{\delta\widetilde{y}^i}{F}\otimes\frac{\delta\widetilde{y}^j}{F}.$$

With respect to this metric we have an orthonormal basis $\{\widehat{e}_a, \widehat{e}_{m+a}\}$ for $TS\widetilde{U}$ and its dual frame $\{\omega^a, \omega^{m+a}\}$ for $T^*S\widetilde{U}$, where $\widehat{e}_a = \widetilde{u}^i_a\frac{\delta}{\delta\widetilde{x}^i}$, $\widehat{e}_{m+a} = \widetilde{u}^i_a F\frac{\partial}{\partial\widetilde{x}^i}$, $\omega^a = \widetilde{v}^a_i d\widetilde{x}^i$ and $\omega^{m+a} = \widetilde{v}^a_i\frac{\delta\widetilde{y}^i}{F}$ (see [4], p. 35).

**Theorem 2.4** ([4])**.** *Let $(\widetilde{U}, F)$ be a Finsler manifold. Then there exists a unique connection on the pull back bundle $\pi^*T\widetilde{U}$, called the Chern connection, whose connection forms are characterized by the structure equations:*

$$d\omega^a = \omega^b\wedge\omega^a_b, \qquad \omega_{ab}+\omega_{ba} = -2A_{abc}\omega^{m+c}.$$

*where $\omega_{ab}=\omega^c_a\delta_{cb}$, and $A_{abc}=\frac{1}{4}F(F^2)_{\widetilde{y}^i\widetilde{y}^j\widetilde{y}^k}\widetilde{u}^i_a\widetilde{u}^j_b\widetilde{u}^k_c$. In particular, $\omega_{mb}+\omega_{bm}=0$.*

Note that in the above theorem we have $\omega^{m+m} = \mathrm{d}(\log F)$ (see [4], p. 36, Exercise 2.15 (a)), and $\omega^\alpha_m = \omega^{m+\alpha}$ (see [4], p. 43, Exercise 2.4.7 (c)).

## 3. Finsler metrics on orbifolds

In this section we consider Finsler metrics on orbifolds. We will define the Chern connection forms on orbifolds, which will be useful in our study of the Gauss–Bonnet–Chern formula.

Let $M$ be an orbifold with an orbifold charts family $\mathcal{F}$ as in Section 2. For each $\{\widetilde{U}, G, \varphi\}\in\mathcal{F}$, let $T\widetilde{U}$ be the tangent vector bundle. Let $S\widetilde{U}$ be the quotient of $T\widetilde{U}\setminus 0$ under the following equivalence relation: $(\widetilde{x},\widetilde{y})\sim(\widetilde{x},\widetilde{y}')$ if and only if there exists a positive constant $a$ such that $\widetilde{y} = a\widetilde{y}'$. For each injection

$\lambda : \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}$, let $\lambda^* : S\widetilde{U} \to S\widetilde{U}'$ be the corresponding bundle map. Let $\varphi^*$ be the restriction to $S\widetilde{U}$ of the bundle map $T\widetilde{U} \to TM$. Set $SM = \bigcup \varphi^*(S\widetilde{U})$. Then we obtain an orbifold $SM$ with an orbifold charts family $\mathcal{F}^* = (\{S\widetilde{U}, G^*, \varphi^*\})$. We call $SM$ the projective orbifold sphere bundle and denote the projection of $SM$ onto $M$ by $\pi$.

*Definition 3.1* ([8], the pull back orbifold tangent bundle). Let $Pr : TM \to M$ be the orbifold tangent bundle over an orbifold $M$ and $\pi : SM \to M$ be the $C^\infty$ projective orbifold sphere bundle. By a pull back orbifold tangent bundle $\pi^*TM$ over $SM$ via $\pi$ we mean an orbifold vector bundle $\widetilde{\pi} : E \to SM$ with a $C^\infty$ map $\overline{f} : E \to TM$ such that each local lifting of $\overline{f}$ is an isomorphism restricted to each fiber, and $\overline{f}$ covers the $C^\infty$ map $\pi$ between $SM$ and $M$.

The notions related to the dual $\pi^*T^*M$ can be defined similarly.

The fiber of $\pi^*TM$ at a point $(x, [y]) \in SM$ is defined by

$$\pi^*TM|_{(x,[y])} = \{(x, [y], v) \mid v \in \widetilde{\pi}^{-1}(x, [y])\} \cong Pr^{-1}(x).$$

Let $\{\widetilde{U}, G, \varphi\} \in \mathcal{F}$ be a local orbifold chart over a uniformized open subset $U$ of $M$ and $(\widetilde{x}_1, \ldots, \widetilde{x}_m) = (\widetilde{x}^i) : \widetilde{U} \to \mathbb{R}^m$ be a local coordinate system on the open subset $\widetilde{U}$. Let $(\frac{\partial}{\partial \widetilde{x}^i})$ and $d\widetilde{x}^i$ be a coordinate bases for the tangent space $T_{\widetilde{x}}\widetilde{U}$ and cotangent space $T^*_{\widetilde{x}}\widetilde{U}$, respectively. The nature local coordinate $(\widetilde{x}^i, \widetilde{y}^i)$ of $T\widetilde{U}$ is given by

$$\widetilde{y} = \widetilde{y}^i \frac{\partial}{\partial \widetilde{x}^i}.$$

A Finsler metric on $U$ is defined to be a $G^*-$invariant Finsler metric on $\widetilde{U}$, i.e., a Finsler metric $F^{\widetilde{U}}$ satisfying the condition

$$F^{\widetilde{U}}(\widetilde{x}, \widetilde{y}) = F^{\widetilde{U}}(\sigma(\widetilde{x}), d\sigma_{\widetilde{x}}(\widetilde{y})),$$

for any $\sigma \in G$ and for any point $(\widetilde{x}, \widetilde{y}) \in T\widetilde{U}$.

*Definition 3.2* (Finsler metric on orbifold). A Finsler metric $F$ on an orbifold $M$ is a $C^\infty$ function $F : TM \to [0, +\infty)$ such that for any uniformized open subset $U \subset M$, the restriction of $F$ to $T\widetilde{U}$ is a Finsler metric on $U$.

According to the definition, to give a Finsler metric $F$ on an orbifold $M$ is to define a Finsler metric $F^{\widetilde{U}}$ on each $\widetilde{U}$ such that

$$F^{\widetilde{U}}(\widetilde{x}, \widetilde{y}) = F^{\widetilde{U}'}(\lambda\widetilde{x}, d\lambda_{\widetilde{x}}\widetilde{y}),$$

for any $(\widetilde{x}, \widetilde{y}) \in T\widetilde{U}$ and for any injection

$$\lambda : \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U}', G', \varphi'\}.$$

Following the idea of [2], we now prove

**Lemma 3.3.** *There always exists a Finsler metric on any orbifold.*

Proof. Let $\{U_i\}$ be a locally finite covering of an orbifold $M$, and $\{\widetilde{U}_i, G_i, \varphi_i\} \in \mathcal{F}$ be the corresponding orbifold charts such that $\varphi_i : \widetilde{U}_i/G_i \to U_i$ is a homeomorphism. Let $T\widetilde{U}_i$ be the tangent bundle over $\widetilde{U}_i$ and suppose that $F'^{\widetilde{U}_i}$ is a Finsler metric on $T\widetilde{U}_i$.

Define

$$F^{*\widetilde{U}^i}(\widetilde{x}, \widetilde{y}) = \frac{1}{|G_i|} \sum_{\sigma \in G_i} F'^{\widetilde{U}^i}(\sigma\widetilde{x}, d\sigma_{\widetilde{x}}\widetilde{y}).$$

Let

$$F^{\widetilde{U}^i}(\widetilde{x}, \widetilde{y}) = \sum_j F^{*\widetilde{U}_i}(\widetilde{x}_{ij}, \lambda_{ij}\widetilde{y}), \quad \widetilde{x} \in \widetilde{U}_i,\ \widetilde{y} \in T_{\widetilde{x}}\widetilde{U_i},$$

where the sum is taken over all the indices $j$ such that $\varphi_j^{-1} \circ \varphi_i(\widetilde{x}) \neq \phi$, here $\widetilde{x}_{ij} \in \varphi_j^{-1} \circ \varphi_i(\widetilde{x})$, and $\lambda_{ij}$ are defined by $\lambda_{ij} = (\lambda_i)_*(\lambda_j^{-1})_*$, where $\varphi_i(\widetilde{x}) \in U \subset U_i \bigcap U_j$, $\{\widetilde{U}, G, \varphi\} \in \mathcal{F}$, and $(\lambda_i)_* : T\widetilde{U} \to T\widetilde{U}_i|_{\lambda_i(U)}$, $(\lambda_j)_* : T\widetilde{U} \to T\widetilde{U}_j|_{\lambda_j(U)}$. Now suppose $\{\psi_i\}$ is a partition of unity subordinated to the covering $\{U_i\}$ and define $F = \sum_i \psi_i F^{\widetilde{U}_i}$. Then it is easy to check that $F$ is a Finsler metric on $M$. $\square$

**Lemma 3.4.** *Let $M$ be an orbifold. Define*

$$g = g_{ij} d\widetilde{x}^i \otimes d\widetilde{x}^j, \quad A = A_{ijk} d\widetilde{x}^i \otimes d\widetilde{x}^j \otimes d\widetilde{x}^k.$$

*Then $g$ is a Riemannian metric on the pull back orbifold tangent bundle $\pi^*TM$ and $A$ is a symmetric section of $\pi^*TM \otimes \pi^*TM \otimes \pi^*TM$.*

Proof. Let $\{U_i\}$ be a locally finite covering of $M$, and $\{\widetilde{U}_i, G_i, \varphi_i\} \in \mathcal{F}$ be the corresponding orbifold charts on $M$. Since $F$ is a Finsler metric on the orbifold, we have

$$F^{\widetilde{U}}(\widetilde{x}, \widetilde{y}) = F^{\widetilde{U}'}(\lambda(\widetilde{x}), d\lambda_{\widetilde{x}}(\widetilde{y})) = F^{\widetilde{U}'}(x', y'),$$

for any injection $\lambda : \{\widetilde{U}, G, \varphi\} \to \{\widetilde{U'}, G', \varphi'\}$, where $\lambda(\widetilde{x}) = x'$, $d\lambda_{\widetilde{x}} y = y'$. Thus

$$F^{\widetilde{U}}_{\widetilde{y}^i}(\widetilde{x}, \widetilde{y}) = F^{\widetilde{U}'}_{\widetilde{x}^s}(x', y') \frac{\partial \widetilde{x}'^s}{\partial \widetilde{y}^i} + F^{\widetilde{U}'}_{\widetilde{y}'^s}(x', y') \frac{\partial \widetilde{y}'^s}{\partial \widetilde{y}^i} = F^{\widetilde{U}'}_{\widetilde{y}'^s}(x', y') \frac{\partial \widetilde{x}'^s}{\partial \widetilde{x}^i}.$$

Similarly, it is easy to check that

$$F^{\widetilde{U}}_{\widetilde{y}^i \widetilde{y}^j}(\widetilde{x}, \widetilde{y}) = F^{\widetilde{U}'}_{\widetilde{y}'^s \widetilde{y}'^t}(x', y') \frac{\partial \widetilde{x}'^s}{\partial \widetilde{x}^i} \frac{\partial \widetilde{x}'^t}{\partial \widetilde{x}^j},$$

and that

$$F^{\widetilde{U}}_{\widetilde{y}^i \widetilde{y}^j \widetilde{y}^k}(\widetilde{x}, \widetilde{y}) = F^{\widetilde{U}'}_{\widetilde{y}'^s \widetilde{y}'^t \widetilde{y}'^n}(x', y') \frac{\partial \widetilde{x}'^s}{\partial \widetilde{x}^i} \frac{\partial \widetilde{x}'^t}{\partial \widetilde{x}^j} \frac{\partial \widetilde{x}'^n}{\partial \widetilde{x}^k}.$$

Equivalently, we have

$$g_{ij}^{\widetilde{U}}(\widetilde{x},\widetilde{y}) = g_{st}^{\widetilde{U}'}(\lambda(\widetilde{x}), d\lambda_x(\widetilde{y}))\frac{\partial \widetilde{x}'^s}{\partial \widetilde{x}^i}\frac{\partial \widetilde{x}'^t}{\partial \widetilde{x}^j},$$

and

$$A_{ijk}^{\widetilde{U}} = A_{stn}^{\widetilde{U}'}\frac{\partial \widetilde{x}'^s}{\partial \widetilde{x}^i}\frac{\partial \widetilde{x}'^t}{\partial \widetilde{x}^j}\frac{\partial \widetilde{x}'^n}{\partial \widetilde{x}^k}.$$

Therefore we have

$$g_{ij}^{\widetilde{U}}(\widetilde{x},\widetilde{y})d\widetilde{x}^i \otimes d\widetilde{x}^j = g_{st}^{\widetilde{U}'}(\lambda(\widetilde{x}), d\lambda_x(\widetilde{y}))d\widetilde{x}'^s \otimes d\widetilde{x}'^t,$$

and

$$A_{ijk}^{\widetilde{U}}(\widetilde{x},\widetilde{y})d\widetilde{x}^i \otimes d\widetilde{x}^j \otimes d\widetilde{x}^k = A_{stn}^{\widetilde{U}'}(\lambda(\widetilde{x}), d\lambda_x(\widetilde{y}))d\widetilde{x}'^s \otimes d\widetilde{x}'^t \otimes d\widetilde{x}'^n.$$

This proves the lemma. □

**Corollary 3.5** ([7], [8], [17])**.** *There always exists a Riemannian metric on any orbifold.*

*Remark.* Similarly as in classical Finsler geometry, we call $g$ and $A$ the fundamental tensor and Cartan tensor of the Finsler metric, respectively.

At any point $p = (x, [y]) \in SM$, choose $(S\widetilde{U}, G^*, \varphi^*) \in \mathcal{F}^*$ such that $\widetilde{p} = (\widetilde{x}, [\widetilde{y}]) \in S\widetilde{U}$, $\varphi^*(\widetilde{p}) = p$. Let

$$\ell_{S\widetilde{U}} = \ell_{\widetilde{p}} = \frac{\widetilde{y}^k}{F(\widetilde{x},\widetilde{y})}\frac{\partial}{\partial \widetilde{x}^k} := \widetilde{u}_m^k \frac{\partial}{\partial \widetilde{x}^k}.$$

Then $\ell_{S\widetilde{U}}$ is a section of the uniformized subset $SU$. In fact, let $\lambda^* : \{S\widetilde{U}, G^*, \varphi^*\} \to \{S\widetilde{U}', G^{*\prime}, \varphi^{*\prime}\}$ be an injection. Then one verifies easily that

$$\begin{aligned}\ell_{S\widetilde{U}'} = \ell_{(\widetilde{x}',\widetilde{y}')} = \ell(\lambda\widetilde{x}, d\lambda_{\widetilde{x}}\widetilde{y}) &= \frac{\widetilde{y}'^k}{F(\widetilde{x}',\widetilde{y}')}\frac{\partial}{\partial \widetilde{x}'^k} \\ &= \frac{\widetilde{y}'^k}{F(\widetilde{x},\widetilde{y})}\frac{\partial \widetilde{x}^j}{\partial \widetilde{x}'^k}\frac{\partial}{\partial \widetilde{x}^j} = \frac{\widetilde{y}^j}{F(\widetilde{x},\widetilde{y})}\frac{\partial}{\partial \widetilde{x}^j} = \ell_{(\widetilde{x},\widetilde{y})} = \ell_{S\widetilde{U}}.\end{aligned}$$

Thus $\ell = \sum_i \psi_i \ell_{S\widetilde{U}^i}$ is a globally defined section of $SM$, where $\{\psi_i\}$ is a partition of unity subordinated to the covering $\{U_i\}$.

For the pull back tangent bundle $\pi^* T\widetilde{U}$, define an orthonormal basis $\{e_1, e_2, \dots, e_m\}$ with $e_m = \ell_{S\widetilde{U}} = \widetilde{u}_m^k \frac{\partial}{\partial \widetilde{x}^k}$ and $e_c = \widetilde{u}_c^k \frac{\partial}{\partial \widetilde{x}^k}$, where $\widetilde{u}_c^k$ are locally defined functions on $S\widetilde{U}$.

Let $(\widetilde{v}_k^c) = (\widetilde{u}_c^k)^{-1}$, we can define a 1-form $\{\omega_{S\widetilde{U}}^c\}$ over $S\widetilde{U}$ by

$$\omega_{S\widetilde{U}}^c = \widetilde{v}_k^c d\widetilde{x}^k.$$

It follows from the equality $\omega^c_{S\widetilde{U}}(e_m) = \delta_{cm}$ that $\widetilde{v}^m_k = F_{\widetilde{y}^k}$, that is, $\omega^m_{S\widetilde{U}} = F_{\widetilde{y}^i} d\widetilde{x}^i$. Let $\lambda^* : \{S\widetilde{U}, G^*, \varphi^*\} \to \{S\widetilde{U}', G^{*\prime}, \varphi^{*\prime}\}$ be an injection. For $\widetilde{p} \in S\widetilde{U}$, let $\widetilde{p}^* = (\widetilde{p}; e_1, e_2, \ldots, e_m)$ and

$$\lambda^*(\widetilde{p}^*) = (\lambda(\widetilde{p}); d\lambda_{\widetilde{p}} e_1, d\lambda_{\widetilde{p}} e_2, \ldots, d\lambda_{\widetilde{p}} e_m),$$

let $\{\widetilde{x}'^i\}$ be a coordinate system in $\widetilde{U}'$ and denote $e'_j = \widetilde{u}'^i_j \frac{\partial}{\partial \widetilde{x}^i}$, $(\widetilde{v}'^j_i) = (\widetilde{u}'^i_j)^{-1}$. Then we have $\omega^c_{S\widetilde{U}'}(\lambda^*(\widetilde{p}^*)) = \widetilde{v}'^c_k d\widetilde{x}'^k = \widetilde{v}'^c_k \frac{\partial \widetilde{x}'^k}{\partial \widetilde{x}^j} d\widetilde{x}^j = \widetilde{v}^c_j d\widetilde{x}^j$. Denote $\omega^c = \sum_i \psi_i \omega^c_{S\widetilde{U}^i}$, where $\{\psi_i\}$ is defined as above. Then $\{\omega^c\}$ defines a dual coframe over $SM$.

Let $\omega^b_{\widetilde{U}a}$ be the Chern connection forms on the Finsler manifold $(\widetilde{U}, F^{\widetilde{U}})$. From the equalities $\omega^a(\lambda^* \cdot) = \omega^a$, $A(\lambda^* \cdot) = A$ and the uniqueness of the forms $\omega^b_{\widetilde{U}a}$ by Theorem 2.4, it follows that $\omega^b_{\widetilde{U}a}(\lambda^* \cdot) = \omega^b_{\widetilde{U}'a}$. Using the partition of unity, one can prove that on a Finsler orbifold, there always exists a unique connection whose connection forms $\omega^b_a$ verify the same structural equations as that of the Chern connection of a Finsler metric on a manifold. We will call this connection the Chern connection in the following.

According to the above remark, we can define $\dot{A}$ to be the covariant differential of the Cartan tensor $A$ on orbifolds as in section 2. In the following sections we will denote $\dot{A}$ the covariant differential of the Cartan tensor $A$ on orbifolds. Similarly to Definition 2.2, we give

*Definition 3.6.* If $\dot{A} = 0$, then the Finsler orbifold $(M, F)$ is called a Landsberg orbifold.

From Proposition 2.3 and Lemma 3.3, we conclude that

**Proposition 3.7.** *If $M$ is a Landsberg orbifold, then the function $V(\widetilde{x})$ defined by* (1) *is a constant on each orbifold chart of $M$.*

Similarly as in the case of a Finsler manifold, we have a Sasaki type metric on $SM$ defined by

$$\delta_{ab}\omega^a \otimes \omega^b + \delta_{ab}\omega^{m+a} \otimes \omega^{m+b}.$$

The curvature two-forms $\Omega^b_a$ of the Chern connection can also be defined by the following structure equations:

$$\Omega^b_a = d\omega^b_a - \omega^c_a \wedge \omega^b_c.$$

Taking the exterior differentiation, one obtains the second Bianchi identity

$$d\Omega^b_a = \Omega_{cb} \wedge \omega^c_a - \Omega^c_a \wedge \omega_{cb}.$$

**Theorem 3.8.** *The Chern curvature two-forms $\Omega_a^b$ on the orbifold $SM$ can be written as*

$$\Omega_a^b = \frac{1}{2} R_{a\ cd}^{b} \omega^c \wedge \omega^d + P_{a\ cd}^{b} \omega^c \wedge \omega^{m+d}.$$

PROOF. For the case of Finslerian manifold, see [4]. The proof for the general case is similar. □

## 4. The Gauss–Bonnet–Chern formula for a compact Finsler orbifold surface

In this section, we turn our attention to orientable Finsler orbifold surfaces without boundary. Let $(M, F)$ be a Finsler orbifold surface. Let $SM$ be the projective orbifold sphere bundle of $(M, F)$. By section 3, we know that the pull back orbifold tangent bundle $\pi^*TM$ has a global section

$$\ell = \frac{\widetilde{y}^1}{F(\widetilde{x}, \widetilde{y})} \frac{\partial}{\partial \widetilde{x}^1} + \frac{\widetilde{y}^2}{F(\widetilde{x}, \widetilde{y})} \frac{\partial}{\partial \widetilde{x}^2}$$

and a natural Riemannian metric

$$g = g_{ij} d\widetilde{x}^i \otimes d\widetilde{x}^j.$$

By Euler's theorem, we have $g(\ell, \ell) = 1$ .

Fix a positively oriented $g-$orthonormal frame $\{e_1, e_2\}$ for $\pi^*TM$, such that $e_2 = \ell$, that is,

$$e_1 = \frac{F_{\widetilde{y}^2}}{\sqrt{g}} \frac{\partial}{\partial \widetilde{x}^1} - \frac{F_{\widetilde{y}^1}}{\sqrt{g}} \frac{\partial}{\partial \widetilde{x}^2}, \qquad e_2 = \frac{\widetilde{y}^1}{F} \frac{\partial}{\partial \widetilde{x}^1} + \frac{\widetilde{y}^2}{F} \frac{\partial}{\partial \widetilde{x}^2},$$

where $g = \det(g_{ij})$. Let $\{\omega^1, \omega^2\}$ be the dual base of $\{e_1, e_2\}$. Then one easily checks that

$$\omega^1 = \frac{\sqrt{g}}{F} (\widetilde{y}^2 d\widetilde{x}^1 - \widetilde{y}^1 d\widetilde{x}^2), \qquad \omega^2 = F_{\widetilde{y}^1} d\widetilde{x}^1 + F_{\widetilde{y}^2} d\widetilde{x}^2.$$

Let $\omega^3 = \omega_2^1 = \frac{\sqrt{g}}{F^2} (\widetilde{y}^2 \delta \widetilde{y}^1 - \widetilde{y}^1 \delta \widetilde{y}^2)$. It is easily seen that $\omega^1, \omega^2, \omega^3$ are globally defined on the projective orbifold sphere bundle $SM$.

The three-dimensional projective orbifold sphere bundle $SM$ has a natural Sasaki type metric

$$\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3,$$

Let $A = A_{abc}\omega^a \otimes \omega^b \otimes \omega^c$. By Euler's theorem, we have $A_{abc} = 0$, whenever one of the $a$, $b$, $c$ is two, i.e., $A_{ab2} = 0$. Then the Landsberg tensor becomes $\dot{A} = \dot{A}(e_1, e_1, e_1)\omega^1 \otimes \omega^1 \otimes \omega^1$.

Denote $I := A_{111} = A(e_1, e_1, e_1)$. It follows from the equations

$$\omega_{ab} + \omega_{ba} = -2A_{abc}\omega^{m+c}$$

of Theorem 2.4 that

$$\omega_{11} = -I\omega^3, \omega_{12} + \omega_{21} = 0, \omega_{22} = 0.$$

Moreover, we have

$$\omega_{21} = \omega_2^c \delta_{c1} = \omega_2^1 = \omega^3.$$

So the Chern connection matrix becomes

$$\begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} \omega_1^1 & \omega_1^2 \\ \omega_2^1 & \omega_2^2 \end{pmatrix} = \begin{pmatrix} -I\omega^3 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix}.$$

It is easily seen that

$$d\omega^1 = -I\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3, \qquad d\omega^2 = -\omega^1 \wedge \omega^3.$$

Let

$$d\omega^3 = K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3 + P\omega^2 \wedge \omega^3.$$

Taking the exterior differential of $d\omega^2 = -\omega^1 \wedge \omega^3$, we obtain

$$P\omega^1 \wedge \omega^2 \wedge \omega^3 = 0.$$

Thus $P = 0$ and $d\omega^3 = K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3$. On the other hand, similarly as Exercise 4.4.7(d) of [4], one can prove that $J = \dot{A}_{111}$. This implies that if $J = 0$, then $(M, F)$ is a Landsberg orbifold surface. In particular, from Proposition 3.7 we have

**Proposition 4.1.** *If $J = 0$, then the volume function* $\mathrm{Vol}(\widetilde{x})$ *defined by* (1) *is a constant.*

We denote by $L = \mathrm{Vol}(\widetilde{x})$ the Riemannian arc length of the indicatrix.

**Theorem 4.2.** *Let $(M, F)$ be a compact, connected, oriented Landsberg orbifold surface without boundary. Then*

$$\frac{1}{L}\int_M K\omega^1 \wedge \omega^2 = \chi_V(M).$$

Proof. We follow the idea of [3], [4]. Let $U$ be a vector field on the orbifold surface $M$, with zeros at $x_i, i = 1, 2, \dots, p$. Denote by $I_{x_i}$ the index of $U$ at $x_i$, $i = 1, 2, \dots, p$.

Let $S_{\varepsilon,x_i}$ be a geodesic circle of sufficiently small radius $\varepsilon$ centered at $x_i$. Remove from $M$ the interior of $S_{\varepsilon,x_i}$, and denote the resulting orbifold with boundary by $M_\varepsilon$. Let

$$V = \frac{U}{F(U)} : M_\varepsilon \to V(M_\varepsilon) \subset SM, \quad x \longmapsto \frac{U(x)}{F(U(x))}.$$

Recall that for a Finsler orbifold surface, we have

$$-\Omega_1^2 = -d\omega_1^2 = K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3.$$

It can be checked that $K = R^1_{212}$ is the Gauss curvature of the orbifold charts. We integrate the above formula over the two dimensional orbifold $V(M_\varepsilon)$. Applying Strokes's theorem and taking the limit as $\varepsilon \to 0$, we obtain

$$\int_M V^*(-K\omega^1 \wedge \omega^2 + J\omega^1 \wedge \omega^3) = \sum_{i=1}^{p} \lim_{\varepsilon \to 0} \int_{V(S_\varepsilon, x_i)} \omega_1^2,$$

here the boundary cycle of each $V(S_\varepsilon, x_i)$ is in a clockwise manner.

On the other hand, we have

$$\int_{V(S_\varepsilon,x_i)} \omega_1^2 \to -I_{x_i} \int_{\widetilde{S}_{\widetilde{x}_i}} \omega^2_{\widetilde{U}_i 1}, \quad \text{as } \varepsilon \to 0,$$

here the circles $S_{x_i}$ are given in a counter clockwise orientation, and $\widetilde{S}_{\widetilde{x}_i}$ are the corresponding circles in the orbifold charts. In fact, let $\{\widetilde{U}_i, G_i, \varphi_i\} \in \mathcal{F}$, $\widetilde{x}_i \in \widetilde{U}$ be an orbifold chart such that $x_i \in \varphi_i(\widehat{U}_i)$ and $\varphi_i(\widetilde{x}_i) = x_i$. Let $\widetilde{V}$ be the corresponding unit vector field on $\widetilde{U}_i$. Then $\widetilde{x}_i$ are zeros of $\widetilde{V}$. From the degree theorem, we have

$$\int_{V(S_\varepsilon,x_i)} \omega_1^2 = \frac{1}{|G_{\widetilde{x}_i}|} \int_{\widetilde{V}(\widetilde{S}_\varepsilon,\widetilde{x}_i)} \omega^2_{\widetilde{U}_i 1} \to -\frac{I_{\widetilde{x}_i}}{|G_{\widetilde{x}_i}|} \int_{\widetilde{S}_{\widetilde{x}_i}} \omega^2_{\widetilde{U}_i 1} = -I_{x_i} \int_{\widetilde{S}_{\widetilde{x}_i}} \omega^2_{\widetilde{U}_i 1}, \quad \text{as } \varepsilon \to 0.$$

Moreover, we have

$$\int_M V^*(K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3) = \sum_{i=1}^{p} I_{x_i} \int_{\widetilde{S}_{\widetilde{x}_i}} \omega^2_{\widetilde{U}_i 1}.$$

Note that

$$\omega^2_{\widetilde{U}_i 1} = \frac{\sqrt{g}}{F}\left(\widetilde{y}^1 \frac{\delta\widetilde{y}^2}{F} - \widetilde{y}^2 \frac{\delta\widetilde{y}^1}{F}\right),$$

where $\delta\widetilde{y}^i = d\widetilde{y}^i + N^i_j d\widetilde{x}^j$. Meanwhile, as $\varepsilon \to 0$, the circle $\widetilde{S}_{\varepsilon,\widetilde{x}_i}$ shrinks to $\widetilde{x}_i$. Therefore the $d\widetilde{x}$ terms in the above integral do not contribute. Hence

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{V(S_\varepsilon,x_i)} \omega^2_1 &= \lim_{\varepsilon\to 0}\int_{V(S_\varepsilon,x_i)} \frac{\sqrt{g}}{F}\left(\widetilde{y}^1 \frac{\delta\widetilde{y}^2}{F} - \widetilde{y}^2 \frac{\delta\widetilde{y}^1}{F}\right)\\ &= -I_{x_i}\int_{\widetilde{S}_{\widetilde{x}_i}} \frac{\sqrt{g}}{F}\left(\widetilde{y}^1 \frac{d\widetilde{y}^2}{F} - \widetilde{y}^2 \frac{d\widetilde{y}^1}{F}\right) = -I_{x_i}\,\mathrm{Vol}(\widetilde{x}_i).\end{aligned}$$

Therefore we have

$$\int_M V^*(K\omega^1\wedge\omega^2 - J\omega^1\wedge\omega^3) = \sum_{i=1}^{p}(I_{x_i}\,\mathrm{Vol}(\widetilde{x}_i)).$$

Since $M$ is a Landsberg orbifold surface, we have $J = 0$. Thus $\mathrm{Vol}(\widetilde{x}_i) = L$ is a constant by Proposition 4.1, and the above integral becomes

$$\int_M V^*(K\omega^1\wedge\omega^2) = \sum_{i=1}^{p}(I_{x_i}\,\mathrm{Vol}(\widetilde{x}_i)) = L\sum_{i=1}^{p} I_{x_i} = L\cdot\chi_V(M).$$

Moreover,

$$K\omega^1\wedge\omega^2 = K\sqrt{g}d\widetilde{x}^1\wedge d\widetilde{x}^2 = -d\omega^2_1.$$

Taking the exterior differentiation of the above equation on $SM$, one gets

$$\frac{\partial}{\partial\widetilde{y}^i}(K\sqrt{g}) = 0.$$

This means that $K\sqrt{g}$ lives on the orbifold $M$. Consequently we obtain the Gauss–Bonnet–Chern Formula in the orbifold case:

$$\frac{1}{L}\int_M K\omega^1\wedge\omega^2 = \chi_V(M).$$

This completes the proof of the theorem. □

Next we generalize the above Gauss–Bonnet–Chern theorem to $m\,(m > 2)$ dimensional Finsler orbifold $M$ with the volume function $\mathrm{Vol}(\widetilde{x})$ being constant, following the idea of [3]. We consider the following polynomials on the projective orbifold sphere bundle $SM$:

$$\Phi_k = \varepsilon^{\alpha_1\dots\alpha_{m-1}}\Omega_{\alpha_1\alpha_2}\wedge\cdots\wedge\Omega_{\alpha_{2k-1}\alpha_{2k}}\wedge\omega^m_{\alpha_{2k+1}}\wedge\omega^m_{\alpha_{2k+2}}\wedge\cdots\wedge\omega^m_{\alpha_{m-1}}, \quad \left(0\le k\le\left[\frac{m-1}{2}\right]\right),$$

$$\Psi_k = (2k+2)\varepsilon^{\alpha_1\dots\alpha_{m-1}}\Omega_{\alpha_1\alpha_2}\wedge\cdots\wedge\Omega_{\alpha_{2k-1}\alpha_{2k}}\wedge\Omega^m_{\alpha_{2k+1}}\wedge\omega^m_{\alpha_{2k+2}}\wedge\cdots\wedge\omega^m_{\alpha_{m-1}}, \quad \left(0\le k\le\left[\frac{m}{2}\right]-1\right),$$

$$\Psi_{-1}=0.$$

By taking exterior differential and using the equations $\omega_{ab}+\omega_{ba}=-2A_{abc}\omega^{m+c}$, we have

$$d\Phi_k = \Psi_{k-1}+\frac{m-2k-1}{2k+2}\Psi_k+Q_k,$$

where $Q_0=0$ and

$$\begin{aligned} Q_k = k\varepsilon^{\alpha_1\dots\alpha_{m-1}}&[\Omega_{\alpha_1\alpha_2}\wedge(\omega_{\alpha_1\alpha_1}-\omega_{\alpha_2\alpha_2})+\Omega_{\alpha_2\alpha_2}\wedge(\omega_{\alpha_1\alpha_2}+\omega_{\alpha_2\alpha_1}) \\ &+(k-1)\Omega_{\alpha_2\alpha_3}\wedge(\omega_{\alpha_1\alpha_3}+\omega_{\alpha_3\alpha_1}) \\ &+(m-2k-1)\{\Omega_{\alpha_2\alpha_{2k+1}}\wedge(\omega_{\alpha_1\alpha_{2k+1}}+\omega_{\alpha_{2k+1}\alpha_1}) \\ &+\frac{1}{k}\Omega_{\alpha_1\alpha_2}\wedge(\omega_{\alpha_{2k+1}\alpha_{2k+1}})\}]\wedge\Omega_{\alpha_3\alpha_4}\wedge\dots \\ &\wedge\Omega_{\alpha_{2k-1}\alpha_{2k}}\wedge\omega^m_{\alpha_{2k+1}}\wedge\omega^m_{\alpha_{2k+2}}\wedge\cdots\wedge\omega^m_{\alpha_{m-1}}. \end{aligned}$$

Define

$$\Pi = \begin{cases} \dfrac{1}{\pi^p 2^p}\displaystyle\sum_{k=0}^{p-1}\frac{(-1)^k}{1\cdot3\cdot\cdots\cdot(2p-2k-1)k!2^k}\Phi_k, & m=2p, \\ \dfrac{1}{\pi^q 2^{2q+1}q!}\displaystyle\sum_{k=0}^{q}(-1)^{k+1}C_q^k\Phi_k, & m=2q+1 \end{cases}$$

and

$$\Omega = \begin{cases} \dfrac{(-1)^{p-1}}{2^{2p}\pi^p p!}\varepsilon^{i_1\cdots i_m}\Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{m-1}i_m}, & m=2p, \\ 0, & m=2q+1, \end{cases}$$

where $C_q^k$ is the binomial coefficient given by $\frac{q!}{k!(q-k)!}$. Then a direct computation shows that

$$d\Pi = \Omega + Q,$$

where

$$Q = \begin{cases} \dfrac{1}{\pi^p 2^p} \displaystyle\sum_{k=0}^{p-1} \dfrac{(-1)^k}{1 \cdot 3 \cdot \cdots \cdot (2p-2k-1)k!2^k} Q_k, & m = 2p, \\ \dfrac{1}{\pi^q 2^{2q+1} q!} \displaystyle\sum_{k=0}^{q} (-1)^{k+1} C_q^k Q_k, & m = 2q+1. \end{cases}$$

Similarly as Theorem 4.2, we can prove

**Theorem 4.3.** *Let $(M, F)$ be a compact, connected, oriented boundaryless $m$-dimensional Finsler orbifold. Suppose the volume function $\mathrm{Vol}(\widetilde{x}) = L$ defined by (1) is a constant. Then for any unit vector field $V$ on $M$, we have*

$$-\frac{\mathrm{Vol}(S^{m-1})}{L} \int_M V^*(\Omega + Q) = \chi_V(M),$$

*where $\mathrm{Vol}(S^{m-1})$ is the Riemannian volume of $m-1$ dimensional sphere $S^{m-1}$.*

## 5. The case of an orbifold with boundary

Let $N$ be an orbifold with boundary $\partial N$. Let $X : \partial N \longmapsto SN$ be a unit vector field on $N$ with finite number of zeros in the interior of $N$ such that the restriction of $X$ to $\partial N$ is the inner unit Finslerian normal vector field on $\partial N$.

Similarly as the proof of the Theorem 4.2, we have

**Theorem 5.1.** *Let $(N, F)$ be a compact, connected, oriented Finsler orbifold with boundary $\partial N$. Suppose the volume function $V(\widetilde{x}) = L$ defined as above is a constant. Then for any unit vector field $X$ on $N$ which coincides with the inner normal vector field on $\partial N$, we have*

$$-\int_N X^*(\Omega + Q) = \frac{L}{\mathrm{Vol}(S^{m-1})} \chi'(N) - \int_{\partial N} (X)^*(\Pi),$$

*where the orientation of $\partial N$ is induced with respect to the outer normal vector field on $N$ and $\mathrm{Vol}(S^{m-1})$ the Riemannian volume of $m-1$ dimensional unit sphere $S^{m-1}$.*

Note that $\chi'(N)$ is called the inner orbifold Euler characteristic of $N$ by Satake [17], and $\chi'(N)$ can be given by the triangulation in the interior of $N$ (see [17]).

In particular, when $(N, F)$ is two dimensional Landsberg orbifold with boundary, we have

**Theorem 5.2.** *Let $(N, F)$ be a compact, connected, oriented Landsberg orbifold surface with boundary. Let $X : \partial N \longmapsto SN$ be a unit vector field on $N$ with finite number of zeros in the interior of $N$ such that the restriction of $X$ to $\partial N$ is the inner unit Finslerian normal vector field on $\partial N$.Then*

$$\frac{1}{L}\int_N K\omega^1 \wedge \omega^2 = \chi'(N) - \frac{1}{L}\int_{\partial N} (X)^*(\omega_1^2).$$

*Where $L = V(\widetilde{x})$ is the Riemannian arc length of the indicatrix and $\partial N$ is the induced orientation with respect to the outer normal vector field on $N$.*

*Remark.* When the orbifold $N$ is two-dimensional, the boundary $\partial N$ become a curve. In this case, we can extend Theorem 5.2 to nonconstant indicatrix volume with regular piecewise $C^\infty$ curve using the method of [9], [12].

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