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On Randers metrics of isotropic scalar curvature

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Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. In this paper, we study the scalar curvature defined by H. Akbar-Zadeh in Finsler geometry and obtain the formula of scalar curvature for Randers metrics. We prove that a Randers metric of isotropic scalar curvature must be of isotropic S-curvature. Further, we consider Yamabe problem on Randers manifolds and give a negative answer to Yamabe problem on Randers manifolds with isotropic S-curvature.

1. Introduction

In 1960, in order to solve Poincare conjecture, H. Yamabe considered conformal metrics and the following question as the first step (see [3], [10]):

For a Riemannian metric α on a compact manifold M of dimension $n \geq 3$, is there a non-constant smooth real-valued function $\sigma = \sigma(x)$ on M such that the Riemannian metric $\bar{\alpha} := e^{\sigma}\alpha$ is of constant scalar curvature?

H. Yamabe attempted to solve this question using techniques of calculus of variations and elliptic partial differential equations ([10]). He claimed that every compact Riemannian n-manifold M has a conformal metric of constant scalar curvature. Unfortunately, his proof contained an error, discovered by N. Trudinger in 1968 ([9]). Later, because of the outstanding contributions made by N. Trudinger, T. Aubin and R. Schoen, etc., the solution of the Yamabe

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problem has been completed in 1984 (see [3], [7]). The solution of the Yamabe problem marks a milestone in the development of differential geometry and the theory of nonlinear partial differential equations.

Finsler geometry is just Riemannian geometry without the quadratic restriction. Hence, it is natural to extend Yamabe problem in Riemannian geometry to Finsler geometry. To study Yamabe problem in Finsler geometry, the first work that we have to do is to define scalar curvature in Finsler geometry. However, there is no unified definition of scalar curvature in Finsler geometry, although several geometers have offered several versions of the definition of scalar curvature. Here, we adopt the definition of scalar curvature introduced by H. Akbar-Zadeh ([1], [2]). For a Finsler metric F on an n-dimensional manifold M, let $\mathbf{Ric}$ denotes the Ricci curvature of F (see section 2 for the details). The *scalar curvature* $\mathbf{r}$ of F is defined as follows ([1], [2]):

$$\mathbf{r} := g^{ij}\mathbf{Ric}_{ij}, \tag{1}$$

where

$$\mathbf{Ric}_{ij} := \frac{1}{2}\mathbf{Ric}_{y^iy^j}, \quad (g^{ij}) := (g_{ij})^{-1}$$

and $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$. We say a Finsler metric F to be of *isotropic scalar curvature* if there exists a scalar function $\mu(x)$ on M such that $\mathbf{r} = n(n-1)\mu(x)$.

Randers metrics were introduced by physicist G. Randers in 1941 in the context of general relativity. Later on, these metrics were used in the theory of the electron microscope by R. S. Ingarden in 1957, who first named them Randers metrics. Randers metrics form an important and ubiquitous class of Finsler metrics with a strong presence in both the theory and applications of Finsler geometry, and studying Randers metrics is an important step to understand general Finsler metrics (see [5]). A Randers metric on a manifold M is a Finsler metric that can be expressed in the following special form:

$$F = \alpha + \beta,$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M such that the norm of β with respect to α satisfies that $\|\beta\|_\alpha(x) < 1$. In this paper, our main focus is on the Yamabe problem of Randers metrics. Firstly, we compute the scalar curvature of Randers metrics and get the following theorem.

Theorem 1.1. *Let F be a Randers metric on an n-dimensional manifold M. If the scalar curvature of F is isotropic, that is, $\mathbf{r} = n(n-1)\mu(x)$, then F is of isotropic S-curvature.*

Based on Theorem 1.1, we give a negative answer to Yamabe problem on Randers metrics with isotropic S-curvature and obtain the following theorem.

Theorem 1.2. *Let F be a non-Riemannian Randers metric with isotropic S-curvature on an n-dimensional manifold $M(n \geq 3)$. Then there is no non-constant scalar function $\sigma = \sigma(x)$ such that $\bar{F} := e^{\sigma} F$ is of isotropic scalar curvature.*

A Finsler metric is called *conformally flat Finsler metric* if it is conformally related to a Minkowski metric. Note that any Minkowski metric has zero scalar curvature. According to Theorem 1.1 and Theorem 1.2, it is easy to reach the following result.

Corollary 1.3. *Let F be a conformally flat non-Riemannian Randers metric on an n-dimensional manifold $M(n \geq 3)$. If the scalar curvature of F is isotropic, that is, $\mathbf{r} = n(n-1)\mu(x)$, then F must be Minkowskian.*

2. Preliminaries

Let M be an n-dimensional smooth manifold and TM be the tangent bundle. A Finsler metric on M is a continuous function $F : TM \to [0, \infty)$ with the following properties:

(1) **Smoothness**: $F(x, y)$ is C^{∞} on $TM \backslash \{0\}$.

(2) **Homogeneity**: $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$.

(3) **Strong convexity/Regularity**: the fundamental tensor $\big(g_{ij}(x, y)\big)$ is positive definite, where

$$g_{ij}(x, y) := \frac{1}{2}\left[F^2\right]_{y^i y^j}(x, y).$$

For a given Finsler $F = F(x, y)$, the geodesics of F are characterized locally by a system of 2nd ODEs:

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i = \frac{1}{4} g^{il}\Big\{[F^2]_{x^m y^l} y^m - [F^2]_{x^l}\Big\}.$$

G^i are called the *geodesic coefficients* of F.

For any $x \in M$ and $y \in T_x M \backslash \{0\}$, the *Riemann curvature* $\mathbf{R}_y = R^i{}_k(x, y) \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k(x, y) := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}. \tag{2}$$

The *Ricci curvature* $\mathbf{Ric}(x,y)$ is the trace of the Riemmann curvature defined by

$$\mathbf{Ric}(x,y) := R^m_{\ m}(x,y). \tag{3}$$

Obviously, the Ricci curvature is a positive homogeneous function of degree two in y. The *Ricci tensor* is defined by

$$\mathbf{Ric}_{ij} := \frac{1}{2}\mathbf{Ric}_{y^iy^j}.$$

Then $\mathbf{Ric}(x,y) = \mathbf{Ric}_{ij}y^iy^j$. A Finsler metric F is called an *Einstein metric* if there is a scalar function $\mu = \mu(x)$ on M such that F satisfies

$$\mathbf{Ric} = (n-1)\mu F^2. \tag{4}$$

The scalar curvature of F introduced by *H. Akbar-Zadeh* ([1], [2]) is defined by (1), that is, $\mathbf{r} := g^{ij}\mathbf{Ric}_{ij}$. By the definition, a Finsler metric F on an n-dimensional manifold is of isotropic scalar curvature if $\mathbf{r} = n(n-1)\mu(x)$, where $\mu(x)$ is a scalar function. Obviously, Einstein metric must be of isotropic scalar curvature. However, the converse may not be true.

Define the Busemann–Hausdorff volume form of F by

$$dV_F = \sigma_{BH}(x)dx^1 \wedge \cdots \wedge dx^n,$$

where

$$\sigma_{BH}(x) := \frac{\mathrm{Vol}\big(\mathbf{B}^n(1)\big)}{\mathrm{Vol}\big((y^i) \in R^n | F(x, y^i\frac{\partial}{\partial x^i}) < 1\big)}.$$

Here $\mathrm{Vol}(\cdot)$ denotes the Euclidean volume function on subsets in R^n. Further, the *distortion* of F is defined by

$$\tau(x,y) := \ln\frac{\sqrt{\det\big(g_{ij}(x,y)\big)}}{\sigma_{BH}(x)}.$$

The distortion τ is a basic invariant which characterizes Riemannian metrics among Finsler metrics, namely, $\tau = 0$ if and only if the Finsler metric is Riemannian. The vertical derivative of τ on tangent spaces gives rise to the *mean Cartan torsion* $\mathbf{I} = I_idx^i$, $I_i = \tau_{y^i}$. The horizontal derivative of τ along geodesics is the so-called *S-curvature*, $\mathbf{S}(x,y) := \tau_{|m}(x,y)y^m$. In a standard local coordinate system, we have the following formula for S-curvature:

$$\mathbf{S}(x,y) := \frac{\partial G^m}{\partial y^m}(x,y) - y^m\frac{\partial}{\partial x^m}\big(\ln \sigma_{BH}(x)\big).$$

We say that F is of *isotropic S-curvature* if

$$\mathbf{S}(x,y) = (n+1)cF(x,y),$$

where $c = c(x)$ is a scalar function on M.

For a Randers metric $F = \alpha + \beta$ on an n-dimensional manifold M, we have ([5], [8])

$$g^{ij} = \frac{\alpha}{F}a^{ij} - \frac{\alpha}{F^2}(b^i y^j + b^j y^i) + \frac{b^2\alpha + \beta}{F^3} y^i y^j, \tag{5}$$

where $b := \|\beta\|_\alpha$ denotes the norm of β with respect to α. The mean Cartan tensor $\mathbf{I} = I_i dx^i$ of $F = \alpha + \beta$ is given by ([6])

$$I_i = \frac{n+1}{2F}\left(b_i - \frac{\beta y_i}{\alpha^2}\right), \tag{6}$$

where $y_i := a_{ij}y^j$.

Let "|" denote the horizontal covariant derivative with respect to α. Denote

$$\begin{gathered}
r_{ij} := (b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \\
r^i{}_j := a^{il} r_{lj}, \quad r_i := b^j r_{ji}, \quad r := b^i b^j r_{ij}, \\
s^i{}_j := a^{il} s_{lj}, \quad s_i := b^j s_{ji}, \\
e_{ij} := r_{ij} + s_i b_j + s_j b_i, \\
q_{ij} := r_{im} s^m{}_j, \quad t_{ij} := s_{im} s^m{}_j, \quad q^i{}_j := a^{il} q_{lj}, \quad t^i{}_j := a^{il} t_{lj}, \\
q_i := b^j q_{ji}, \quad t_i := b^j t_{ji}, \quad t := t_i b^i, \\
w_{ij} := r_{im} r^m{}_j, \quad p_j := r_{ji} s^i,
\end{gathered} \tag{7}$$

and $s_0 := s_i y^i$, $r_{00} := r_{ij} y^i y^j$, $e_{00} := e_{ij} y^i y^j$, $q_{00} := q_{ij} y^i y^j$, etc..

According to [5], we have the following lemma.

Lemma 2.1 ([5]). *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M, then F is of isotropic S-curvature, $\mathbf{S} = (n+1)c(x)F$, where $c(x)$ is a scalar function on M, if and only if*

$$e_{00} = 2c(x)(\alpha^2 - \beta^2).$$

3. Scalar curvature of Randers metrics

In this section, we compute the scalar curvature of Randers metrics. For a Randers metric $F = \alpha + \beta$, the Ricci curvature of F is given by (see [5])

$$\mathbf{Ric} = \mathbf{Ric}_\alpha + (2\alpha s^m{}_{0|m} - 2t_{00} - \alpha^2 t^m{}_m) + (n-1)\Xi, \tag{8}$$

where $\mathbf{Ric}_\alpha$ denotes the Ricci curvature of α and

$$\Xi := \frac{2\alpha}{F}(q_{00} - \alpha t_0) + \frac{3}{4F^2}(r_{00} - 2\alpha s_0)^2 - \frac{1}{2F}(r_{00|0} - 2\alpha s_{0|0}).$$

Then the scalar curvature of F is expressed as

$$\mathbf{r} = g^{ij}\mathbf{Ric}_{ij} = (\mathbf{Ric}_\alpha)_{ij} g^{ij} + \frac{1}{2}E_{ij}g^{ij} + \frac{1}{2}(n-1)\Xi_{ij}g^{ij}, \tag{9}$$

where $(\mathbf{Ric}_\alpha)_{ij}$ denote the Ricci tensor of α and

$$E := 2\alpha s^m{}_{0|m} - 2t_{00} - \alpha^2 t^m{}_m,$$
$$E_{ij} := E_{y^i y^j}, \quad \Xi_{ij} := \Xi_{y^i y^j}.$$

Now we calculate each term on the right side of (9) as follows. Firstly, we can get

$$(\mathbf{Ric}_\alpha)_{ij} g^{ij} = \frac{\alpha}{F}\mathbf{r}_\alpha - \frac{2\alpha}{F^2}(\mathbf{Ric}_\alpha)_{ij} b^i y^j + \frac{b^2\alpha + \beta}{F^3}\mathbf{Ric}_\alpha, \tag{10}$$

where $\mathbf{r}_\alpha$ denotes scalar curvature of Riemannian metric α. Further, we obtain the following

$$\begin{aligned} E_{ij}g^{ij} = 2\Bigg\{ & \frac{\alpha}{F}\left[\frac{n+1}{\alpha} s^m{}_{0|m} - (n+2)t^m{}_m\right] \\ & - \frac{4\alpha}{F^2}\left[s^m{}_{0|m} s + \alpha b^i s^m{}_{i|m} - 2t_0 - \beta t^m{}_m\right] + 2\frac{b^2\alpha + \beta}{F^3}E \Bigg\}, \end{aligned} \tag{11}$$

where $s := \beta/\alpha$.

In order to determine $\Xi_{ij}g^{ij}$, let

$$A := r_{00} - 2\alpha s_0, \quad B := r_{00|0} - 2\alpha s_{0|0},$$
$$D_1 := q_{00} - \alpha t_0, \quad D := \alpha D_1.$$

Then $\Xi = 2\left(\frac{D}{F}\right) + \left(\frac{3}{4}\right)\left(\frac{A^2}{F^2}\right) - \left(\frac{1}{2}\right)\left(\frac{B}{F}\right)$. Write

$$\Xi_{ij} := \Xi^1_{ij} + \Xi^2_{ij} + \Xi^3_{ij}, \tag{12}$$

where

$$\Xi^1_{ij} := 2\left[\frac{\alpha}{F}(q_{00}-\alpha t_0)\right]_{y^iy^j} = 2\left(\frac{D}{F}\right)_{y^iy^j},$$

$$\Xi^2_{ij} := \left[\frac{3}{4F^2}(r_{00}-2\alpha s_0)^2\right]_{y^iy^j} = \left(\frac{3}{4}\right)\left(\frac{A^2}{F^2}\right)_{y^iy^j},$$

$$\Xi^3_{ij} := -\left[\frac{1}{2F}(r_{00|0}-2\alpha s_{0|0})\right]_{y^iy^j} = -\left(\frac{1}{2}\right)\left(\frac{B}{F}\right)_{y^iy^j}.$$

We have

$$\begin{aligned}
\Xi^1_{ij}g^{ij} = {} & \frac{2}{F}\left\{\frac{\alpha}{F}\left[(n+3)\frac{D_1}{\alpha}+2\alpha q^m{}_m-(n+1)t_0\right]\right.\\
& \left.-\frac{4\alpha}{F^2}\big[sD_1+\alpha(q_{00\cdot i}b^i-st_0-\alpha t)\big]+6\frac{b^2\alpha+\beta}{F^3}\right\}\\
& +\frac{4}{F^2}\left\{\frac{\alpha}{F}\big[(3+s)D_1+\alpha(q_{00\cdot i}b^i-st_0-\alpha t)\big]\right.\\
& \left.-\frac{\alpha}{F^2}\big[F(sD_1+\alpha(q_{00\cdot i}b^i-st_0-\alpha t)+3D(s+b^2))\big]+\frac{b^2\alpha+\beta}{F^3}\right\}\\
& -2(n-1)\frac{D}{F^3}+4\frac{D}{F^3}\left[\frac{\alpha}{F}(1-b^2)+\frac{b^2\alpha+\beta}{F}\right]
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\Xi^2_{ij}g^{ij} = {} & \frac{6}{F^2}\left\{\frac{\alpha}{F}\left[w_{00}-2\frac{r_{00}s_0}{\alpha}-2\alpha p_0+3s_0^2-t\alpha^2\right]\right.\\
& \left.-\frac{2\alpha}{F^2}A(r_0-ss_0)+\frac{b^2\alpha+\beta}{F^3}A^2\right\}+\frac{3A}{F^2}\left\{\frac{\alpha}{F}\big[r^m{}_m-(n+1)\frac{s_0}{\alpha}\big]\right.\\
& \left.-\frac{2\alpha}{F^2}(r_0-ss_0)+\frac{b^2\alpha+\beta}{F^3}A\right\}-\frac{12A}{F^3}\left\{\frac{\alpha}{F}\left[\frac{r_{00}}{\alpha}+r_0-(2+s)s_0\right]\right.\\
& \left.-\frac{\alpha}{F^2}\big[F(r_0-ss_0)+A(s+b^2)\big]+\frac{b^2\alpha+\beta}{F^2}\right\}-\frac{3(n-1)A^2}{2F^4}\\
& +\frac{9A^2}{2F^4}\left[\frac{\alpha}{F}(1-b^2)+\frac{b^2\alpha+\beta}{F}\right]
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\Xi^3_{ij}g^{ij} = {} & \frac{1}{2F}\left\{\frac{2\alpha}{F}\left[2r^m{}_{0|m}+r^m{}_{m|0}-(n+3)\frac{s_{0|0}}{\alpha}-2\alpha s^m{}_{|m}\right]\right.\\
& \left.-\frac{4\alpha}{F^2}\big[r_{00|0\cdot i}b^i-2ss_{0|0}-2\alpha s_{0|0\cdot i}b^i\big]+6\frac{b^2\alpha+\beta}{F^3}B\right\}\\
& +\frac{1}{F^2}\left\{\frac{\alpha}{F}\left[\frac{3r_{00|0}}{\alpha}-2(3+s)s_{0|0}+r_{00|0\cdot i}b^i-2\alpha s_{0|0\cdot i}b^i\right]\right.
\end{aligned}$$

$$-\frac{\alpha}{F^2}\big[F(r_{00|0\cdot i}b^i-2s_0s_{0|0}-2\alpha s_{0|0.}b^i)+3B(s+b^2)\big]+3\frac{b^2\alpha+\beta}{F^3}B\Big\}$$
$$-\frac{n-1}{2F^3}B+\frac{2B}{F^3}\left[\frac{\alpha}{F}(1-b^2)+\frac{b^2\alpha+\beta}{F}\right]. \tag{15}$$

Plugging (10)–(15) into (9) yields the expression of scalar curvature for Randers metric $F=\alpha+\beta$ as follows,

$$\mathbf{r}=\frac{\alpha}{F}\mathbf{r}_\alpha+\frac{1}{4F^5}\{\Sigma_1+\Sigma_2\alpha\}, \tag{16}$$

where

$$\begin{aligned}\Sigma_1:=&\Big\{\big[-12s^m{}_{|m}+24q^m{}_m-48s^m{}_{i|m}b^i+16t-8(2n+3+2b^2)t^m{}_m\big]\beta\\&-4(n-7+6b^2)t_0+4r^m{}_{0|m}+2r^m{}_{m|0}+4(n+1+4b^2)s^m{}_{0|m}\\&-8(\mathbf{Ric}_\alpha)_{ij}b^iy^j-12s_0r^m{}_m-16q_{00\cdot i}b^i-24p_0+8s_{0|0\cdot i}b^i\Big\}\alpha^4\\&+\Big\{\big[8(1-2n)t^m{}_m-16s^m{}_{i|m}b^i+8q^m{}_m-4s^m{}_{|m}\big]\beta^3\\&+\big[4(29-5n)t_0-24q_{00\cdot i}b^i+16b^2s^m{}_{0|m}+12r^m{}_{0|m}+6r^m{}_{m|0}-24p_0\\&+24s^m{}_{0|m}n-24(\mathbf{Ric}_\alpha)_{ij}b^iy^j-12s_0r^m{}_m+8s^m{}_{0|m}+8s_{0|0\cdot i}b^i\big]\beta^2\\&+\big[2(n+5+6b^2)(2q_{00}-s_{0|0})+12r_{00}r^m{}_m+24w_{00}+72s_0r_0\\&-16(1+2b^2)t_{00}+12(n+20)s_0^2+8\mathbf{Ric}_\alpha b^2-8r_{00|0\cdot i}b^i+4\mathbf{Ric}_\alpha\big]\beta\\&-(n+3-6b^2)r_{00|0}+6(n-29-12b^2)r_{00}s_0-36r_{00}r_0\Big\}\alpha^2\\&+4(n-3)s^m{}_{0|m}\beta^4+[4(n-3)q_{00}-16t_{00}+4\mathbf{Ric}_\alpha+2(1-n)s_{0|0}]\beta^3\\&+[(3-n)r_{00|0}+6(1-n)r_{00}s_0]\beta^2+3(18-n)r_{00}^2\beta\end{aligned}$$

and

$$\begin{aligned}\Sigma_2:=&\Big\{8q^m{}_m-4(n+2+2b^2)t^m{}_m-4s^m{}_{|m}+4t+16s^m{}_{i|m}b^i\Big\}\alpha^4\\&+\Big\{\big[24q^m{}_m+12t-12s^m{}_{|m}-8(3n+2+b^2)t^m{}_m-48s^m{}_{i|m}b^i\big]\beta^2\\&+\big[-8(2n+3b^2-10)t_0-24(\mathbf{Ric}_\alpha)_{ij}b^iy^j+16(n+1+2b^2)s^m{}_{0|m}\\&+6r^m{}_{m|0}-24s_0r^m{}_m+12r^m{}_{0|m}-40q_{00\cdot i}b^i-48p_0+16s_{0|0\cdot i}b^i\big]\beta\\&-4r_{00|0\cdot i}b^i+12w_{00}+192s_0^2+24b^2q_{00}+6r_{00}r^m{}_m+72s_0r_0+4\mathbf{Ric}_\alpha b^2\\&-12b^2s_{0|0}-16b^2t_{00}+72b^2s_0^2\Big\}\alpha^2+\big[4(2-n)t^m{}_m\big]\beta^4\\&+\big[8(8-n)t_0+16(n-1)s^m{}_{0|m}+2r^m{}_{m|0}+4r^m{}_{0|m}-8(\mathbf{Ric}_\alpha)_{ij}b^iy^j\big]\beta^3\end{aligned}$$

$$+\big[8(n+1)q_{00}-16(2+b^2)t_{00}+4\mathbf{Ric}_\alpha b^2-4(n+2)s_{0|0}$$
$$+6r_{00}r^m_{\ m}+12(n-2)s_0^2+12w_{00}+8\mathbf{Ric}_\alpha-4r_{00|0\cdot i}b^i\big]\beta^2$$
$$+\big[-240r_{00}s_0+6b^2r_{00|0}-36r_{00}r_0-2nr_{00|0}\big]\beta-3(n-12-6b^2)r_{00}^2.$$

It is clear that Σ_1 and Σ_2 are homogeneous polynomials of degree 5 and 4 in y, respectively.

4. Proof of Theorems

In this section, we will prove Theorem 1.1 and Theorem 1.2 respectively.

Proof of Theorem 1.1. Assume $F=\alpha+\beta$ is a Randers metric with isotropic scalar curvature.

Note that

$$r_{00}=e_{00}-2s_0\beta, \tag{17}$$

and then

$$r_{00|0}=e_{00|0}-2s_{0|0}\beta-2s_0e_{00}+4s_0^2\beta. \tag{18}$$

Then plugging (17), (18) into (16) and multiplying (16) by $(\alpha+\beta)^5$, one has

$$(\alpha+\beta)^5\mathbf{r}=\Gamma_1+\Gamma_2\alpha, \tag{19}$$

where Γ_1 and Γ_2 are polynomials in y.

By direct computation, we can find that

$$\Gamma_2\beta-\Gamma_1=-18(1-b^2)\beta e_{00}^2+(\alpha^2-\beta^2)H_{000}, \tag{20}$$

here H_{000} is a polynomial of degree 3 in y.

On the other hand, by assumption, we have

$$\mathbf{r}=n(n-1)\mu(x). \tag{21}$$

Then

$$(\alpha+\beta)^5\mathbf{r}=n(n-1)\mu(x)(\Pi_1+\Pi_2\alpha), \tag{22}$$

where

$$\Pi_1:=5\alpha^4\beta+10\alpha^2\beta^3+\beta^5,$$
$$\Pi_2:=\alpha^4+10\alpha^2\beta^2+5\beta^4.$$

It's easy to obtain

$$\Pi_2\beta - \Pi_1 = -4(\alpha^2 - \beta^2)(\alpha^2 + \beta^2)\beta. \tag{23}$$

Comparing (19) and (22), we have the following

$$\Gamma_1 = n(n-1)\mu(x)\Pi_1, \quad \Gamma_2 = n(n-1)\mu(x)\Pi_2.$$

Then by (20) and (23), we have

$$18(1-b^2)\beta e_{00}^2 = (\alpha^2 - \beta^2)\{H_{000} + 4n(n-1)\mu(x)(\alpha^2 + \beta^2)\beta\}. \tag{24}$$

It is well-known that $b^2 := \|\beta\|_\alpha^2 < 1$ for Randers metric $F = \alpha + \beta$. Because $\alpha^2 - \beta^2$ is an irreducible polynomial in y, e_{00} must be divided by $\alpha^2 - \beta^2$. Therefore, there is a scalar function $c = c(x)$ such that

$$e_{00} = 2c(\alpha^2 - \beta^2). \tag{25}$$

By Lemma 2.1, F is of isotropic S-curvature. □

As we mentioned in Section 2, Einstein metrics must be of isotopic scalar curvature but the converse may not be true. Hence Theorem 1.1 generalizes a result given by D. Bao and C. Robles in 2004 which says that any Einstein–Randers metric is of isotopic S-curvature.

Next, we prove Theorem 1.2.

Proof of Theorem 1.2. Assume that F is a Randers metrics of isotropic S-curvature,

$$\mathbf{S} = (n+1)\mu(x)F. \tag{26}$$

If Randers metric $\bar{F} := e^\sigma F$ is of isotropic scalar curvature, where $\sigma = \sigma(x)$ is a scalar function, then by Theorem 1.1, $\bar{F}$ is of isotropic S-curvature,

$$\bar{\mathbf{S}} = (n+1)\lambda(x)\bar{F}. \tag{27}$$

According to [4], for any Finsler metrics F and $\bar{F} := e^\sigma F$, their S-curvatures have the following relationship

$$\bar{\mathbf{S}} = \mathbf{S} + F^2\sigma^r I_r, \tag{28}$$

where $\sigma^r := g^{rj}\sigma_j$, $\sigma_j := \frac{\partial\sigma}{\partial x^j}$ and I_r is the mean Cartan tensor of F.

Substituting (5), (6), (26) and (27) into (28), we have

$$-\frac{n+1}{2(\alpha+\beta)}\Big\{(-\tau + 2\lambda e^\sigma - 2\mu)\alpha^2 + (-\tau\beta + 4\lambda e^\sigma\beta - 4\mu\beta + \sigma_0 b^2)\alpha + (2\lambda e^\sigma\beta^2 - 2\mu\beta^2 + \sigma_0\beta)\Big\} = 0, \tag{29}$$

where $\sigma_0 := \sigma_i y^i$, $\tau := a^{ij}\sigma_i b_j$.

From (29), we obtain

$$(-\tau + 2\lambda e^{\sigma} - 2\mu)\alpha^2 + (2\lambda e^{\sigma}\beta - 2\mu\beta + \sigma_0)\beta = 0, \tag{30}$$

and

$$-\tau\beta + 4\lambda e^{\sigma}\beta - 4\mu\beta + \sigma_0 b^2 = 0. \tag{31}$$

Since α^2 is not reducible, by (30), we have

$$2\lambda e^{\sigma} = \tau + 2\mu \tag{32}$$

and

$$2\lambda e^{\sigma}\beta - 2\mu\beta + \sigma_0 = 0. \tag{33}$$

Plugging (32) into (33) and (31), we get

$$\tau\beta = -\sigma_0, \quad \tau\beta = -b^2\sigma_0.$$

Then $(1-b^2)\sigma_0 = 0$. However, we know that, for regular Randers metrics, $b^2 < 1$. Therefore, $\sigma_0 = 0$, which implies that σ is a constant. □

References

[1] H. Akbar-Zadeh, Sur les espaces de Finsler a courbures sectionnelles constantes, *Acad. Roy. Belg. Bull. Cl. Sci.* **74** (1988), 281–322.

[2] H. Akbar-Zadeh, Initiation to Global Finslerian Geometry, *Elsevier*, 2006.

[3] T. Aubin, Some Non-Linear Problems in Riemannian Geometry, *Springer-Verlag*, 1998.

[4] S. Bácsó and X. Cheng, Finsler conformal transformations and the curvature invariance, *Publ. Math. Debrecen* **70** (2007), 221–231.

[5] X. Cheng and Z. Shen, Finsler Geometry – An Approach via Randers Spaces, *Springer and Science Press*, *Heidelberg, New York, Dordrecht, London, Beijing*, 2012.

[6] S. S. Chern and Z. Shen, Rieman–Finsler Geometry, Nankai Tracts in Mathematics, Vol. 6, *World Scientific*, *New Jersey, London, Singapore, Beijing*, 2005.

[7] J. M. Lee and T. H. Parker, The Yamabe problem, *Bull. Amer. Math. Soc.* **17** (1987), 37–91.

[8] Z. Shen, Differential Geometry of Spray and Finsler Spaces, *Kluwer Academic Publishers*, *Dordrecht*, 2001.

[9] N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, *Ann. Scuola Norm. Sup. Pisa* **22** (1968), 265–274.

[10] H. Yamabe, On the deformation of Riemannian structures on compact manifolds, *Osaka Math. J.* **12** (1960), 21–37.

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