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On some band decompositions of semigroups

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Abstract. M. S. Putcha in [5] described semigroups which are bands of r-Archimedean or t-Archimedean semigroups. L. N. Ševrin, J. L. Galbiati, M. L. Veronesi, S. Bogdanović and M. Ćirić described rectangular bands of π-groups. In this paper we characterize some bands of r-Archimedean semigroups.

Let $\mathbb{N}$ be the set of all positive integers. A semigroup S is *right Archimedean* (or *r-Archimedean*) if, for every $a, b \in S$, there exists $n \in \mathbb{N}$ such that $a^n \in bS$. The dual of a right Archimedean semigroup is a *left Archimedean* (or *l-Archimedean*) one. A semigroup S is *t-Archimedean* if, for every $a, b \in S$, there exists $n \in \mathbb{N}$ for which $a^n \in bS \cap Sb$.

A semigroup B is a *band* if for each $x \in B$, $x^2 = x$ holds.

A semigroup S is a *band* Y *of semigroups* S_α if $S = \bigcup_{\alpha \in Y} S_\alpha$, Y is a band, $S_\alpha \cap S_\beta = \emptyset$ for $\alpha, \beta \in Y$ with $\alpha \neq \beta$ and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

A congruence ϱ on S is called a *band congruence* if S/ϱ is a band.

Theorem 1 [5]. *A semigroup S is a band of r-Archimedean semigroups if and only if*

$$(1) \qquad (\forall a \in S)(\forall x, y \in S^1)(\exists i, j \in \mathbb{N})(xay)^i \in xa^2yS,\ (xa^2y)^j \in xayS.$$

In this theorem, it is proved that if (1) holds then the relation ϱ defined on S by

$$(2) \qquad a\varrho b \iff (\forall x, y \in S^1)(\exists i, j \in \mathbb{N})(xay)^i \in xbyS,\ (xby)^j \in xayS$$

is a band congruence on S.

For undefined notions and notations we refer to [1] and [3].

Recall that a band B is a *right regular band* if $ef = fef$ for every e, $f \in B$.

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Theorem 2. *A semigroup S is a right regular band of r-Archimedean semigroups if and only if*

$$(\forall u, v \in S)(\exists n \in \mathbb{N})\ (uv)^n \in vS. \tag{3}$$

Proof. Let S be a right regular band Y of r-Archimedean semigroups S_α. If $u \in S_\alpha$, $v \in S_\beta$ then $uv \in S_{\alpha\beta} = S_{\beta\alpha\beta}$, $vuv \in S_{\beta\alpha\beta}$. Since $S_{\beta\alpha\beta}$ is r-Archimedean, then there exists $n \in \mathbb{N}$ such that $(uv)^n \in vuvS_{\beta\alpha\beta} \subseteq vS$, and so (3) holds.

Conversely, let statement (3) hold on a semigroup S and let $a \in S$, $x, y \in S^1$. Then, for $u = a$, $v = ayx$, there exists $n \in \mathbb{N}$ such that

$$(xa^2y)^{n+1} = (xaay)^{n+1} = x(aayx)^n a^2 y \in xayxSa^2y \subseteq xayS.$$

Also, for $u = yxayx$, $v = ayxa$ there exists $m \in \mathbb{N}$ such that

$$\begin{aligned}(xay)^{3(m+1)} &= (xayxayxay)^{m+1} = xa(yxayxayxa)^m yxayxay \\ &\in xaayxaSyxayxay \subseteq xa^2yS.\end{aligned}$$

Now, by Theorem 1 we have that S is a band Y of r-Archimedean semigroups S_α.

Let $a, b \in S$. Then, by (3), $(ab)^n = bt$ for some $t \in S$ and $n \in \mathbb{N}$. If $a \in S_\alpha$, $b \in S_\beta$, $t \in S_\gamma$ then $\alpha\beta = \beta\gamma = \beta\beta\gamma = \beta\alpha\beta$. Hence Y is a right regular band. $\square$

Recall that a band B is a *right zero band* if $e = fe$ for every $e, f \in B$.

Theorem 3. *A semigroup S is a right zero band of r-Archimedean semigroups if and only if*

$$(\forall u, v \in S)(\exists m, n \in \mathbb{N})(uv)^m \in vS,\ v^n \in uvS. \tag{4}$$

Proof. Let S be a right zero band Y of r-Archimedean semigroups S_α, $\alpha \in Y$. If $u \in S_\alpha$, $v \in S_\beta$ then $uv \in S_{\alpha\beta} = S_\beta$. As S_β is r-Archimedean, statement (4) holds.

Conversely, let statement (4) hold on a semigroup S. Then, by Theorem 2, it follows that S is a right regular band Y of r-Archimedean semigroups S_α, $\alpha \in Y$. Let $a \in S_\alpha$, $b \in S_\beta$. Then by (4) there exists $t \in S_\gamma$ such that $b^n = abt$ whence $\beta = \alpha\beta\gamma = \alpha\beta\alpha\beta\gamma = \alpha\beta\beta = \alpha\beta$. Thus Y is a right zero band and so the semigroup S is a right zero band Y of r-Archimedean semigroups S_α, $\alpha \in Y$. $\square$

Recall that a band B is a *left normal band* if $efg = egf$ for every $e, f, g \in B$.

Theorem 4. *A semigroup S is a left normal band of r-Archimedean semigroups if and only if*

$$(\forall u,v,w\in S)(\exists n\in\mathbb{N})\ (uvw)^n\in uwvS. \tag{5}$$

Proof. Let $S=\bigcup_{\alpha\in Y}S_\alpha$ where Y is a left normal band and S_α are r-Archimedean semigroups for every $\alpha\in Y$. If $u\in S_\alpha$, $v\in S_\beta$, $w\in S_\gamma$ then $uvw\in S_{\alpha\beta\gamma}=S_{\alpha\gamma\beta}$. Since $uwv\in S_{\alpha\gamma\beta}$ and since $S_{\alpha\gamma\beta}$ is r-Archimedean we have that (5) holds.

Conversely, let statement (5) hold on a semigroup S. If $a\in S$ and $x,y\in S^1$ then by (5) for $u=xa$, $v=a$, $w=yxa^2y$ there exists $n\in\mathbb{N}$ such that

$$(xa^2y)^{2n}=(xaayxa^2y)^n\in xayxa^2yaS\subseteq xayS.$$

Also, for $u=xa$, $v=yxayx$, $w=ay$ there exists $m\in\mathbb{N}$ such that

$$(xay)^{3m}=(xayxayxay)^m\in xaayyxayxS\subseteq xa^2yS.$$

By Theorem 1 it follows that S is a band of r-Archimedean semigroups. Now we shall prove that the congruence ϱ defined by (2) is a left normal band congruence on S. Let $a,b,c\in S$ and $x,y\in S^1$. For $u=xa$, $v=b$, $w=cy$ by (5) there exists $n\in\mathbb{N}$ such that $(xabcy)^n\in xacybS$. Hence $(xabcy)^n=xacybs$ for some $s\in S$ and $(xabcy)^{n+1}=xacybsxabcy$. By (5) for $u=xac$, $v=ybsxa$, $w=bcy$ there exists $m\in\mathbb{N}$ such that $(xacybsxabcy)^m\in xacbcyybsxaS$. Now, $(xacybsxabcy)^m=xacbcyyt$ for some $t\in bsxaS$. By (5) for $u=xacb$, $v=cy$, $w=yt$ there exists $p\in\mathbb{N}$ such that $(xacbcyyt)^p\in xacbytcyS\subseteq xacbyS$. Hence

$$(xabcy)^{(n+1)mp}=(xacybsxabcy)^{mp}=(xacbcyyt)^p\in xacbyS.$$

Similarly we prove that there exist $q,r,l,\in\mathbb{N}$ such that $(xacby)^{(q+1)rl}\in xabcyS$. Hence $abc\varrho acb$ and ϱ is a left normal band congruence on S. It follows that S is a left normal band of r-Archimedean semigroups. □

Recall that a band B is a *right quasinormal band* if $efg=egfg$ for every $e,f,g\in B$.

Theorem 5. *A semigroup S is a right quasinormal band of r-Archimedean semigroups if and only if*

$$(\forall u,v,w\in S)(\exists n\in\mathbb{N})\ (uvw)^n\in uwvwS. \tag{6}$$

Proof. Let $S=\bigcup_{\alpha\in Y}S_\alpha$ where Y is a right quasinormal band and S_α are r-Archimedean semigroups for each $\alpha\in Y$. If $u\in S_\alpha$, $v\in S_\beta$,

$w \in S_\gamma$, then we have $uvw \in S_{\alpha\beta\gamma} = S_{\alpha\gamma\beta\gamma}$. Since $uwvw \in S_{\alpha\gamma\beta\gamma}$ we have that statement (6) holds.

Conversely, let statement (6) hold on a semigroup S. If $a \in S$, $x, y \in S^1$ then for $u = xa$, $v = a$, $w = yxa^2y$ there exists $n \in \mathbb{N}$ such that

$$(xa^2y)^{2n} = (xaayxa^2y)^n \in xayxa^2yayxa^2yS \subseteq xayS.$$

Also, for $u = xa$, $v = yxayx$, $w = ay$ there exists $m \in \mathbb{N}$ such that

$$(xay)^{3m} = (xayxayxay)^m \in xaayyxayxayS \subseteq xa^2yS.$$

Hence, by Theorem 1 the semigroup S is a band of r-Archimedean semigroups.

Let $a, b, c \in S$, $x, y \in S^1$. By (6), for $u = xa$, $v = b$, $w = cy$, there exists $n \in \mathbb{N}$ such that $(xabcy)^n \in xacybcyS$ and so $(xabcy)^n = xacybcyt$ for some $t \in S$.

Using (6) for $u = xac$, $v = ybcytxacy$, $w = bcyt$, there exists $m \in \mathbb{N}$ such that $(xacybcyt)^{2m} = (xacybcytxacybcyt)^m \in xacbcytybcytxacybcytS \subseteq xacbcyS$. Thus

$$(xabcy)^{2nm} = (xacybcyt)^{2m} \in xacbcyS.$$

Similarly, by (6) for $u = xa$, $v = c$, $w = bcy$ there exists $p \in \mathbb{N}$ such that

$$(xacbcy)^p \in xabcycbcyS \subseteq xabcyS.$$

Hence, by (2) it follows that $abc\varrho acbc$ whence ϱ is a right quasinormal band congruence on S and S is a right quasinormal band of r-Archimedean semigroups. $\square$

Recall that a band B is a *right seminormal band* if $efg = egefg$ for every $e, f, g \in B$.

Theorem 6. *A semigroup S is a right seminormal band of r-Archimedean semigroups if and only if*

(7) $$(\forall u, v, w \in S)(\exists n \in \mathbb{N})(uvw)^n \in uwS.$$

Proof. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ where Y is a right seminormal band and S_α is an r-Archimedean semigroup for every $\alpha \in Y$. Then, for $u \in S_\alpha$, $v \in S_\beta$, $w = S_\gamma$ we have $uvw \in S_{\alpha\beta\gamma} = S_{\alpha\gamma\alpha\beta\gamma}$. Since $uwuvw \in S_{\alpha\gamma\alpha\beta\gamma}$, there exists $n \in \mathbb{N}$ such that

$$(uvw)^n \in uwuvwS \subseteq uwS$$

and so (7) holds.

Conversely, let statement (7) hold on a semigroup S. Let $a \in S$, $x, y \in S^1$. Then for $u = xa$, $v = yxayxay$ and $w = ay$ there exists $n \in \mathbb{N}$ such that

$$(xay)^{3n} = (xayxayxay)^n \in xaayS = xa^2yS.$$

Also, by (7) for $u = xa$, $v = a$ and $w = yxa^2y$ there exists $m \in \mathbb{N}$ such that $(xa^2y)^{2m} = (xaayxa^2y)^m \in xayxa^2yS \subseteq xayS$. By Theorem 1 we have that S is a band of r-Archimedean semigroups. Hence, $S = \bigcup_{\alpha\in Y} S_\alpha$, Y is a band and S_α is an r-Archimedean semigroup for all $\alpha \in S$. If $a \in S_\alpha$, $b \in S_\beta$, $c \in S_\gamma$, then $abc \in S_{\alpha\beta\gamma}$ and by (7) there exists $n \in \mathbb{N}$ such that $(abc)^n \in acS$. Now, there exists $t \in S$ such that $(abc)^n = act$. If $t \in S_\delta$ then $\alpha\beta\gamma = \alpha\gamma\delta = \alpha\gamma\alpha\gamma\delta = \alpha\gamma\alpha\beta\gamma$. Hence, Y is a right seminormal band. $\square$

Recall that a band B is a *rectangular band* if $efg = eg$ for every $e, f, g \in B$.

Theorem 7. *A semigroup S is a rectangular band of r-Archimedean semigroups if and only if*

$$(\forall x, y, z \in S)(\exists n \in \mathbb{N})\ (xyz)^n \in xzS,\ (xz)^n \in xyzS. \tag{8}$$

Proof. Let Y be a rectangular band, $S = \bigcup_{\alpha\in Y} S_\alpha$ and S_α an r-Archimedean semigroup for every $\alpha \in Y$. Then for $x, y, z \in S$ there exists $\alpha, \beta, \gamma \in Y$ such that $x \in S_\alpha$, $y \in S_\beta$, $z \in S_\gamma$ and $xyz \in S_\alpha S_\beta S_\gamma \subseteq S_{\alpha\beta\gamma} = S_{\alpha\gamma}$, $xz \in S_\alpha S_\gamma \subseteq S_{\alpha\gamma}$. Since $S_{\alpha\gamma}$ is an r-Archimedean semigroup, we have that (8) holds.

Conversely, let statement (8) hold on a semigroup S. Let η be the relation on S defined by

$$a\eta b \iff (\exists n \in \mathbb{N})\quad a^n \in bS,\ b^n \in aS. \tag{9}$$

From $a^2 \in aS$ it follows that η is a reflexive relation. Clearly, η is a symmetric relation.

Let $a, b, c \in S$ and

$$a\eta b \iff (\exists n \in \mathbb{N})a^n \in bS,\ b^n \in aS,$$
$$b\eta c \iff (\exists m \in \mathbb{N})b^m \in cS,\ c^m \in bS.$$

For $k = \max\{n, m\}$ we have $a^k \in bS$, $b^k \in aS \cap cS$, $c^k \in bS$. Hence, there exist $u, v, w \in S$ such that $a^k = bu$, $b^k = cv$, $c^k = bw$. Now, by (8) for $x = b$, $z = u$, $y = b^k$ there exists $p \in \mathbb{N}$ such that

$$(a^k)^p = (bu)^p \in bb^kuS \subseteq b^kS \subseteq cS. \tag{10}$$

Similarly, by (8) for $x = b$, $z = w$, $y = b^k$ theren exists $q \in \mathbb{N}$ such that

$$(c^k)^q = (bw)^q \in bb^k wS \subseteq b^k S \subseteq aS. \tag{11}$$

From (10) and (11) for $r = \max\{p, q\}$ we have $a^{kr} \in cS$, $c^{kr} \in aS$ and so $a\eta c$. Hence, η is a transitive relation and it follows that η is an equivalence relation.

To show that η is right compatible, let $a, b, c \in S$ be arbitrary elements such that

$$a\eta b \iff (\exists n \in \mathbb{N}) a^n \in bS,\ b^n \in aS$$

For $x = a$, $z = c$ and $y = a^n$ there exists (by (8)) $p \in \mathbb{N}$ such that $(ac)^p \in aa^n cS$ and so $(ac)^p = aa^n cu$ for some $u \in S$. Now, since $a^n = bv$ for some $v \in S$ we have

$$(ac)^p = aa^n cu = a^n acu = bvacu. \tag{12}$$

From (12) for $x = b$, $y = va$, $z = cu$ there exists $q \in \mathbb{N}$ such that

$$(ac)^{pq} = (bvacu)^q \in bcuS \subseteq bcS. \tag{13}$$

Similarly, for $x = b$, $y = b^n$ and $z = c$ there exists $r \in \mathbb{N}$ such that $(bc)^r \in bb^n cS$ and so $(bc)^r = bb^n cv = b^n bcv$ for some $v \in S$. Now, from $b^n = aw$, for some $w \in S$ we have

$$(bc)^r = b^n bcv = awbcv. \tag{14}$$

From (8) and (14) for $x = a$, $y = wb$ and $z = cv$ there exists $j \in \mathbb{N}$ such that

$$(bc)^{rj} = (awbcv)^j \in acvS \subseteq acS. \tag{15}$$

From (13) and (15) for $i = \max\{pq, rj\}$ we have $ac\eta bc$ and so η is right compatible.

From $a\eta b$ we have $a^n = bs$ for some $s \in S$, and by (8) for $x = c$, $y = a^n$, $z = a$ and some $m \in \mathbb{N}$ it follows that $(ca)^m \in ca^n aS = cbsaS \subseteq cbS$. Similarly, $(cb)^k \in caS$ for some $k \in \mathbb{N}$. For $r = \max\{m, k\}$ it follows that $ca\eta cb$ and so η is left compatible.

Hence, η is a congruence relation.

From $(a^2)^4 \in aS$ and $a^4 \in a^2 S$ we have $a\eta a^2$ and so η is a band congruence relation.

By (8) we conclude that $abc\eta ac$ for every $a, b, c \in S$. Hence it follows that η is a rectangular band congruence on S.

Let $S = \bigcup_{\alpha \in Y} S_\alpha$, where Y is a rectangular band and S_α are η–classes. If $a, b \in S_\alpha$, then $b^2 \in S_\alpha$ and by (8) there exists $n \in \mathbb{N}$ such that

$a^n \in b^2S$. Now, $a^n = b^2u$ for some $u \in S$. If $u \in S_\beta$, then $a^{n+1} = bbua \in bS_\alpha S_\beta S_\alpha \subseteq bS_{\alpha\beta\alpha} = bS_\alpha$. Hence, S_α is an r-Archimedean semigroup and so the semigroup S is a rectangular band Y of r-Archimedean semigroups S_α. □

Similarly, the semigroup S is a rectangular band of l-Archimedean semigroups if and only if for every $x, y, z \in S$ there exists $n \in \mathbb{N}$ such that $(xyz)^n \in Sxz$, $(xz)^n \in Sxyz$. Now, the semigroup S is a rectangular band of t-Archimedean semigroups if and only if for every $x, y, z \in S$ there exists $n \in \mathbb{N}$ such that $(xyz)^n \in xzS \cap Sxz$, $(xz)^n \in xyzS \cap Sxyx$.

We remark that Theorem 7 can be proved by Theorem 1. It is easy to see that $\eta \subseteq \varrho$ where ϱ is defined by (2) on S^1 and η is a congruence on S.

Example 1. Let S be a semigroup defined by the following Cayley table:

	a	e	f	g	h
a	e	e	f	e	f
e	e	e	f	e	f
f	e	e	f	e	f
g	g	g	h	g	h
h	g	g	h	g	h

Then $S = S_\alpha \cup S_\beta$ where $S_\alpha = \{a, e, f\}$, $S_\beta = \{g, h\}$, $S_\alpha S_\beta S_\alpha \subseteq S_\alpha$, $S_\beta S_\alpha S_\beta \subseteq S_\beta$ and S_α and S_β are r-Archimedean semigroups. In this example the semigroup S is a left zero band of r-Archimedean semigroups.

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