Publ. Math. Debrecen
84/1-2 (2014), 75–88
DOI: 10.5486/PMD.2014.5834

A compactness theorem in Finsler geometry

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Dedicated to Professor Dr. Lajos Tamássy on the occasion of his 90th birthday

Abstract. Let (M, F) be a complete Finsler manifold and P be a minimal and compact submanifold of M. The k-Ricci curvature $\mathbf{Ric}_k(x), x \in M$ is a differential invariant that interpolates between the flag curvature and the Ricci scalar. We prove that if the k-Ricci curvature satisfies the condition $\int_0^\infty \mathbf{Ric}_k(t) > 0$ along any geodesic $\gamma : [0, \infty) \to M, t \to \gamma(t)$ emanating orthogonally from P or $\int_{-\infty}^0 \mathbf{Ric}_k(t) > 0$ along any geodesic $\gamma : (-\infty, 0] \to M, t \to \gamma(t)$ arriving orthogonally to P, then M is compact.

Introduction

The classical Gauss–Bonnet Theorem opened a series of results that are extracting topological properties of a differentiable manifold from the various properties of certain differential geometric invariants of the manifold. The basic topics in this framework consist of the Hopf–Rinow Theorem, the theory of Jacobi fields and the relationship between geodesics and curvature, the Theorems of Hadamard, Myers, Synge, the Rauch Comparison Theorem, the Morse Index Theorem and others. In the Finslerian setting the most recent account of results of this type is due to D. Bao, S. S. Chern and *Z. Shen* in [3], Chapter 6–9. For a weakened version of the Myers theorem we refer to [2].

The main differential geometric invariants involved in these results are the flag curvature and the Ricci scalar. Among the many others there exists one

Mathematics Subject Classification: 53C60.
Key words and phrases: Finsler manifolds, Morse theory, minimal submanifolds.

denoted by $\mathbf{Ric}_k$ and called k-Ricci curvature that interpolates between the flag curvature and the Ricci curvature. In this paper we consider an n-dimensional, complete Finsler manifold (M, F), a minimal, compact submanifold P of it and we prove that if the k-Ricci curvature satisfies the condition $\int_0^\infty \mathbf{Ric}_k(t) > 0$ along any geodesic $\gamma : [0, \infty) \to M, t \to \gamma(t)$ emanating orthogonally from P or $\int_{-\infty}^0 \mathbf{Ric}_k(t) > 0$ along any geodesic $\gamma : (-\infty, 0] \to M, t \to \gamma(t)$ arriving orthogonally to P, then M is compact. For the Riemannian case there are many similar results (see [4] and the references therein). By our knowledge our result is the first of this type for general Finsler spaces but the techniques we use here can be adapted to find many others. Some results for Berwald spaces are obtained by Binh and Tamássy (see [5]). The differential invariant $\mathbf{Ric}_k$ was deeply studied by Z. Shen. In [16], he proves various results concerning the vanishing of homotopy groups under the assumption that the k-Ricci curvature satisfies $\mathbf{Ric}_k \geq k$.

We outline the proof of our result. Considering the submanifold P the notion of conjugate points is replaced with that of focal points. The Morse index form written on a geodesic emanating from or arriving in P takes a special form that involves the second fundamental form of P (see [13]). The conditions M complete but non-compact, and P compact imply that any geodesic emanating orthogonally from P or arriving orthogonally to P is free of focal points. But choosing a convenient orthogonal frame along the geodesic emanating from P or arriving to P, we reduce the Jacobi equation to a scalar differential equation of order two that by our hypothesis on $\mathbf{Ric}_k$ admits on $[0, \infty)$ or $(-\infty, 0]$ a solution with at least one zero. In combination with a form of the Index Lemma from [13] one yields that the said geodesic has focal points. The contradiction shows that M has to be compact.

In the Sections 1–3 we prepare all we need for the detailed proof given in the Section 4.

1. Preliminaries

Let M be a real manifold of dimension n and (TM, π, M) its tangent bundle. The vertical bundle of the manifold M is the vector subbundle of the double tangent bundle TTM denoted by $(\mathcal{V}, \tilde{\pi}, TM)$ and defined by $\mathcal{V} = \operatorname{Ker} d\pi \subset T(TM)$, where $d\pi$ is the linear tangent map to π. Let (x^i) denote the local coordinates on an open subset U of M, and (x^i, y^i) the induced coordinates on $\pi^{-1}(U) \subset TM$. The radial vector field ι is the vertical vector field locally given by $\iota(x, y) = y^i \frac{\partial}{\partial y^i}$.

A Finsler metric on M is a function $F : TM \to \mathbb{R}_+$ satisfying the following properties:

(1) F^2 is smooth on $\widetilde{M}$, where $\widetilde{M} = TM \setminus 0$.

(2) $F(u) > 0$ for all $u \in \widetilde{M}$.

(3) $F(\lambda u) = \lambda F(u)$ for all $u \in TM$, $\lambda \in \mathbb{R}^\star_+$.

(4) For any $p \in M$, the indicatrix $I_p = \{u \in T_pM|\ F(u) < 1\}$ is strongly convex.

A manifold M endowed with a Finsler metric F is called a Finsler manifold (M, F).

An important fact related to a Finsler metric is that it may be not reversible, that is, $F(u) \neq F(-u)$. There are a lot of examples of non-reversible Finsler metrics, perhaps the most known being Randers metrics.

From condition (4) it follows that the quantities $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}$ form the entries of a positive definite matrix so a Riemannian metric $\langle \cdot, \cdot \rangle$ can be introduced in the vertical bundle $(\mathcal{V}, \widetilde{\pi}, TM)$.

On a Finsler manifold there is not, in general, a linear metrical connection. However, there are several metrical connections and among them the analogue of the Levi–Civita connection in the vertical bundle of (M, F).

We will use the Cartan connection which is a good vertical connection on $\mathcal{V}$, i.e., an $\mathbb{R}$-linear map

$$\nabla^v : \mathfrak{X}(\widetilde{M}) \times \mathfrak{X}(\mathcal{V}) \to \mathfrak{X}(\mathcal{V})$$

having the usual properties of a covariant derivative, is metrical with respect to $\langle \cdot, \cdot \rangle$, and 'good' in the sense that the bundle map $\Lambda : T\widetilde{M} \to \mathcal{V}$ defined by $\Lambda(Z) = \nabla^v_Z \iota$ restricted to $\mathcal{V}$ is a bundle isomorphism. The latter property induces the horizontal subspaces $H_u = \operatorname{Ker} \Lambda$ for all $u \in \widetilde{M}$, which are direct summands of the vertical subspaces $V_u = \operatorname{Ker}(d\pi)_u$. They define a vector bundle called the horizontal bundle $\mathcal{H}$ such that

$$T\widetilde{M} = \mathcal{H} \oplus \mathcal{V}.$$

For a tangent vector field X on M we have its vertical lift X^V and its horizontal lift X^H to $\widetilde{M}$.

Let be $\delta_i = \left(\frac{\partial}{\partial x^i}\right)^H$. These local vector fields provide a local basis for the distribution $\mathcal{H}$. Define $\Theta : \mathcal{V} \to \mathcal{H}$ as the vector bundle morphism locally given by $\Theta\left(\frac{\partial}{\partial y^i}\right) = \delta_i$. It is in fact the inverse of the mapping Λ and is clearly an isomorphism of vector bundles. It is called the horizontal map associated to the horizontal bundle $\mathcal{H}$.

Using Θ, first we get the radial horizontal vector field $\chi = \Theta \circ \iota$. For a curve σ on M let $\dot\sigma$ be its tangent vector field. Its horizontal lift $\dot\sigma^H$ is just χ in the point $\dot\sigma(t)$ of TM. Locally, $\dot\sigma^H = \frac{d\sigma^i}{dt}\delta_i$.

Secondly we can extend the covariant derivation ∇^v of the vertical bundle to the whole tangent bundle of $\widetilde{M}$. Denoting it with ∇, for horizontal vector fields H we set

$$\nabla_Z H = \Theta(\nabla^v_Z(\Theta^{-1}(H))), \quad \forall\, Z \in \mathfrak{X}(\widetilde{M}).$$

The covariant derivative of an arbitrary vector field $Y \in \mathfrak{X}(\widetilde{M})$ is decomposed into vertical and horizontal parts:

$$\nabla_Z Y = \nabla_Z Y^V + \nabla_Z Y^H.$$

Thus $\nabla : \mathfrak{X}(T\widetilde{M}) \times \mathfrak{X}(T\widetilde{M}) \to \mathfrak{X}(T\widetilde{M})$ is a linear connection on $\widetilde{M}$ induced by a good vertical connection. Its torsion θ and curvature Ω are defined as usual:

$$\theta(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad \Omega(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

and the torsion has the property that for horizontal vectors, $\theta(X,Y)$ is a vertical vector [1].

The Riemannian metric $\langle\cdot,\cdot\rangle$ on $\mathcal{V}$ can be moved to a Riemannian metric on the vector bundle $\mathcal{H}$ and these two Riemannian metrics provide a Riemannian metric on $\tilde{M}$ (a Sasaki type metric) just by stating that $\mathcal{H}$ is orthogonal to $\mathcal{V}$. All these metrics will be denoted with the same symbol whose meaning will be clear from context.

The metrical property of the connection ∇ holds good:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

The sectional curvature of ∇ along a curve σ is given as follows:

$$K_{\dot\sigma}(U^H, U^H) = \langle \Omega(\dot\sigma^H, U^H)\dot\sigma^H, U^H\rangle,$$

for any $U \in \mathfrak{X}(M)$. This is called the horizontal flag curvature in [1].

2. The Morse index form

We recall some facts about the variation of energy and Morse index form, mainly from [13].

Definition 1. [1]. A *regular curve* $\sigma : [a,b] \to M$ is a C^1-curve such that

$$T(t) \equiv \dot{\sigma}(t) = d\sigma_t \left(\frac{d}{dt}\right) \neq 0, \quad \text{for all } t \in [a,b].$$

The length, with respect to the Finsler metric $F : TM \to \mathbb{R}^+$, of the regular curve is given by

$$L(\sigma) = \int_a^b F(\dot{\sigma}(t))dt$$

and the energy is given by

$$E(\sigma) = \int_a^b F^2(\dot{\sigma}(t))dt.$$

The Finsler metric induces naturally the (Finslerian) distance by

$$d(p,q) = \inf_{\sigma \in C(p,q)} L(\sigma),$$

where $C(p,q)$ is the set of piecewise smooth curves from p to q. The properties of a distance, except the symmetry, are verified. The pair (M,d) is called sometimes a *generalized metric space*. For a non-reversible Finsler metric d is not symmetric, because the length of a curve may not coincide with the length of the reverse curve $\tilde{\sigma}(t) = \sigma(a+b-t) \in M$. The non-reversibility property is also reflected in Cauchy sequences. We say that a sequence $(x_i)_{i\in\mathbb{N}}$ in M is *forward* (resp. *backward*) Cauchy if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $i,j > N$ implies $d(x_i, x_j) < \epsilon$ whenever $i \leq j$ (resp. $i \geq j$).

In this setting the classical Hopf–Rinow theorem splits into a forward and a backward version (see [3], [8]).

The non-reversibility of the distance implies the existence of two open balls, forward balls

$$B^+(p,r) = \{x \in M \mid d(p,x) < r\},$$

where $p \in M$ and $r > 0$, and backward balls,

$$B^-(p,r)) = \{x \in M \mid d(x,p) < r\}.$$

A symmetrized distance can be defined as

$$d_s(p,q) = \frac{1}{2}(d(p,q) + d(q,p)).$$

The closed balls will be denoted by a bar, i.e., $\overline{B}^+(p,r)$ and $\overline{B}^-(p,r)$. The topologies induced by these two kind of balls agree with the topology of the manifold. We also denote the associated balls of d_s by $B_s(x,r)$. In [8] (Proposition 2.2) a Hopf–Rinow theorem for symmetrized closed balls is proved, i.e., the symmetrized distance d_s is complete if $\overline{B}_s^+(p,r)$ are compact for all $x \in M$ and $r > 0$ (or equivalently $\overline{B}^+(p,r) \cap \overline{B}^-(p,r)$ is compact for all $x \in M$ and $r > 0$). The conditions here are weaker that those in the theorem involving forward and backward completeness. In the same paper [8] an example of Randers type with compact symmetrized balls is constructed, which fails to be forward or backward complete.

The non-reversibility of the metric also induces two types of geodesic completeness, *forward*, when the domain of the geodesic can be always extended to (a,∞) for some $a \in \mathbb{R}$ and backward when it can be extended to $(-\infty,b)$ for some $b \in \mathbb{R}$.

The critical points of the (length) energy functional are the geodesics σ in the Finsler manifold M, whenever they are parameterized by arc-length, i.e., $F(\dot{\sigma}) = 1$. A geodesic parameterized by the arc-length will be called normal. One proves that the geodesics are characterized also by

Theorem 2 ([1]). *A regular curve σ is a geodesic for F if and only if*

$$\nabla_{T^H} T^H \equiv 0$$

where $T^H(u) = \dot{\sigma}^H = \chi_u(\dot{\sigma}(t)) \in \mathcal{H}_u$ for all $u \in \widetilde{M}_{\sigma(t)}$.

The second variation formula provides the Jacobi fields and suggests the consideration of the index form. It is derived by using a two parameters geodesic variation. For details we refer to [1], [12].

Let $\sigma : [a,b] \to M$ be a normal geodesic in a Finsler manifold M. We will denote by $\mathfrak{X}[a,b]$ the space of piecewise smooth vector fields X along σ such that

$$\langle X^H, T^H \rangle_T \equiv 0.$$

Definition 3 ([1]). The *Morse index form* $I = I_a^b : \mathfrak{X}[a,b] \times \mathfrak{X}[a,b] \to \mathbb{R}$ of a normal geodesic $\sigma : [a,b] \to M$ is the symmetric bilinear form

$$I(X,Y) = \int_a^b [\langle \nabla_{T^H} X^H, \nabla_{T^H} Y^H \rangle_T - \langle \Omega(T^H, X^H)T^H, Y^H \rangle_T] dt$$

for all $X, Y \in \mathfrak{X}[a,b]$.

After some computations one gets another formula for the Morse index form [1]:

$$I(X,Y) = \langle \nabla_{T^H} X^H, Y^H \rangle_T \Big|_a^b - \int_a^b \langle \nabla_{T^H} \nabla_{T^H} X^H + \Omega(T^H, X^H) T^H, Y^H \rangle_T dt.$$

Definition 4 ([1]). A *Jacobi field* along a geodesic $\sigma : [a,b] \to M$ is a vector field J which satisfies the Jacobi equation

$$\nabla_{T^H} \nabla_{T^H} J^H + \Omega(T^H, J^H) T^H \equiv 0$$

where $J^H(t) = \chi_{\dot\sigma(t)}(J(t))$.

$\dot\sigma$ and $t\dot\sigma$ are Jacobi fields; the first one never vanishes, the second one vanishes only at $t = 0$.

Two points $\sigma(t_0)$ and $\sigma(t_1)$, $t_0, t_1 \in [a,b]$ are said to be conjugate along σ if there exists a nonzero Jacobi field J along σ with $J(t_0) = 0$ and $J(t_1) = 0$.

3. Minimal submanifolds. Focal points

Let P be a submanifold of M of dimension $r < n$. We consider the set

$$A = \{(x,v) \mid x \in P,\ v \in T_x M \setminus \{0\}\} = \{\widetilde{x} \in \widetilde{M} \mid \pi(\widetilde{x}) \in P\}.$$

Let $H_{\widetilde{x}} T_x M$ and $H_{\widetilde{x}} T_x P$ be the horizontal lifts of $T_x M$ and $T_x P$ to $\widetilde{x}$ and

$$H_P TM = \bigcup_{\widetilde{x} \in A} H_{\widetilde{x}} T_x M$$

and

$$H_P TP = \bigcup_{\widetilde{x} \in A} H_{\widetilde{x}} T_x P.$$

For horizontal vector fields $X, Y \in H_P TP$ let X^*, Y^* be some prolongations of them to $H_P TM$. The restriction of $\nabla_{X^*} Y^*$ to $\widetilde{P} = TP \setminus 0$ does not depend of the choice of the prolongations.

Let $P_{\widetilde{x}}^{\perp}$ be the $\langle \cdot, \cdot \rangle_{\widetilde{x}}$ orthogonal complement of $H_{\widetilde{x}} TP$ in $H_{\widetilde{x}} TM$. By the orthogonal decomposition

$$H_{\widetilde{x}} T_x M = H_{\widetilde{x}} T_x P \oplus P_{\widetilde{x}}^{\perp}, \quad \widetilde{x} = (x,v) \in A$$

we obtain that

$$\nabla_{X^*} Y^* = \nabla^*_X Y + \mathbb{I}_v(X,Y).$$

We will call $\mathbb{I}_v(X,Y)$ the second fundamental form at X and Y in the direction of v. Note that for $\tilde{x} = (x,v)$ with $v \in T_x M \setminus T_x P$ we have

$$\langle \nabla_{X^*} Y^*, v^H \rangle_v = \mathbb{I}_v(X,Y). \tag{1}$$

Definition 5. Let $P \subset M$ be an r-dimensional submanifold of a Finsler manifold (M, F). The submanifold P is called minimal if for every tangent vector v to M and for any horizontal orthogonal vectors $V_i^H, i = \overline{1, r}$ (i.e., $\langle V_i^H, V_j^H \rangle_v = 0$ for $i \neq j$) we have $\sum_{i=1}^{r} \mathbb{I}_v(V_i^H, V_i^H) = 0$.

The condition of minimality is equivalent with the vanishing of the trace of the linear operator A_{v^H} defined by

$$\langle A_{v^H} X^H, Y^H \rangle_v = \langle \mathbb{I}_T(X^H, Y^H), v^H \rangle_v.$$

For details we refer to [9], [15].

Now let $\sigma : [a, b] \to M$ be a normal geodesic in M with $\sigma(a) \in P$ and $\dot{\sigma}^H(a)$ in the normal bundle of P (i.e., $\dot{\sigma}^H(a) \perp (H_{\dot{\sigma}(a)} T_{\sigma(a)} P)$).

Let $\widetilde{\mathfrak{X}}^P = \mathfrak{X}^P[a, b]$ be the vector space of all piecewise smooth vector fields X along σ such that $X^H(a) \in T_{\dot{\sigma}(a)} \widetilde{P}$ and let $\mathfrak{X}^P$ be the subspace of $\widetilde{\mathfrak{X}}^P$ consisting of these X such that X^H is orthogonal to $\dot{\sigma}^H$ along the curve σ.

We have

$$\begin{aligned}
\langle \nabla_{T^H} X^H, Y^H \rangle_T &= \langle \nabla_{X^H} T^H + [T^H, X^H] + \theta(T^H, X^H), Y^H \rangle_T \\
&= \langle \nabla_{X^H} T^H, Y^H \rangle_T, \qquad (2)
\end{aligned}$$

because $[T^H, X^H]$ and $\theta(T^H, X^h)$ are vertical vectors ([1]). If Y^H is orthogonal to T^H, then

$$0 = X^H \langle T^H, Y^H \rangle_T = \langle \nabla_{X^H} T^H, Y^H \rangle_T + \langle T^H, \nabla_{X^H} Y^H \rangle_T. \qquad (3)$$

By considering the vector fields X^H, Y^H such that $X^H(a), Y^H(a) \in T_{\dot{\sigma}(a)} \widetilde{P}$ and taking account of formulas (1), (2), (3), the Morse index form $I^P : \mathfrak{X}^P \times \mathfrak{X}^P \to \mathbb{R}$ becomes

$$\begin{aligned}
I^P(X, Y) = {} & \langle \nabla_{T^H} X^H, Y^H \rangle_T \big|^b + \langle \mathbb{I}_T(X^H, Y^H), T^H \rangle_T \big|_a \\
& - \int_a^b \langle \nabla_{T^H} \nabla_{T^H} X^H + \Omega(T^H, X^H) T^H, Y^H \rangle_T dt.
\end{aligned}$$

From [13] we know that I^P is symmetric.

Definition 6 ([13]). Let $P \subset M$ be an r-dimensional submanifold of a Finsler manifold (M, F). A P-Jacobi field J is a Jacobi field which satisfies in addition

$$J(a) \in T_{\sigma(a)} P$$

and

$$\langle \nabla_{T^H} J^H + A_{T^H} J^H, Y^H \rangle_T \big|_a = 0,$$

for all $Y \in (T_{\sigma(a)} P)^H$.

The last condition means in fact that

$$\nabla_{T^H} J^H + A_{T^H} J^H \in ((T_{\sigma(a)}P)^H)^{\perp}.$$

The dimension of the vector space of all P-Jacobi fields along σ is equal to the dimension of M and the dimension of the vector space of the P-Jacobi fields satisfying

$$\langle J^H, T^H \rangle = 0$$

is equal to $\dim M - 1$.

If P is a point, then a P-Jacobi field is a Jacobi field J along σ such that $J(a) = 0$.

A point $\sigma(t_0)$, $t_0 \in [a, b]$ is said to be a P-focal point along σ if there exists a non-null P-Jacobi field J along σ with $J(t_0) = 0$.

We shall use the following Lemma from [13].

Lemma 7. *Let (M, F) be a Finsler manifold and $\sigma : [a, b] \to M$ be a geodesic, and $P \subset M$ be a submanifold of M. Suppose that there are no P-focal points along σ. Let $X, J \in \widetilde{\mathfrak{X}}^P$ be vector fields orthogonal to σ with J a P-Jacobi field such that $X(b) = J(b)$. Then*

$$I^P(X, X) \geq I^P(J, J)$$

with equality if and only if $X = J$.

4. Main result

First, we introduce the k-Ricci curvature, following [16]. For a $(k+1)$-dimensional subspace $\mathcal{V} \in T_xM$ the Ricci curvature $\mathbf{Ric}_y\mathcal{V}$ on $\mathcal{V}$ is the trace of the Riemann curvature restricted to $\mathcal{V}$, with flagpole y, and is given by:

$$\mathbf{Ric}_y(\mathcal{V}) = \sum_{i=1}^{k} \langle R_y(b_i), b_i \rangle_y = \sum_{i=1}^{k} \langle \Omega(y, b_i)y, b_i \rangle_y,$$

where $R_y(b_i) \equiv \Omega(y, b_i)y$ and $y, (b_i)_{i=\overline{1,\dots,k}}$ is an arbitrary orthonormal basis for $(\mathcal{V}, \langle\,,\rangle_y)$, with $b_{k+1} = y$. $\mathbf{Ric}_y(\mathcal{V})$ is well-defined and is positively homogeneous of degree two on $\mathcal{V}$,

$$\mathbf{Ric}_{\lambda y}(\mathcal{V}) = \lambda^2 \mathbf{Ric}_y(\mathcal{V}), \quad \text{for } \lambda > 0,\ y \in \mathcal{V}.$$

It is clear from the definition that $\mathbf{Ric}_y(T_xM)$ is nothing but the Ricci curvature $\mathbf{Ric}(y)$ for $y \in T_xM$.

If $\mathcal{V} = P \subset T_xM$ is a tangent plane, the flag curvature is given by

$$K(P,y) = \frac{\langle R_y(u), u\rangle_y}{\langle y, y\rangle_y \langle u, u\rangle_y - \langle u, y\rangle_y^2},$$

where $u \in P\backslash\{0\}$, $\operatorname{span}(y,u) = P$. This is independent of the choice of $u \in P\backslash\{0\}$, and for u being g_y orthogonal to y and of g_y-norm 1 it becomes

$$\mathbf{K}(P,y) = \frac{\mathbf{Ric}_y P}{F^2(y)}, \quad y \in P.$$

Consider the following function on M:

$$\mathbf{Ric}_k(x) := \inf_{\dim(\mathcal{V})=k+1} \inf_{y\in\mathcal{V}} \frac{\mathbf{Ric}_y(\mathcal{V})}{F^2(y)},$$

the infimum being considered over all $(k+1)$-dimensional subspaces $\mathcal{V} \subset T_xM$ and $y \in \mathcal{V} \setminus \{0\}$. From the above definitions it can be seen that

$$\mathbf{Ric}_1 \leq \cdots \leq \frac{\mathbf{Ric}_k}{k} \leq \cdots \leq \frac{\mathbf{Ric}_{n-1}}{n-1},$$

and

$$\mathbf{Ric}_1 = \inf_{(P,y)} \mathbf{K}(P,y) \quad \text{and} \quad \mathbf{Ric}_{(n-1)} = \inf_{F(y)=1} \mathbf{Ric}(y).$$

We will say that the Finsler manifold (M,F) has positive k-Ricci curvature if and only if $\mathbf{Ric}_k > 0$.

Secondly, we recall a result from the theory of differential equations which will be essential in the proof of our main result.

Theorem 8 ([14])**.** *Consider the differential equation*

$$f''(t) + H(t)f(t) = 0, \quad t \in [0,\infty)$$

with continuous H. If

$$\int_0^\infty H(t)dt > 0,$$

then there exists a solution f satisfying the conditions $f(0) = 1$, $f'(0) = 0$, and there exists $t_0 > 0$ for which $f(t_0) = 0$.

Here $\int_0^\infty$ means $\liminf_{l\to\infty} \int_0^l$. The conditions satisfied by the solution f are similar to those met in the definition of focal points. A differential equation $f''(t) + H(t)f(t) = 0$ admitting such a solution f will be called *focal.* There are several other sufficient conditions for a differential equation $f''(t) + H(t)f(t) = 0$ be focal, [10], [11].

Now we state and prove our main result.

Theorem 9. *Let (M, F) be an n-dimensional Finsler manifold which satisfies the condition*

$$B^+(x, r) \cap B^-(x, r) \text{ is precompact for all } x \in M \text{ and } r > 0, \tag{4}$$

and let P be an r-dimensional compact and minimal submanifold of M. If the k-Ricci curvature satisfies the condition

$$\int_0^\infty \mathbf{Ric}_k(t) > 0 \quad (\textit{resp.} \int_{-\infty}^0 \mathbf{Ric}_k(t) > 0)$$

along any geodesic $\gamma : [0, \infty) \to M, t \to \gamma(t)$ emanating orthogonally from P (resp. $\gamma : (-\infty, 0] \to M, t \to \gamma(t)$ arriving orthogonally to P), then M is compact.

Proof. Suppose by contrary that M is not compact.

Then there exists a normal geodesic $\gamma(t)$ emanating from P and orthogonal to P free of focal points, i.e., there exists a sequence (p_i) such that the distance $d(p_i, P)$ (or $d(P, p_i)$) tends to infinity, since it is supposed that M is non-compact. Otherwise, because of the fact that P is compact, M would be contained in an intersection $B^+(x_1, r_1) \cap B^-(x_2, r_2)$ with $x_1, x_2 \in M$ and $r_1, r_2 > 0$ and (see [7], Proposition 2.2) should be compact.

By condition (4) on M and the compactness of P there exists for each p_i a normal geodesic γ_i which realizes the minimum distance $d(p_i, P)$ (or $d(P, p_i)$) since the Palais–Smale condition for the energy functional is satisfied (see [8] and [6]) so we can apply Morse theory for geodesics (see [7]). Suppose now that the minimum distances $d(p_i, P)$ are realized and $\gamma_i : [0, a_i) \to M, t \to \gamma_i(t)$ along any geodesic emanating orthogonally from P (the reverse case is the same via a change of variables in the integral). Denote by x_i the point in P which is joined with p_i by γ_i, $\gamma_i(0) = x_i \in P$, $\gamma_i(a_i) = p_i$. It is known that the geodesic γ_i intersects P orthogonally with respect to the inner product $\langle\ ,\ \rangle_{\gamma_i'(0)}$, that is $T_i = \gamma_i'(0)$ is orthogonal to P with respect to $\langle\ ,\ \rangle_{\gamma_i'(0)}$. By the compactness of P there exists an accumulation point $x \in P$ of the sequence x_i and also $T_i \to T$ with $T \perp P$ with respect to $\langle\ ,\ \rangle_T$ and $F(T) = 1$. It follows that the length of the geodesic γ with initial data (x, T) is equal to $d(x, \gamma(t))$, so $\gamma(t)$ is P-focal point free.

On the other hand from the conditions in the theorem we will show that γ has P-focal points. This contradiction shows that M has to be compact.

The index form along the geodesic γ with variation vector field V is

$$I^P(V, V) = \int_0^l [\langle \nabla_{T^H} V^H, \nabla_{T^H} V^H \rangle_T - \langle \Omega(T^H, V^H) V^H, T^H \rangle_T] dt$$

$$
\begin{aligned}
&= \langle \nabla_{T^H} V^H, V^H \rangle_T \big|^l + \langle \mathbb{I}_T(V^H, V^H), T^H \rangle_T \big|_0 \\
&\quad - \int_0^l \langle \nabla_{T^H} \nabla_{T^H} V^H + \Omega(T^H, V^H) T^H, V^H \rangle_T dt. \qquad (5)
\end{aligned}
$$

We are going to use the parallel transport with reference vector T. We construct a moving frame $V_i(t)$, $i = \overline{1, r}$ along γ such that

- $V_i(0)$ is an orthogonal basis in $T_{\gamma(0)}P$ and $\langle V_i^H(0), T^H(0) \rangle_{T(0)} = 0$,
- $V_i(t)$ are parallel along γ, i.e., $\nabla_{T^H} V_i^H = 0$.

It follows that the vectors $V_i^H(t)$ are orthogonal to each other and to $T^H(t)$ along γ with respect to the inner product $\langle\ ,\ \rangle_T(t)$.

We have, for $i = 1, \ldots, r$,

$$
\begin{aligned}
I^P(V_i, V_i) &= \langle \nabla_{T^H} V_i^H, V_i^H \rangle_T \big|^l + \langle \mathbb{I}_T(V_i^H, V_i^H), T^H \rangle_T \big|_0 \\
&\quad - \int_0^l \langle \nabla_{T^H} \nabla_{T^H} V_i^H + \Omega(T^H, V_i^H) T^H, V_i^H \rangle_T dt. \qquad (6)
\end{aligned}
$$

We sum up from $i = 1$ to r. Since P is minimal, we have

$$\sum_{i=1}^{k} \langle \mathbb{I}_T(V_i^H, V_i^H), T^H \rangle_T \big|_0 = 0$$

and one yields,

$$
\begin{aligned}
\sum_{i=1}^{r} I(V_i, V_i) &= \sum_{i=1}^{r} \langle \nabla_{T^H} V_i^H, V_i^H \rangle_T \big|^l \\
&\quad - \sum_{i=1}^{r} \int_0^l \langle \nabla_{T^H} \nabla_{T^H} V_i^H + \Omega(T^H, V_i^H) T^H, V_i^H \rangle_T dt.
\end{aligned}
$$

Let us take $X_i(t) = f(t) V_i(t)$ with $f : [0, \infty) \to \mathbb{R}$ satisfying $f(0) = 1$, $f'(0) = 0$. Then

$$X_i(0) = V_i(0), \quad X_i'(t) = f'(t) V_i(t), \quad X_i''(t) = f''(t) V_i.$$

It follows that

$$\sum_{i=1}^{r} I(X_i, X_i) = r f(t) f'(t) \big|^l - r \int_0^l (f''(t) + f(t) \frac{1}{r} \mathbf{Ric}_T(\mathcal{V}) f(t) dt, \qquad (7)$$

where $\mathcal{V}$ is the linear space spanned by $T, V_i, i = 1, \ldots, r$.

In our hypothesis on $\mathbf{Ric}_k$, setting $rH = \mathbf{Ric}_T(\mathcal{V})$ it comes out that the equation $f''(t) + f(t)H(t) = 0$ is focal. By Theorem 10, there exists $t_0 > 0$ such that $f(t_0) = 0$. We take $l = t_0$. In the r.h.s. of (7) the first term vanishes because of $f(t_0) = 0$ and the second is null since f is a solution of the focal equation $f''(t) + f(t)H(t) = 0$. Thus (7) reduces to $\sum_{i=1}^{r} I(X_i, X_i) = 0$. It follows that there exists X_i with $I(X_i, X_i) \leq 0$.

Then, Lemma 7 implies that there exists P-focal points on the geodesic γ, which contradicts the assumption that M is not compact. It follows that M has to be compact. □

The observations in the beginning of section 2 and the previous theorem leads to the following:

Theorem 10. *Let (M, F) be a forward (resp. backward) complete Finsler manifold and P an r-dimensional compact and minimal submanifold of M. If the k-Ricci curvature satisfies both conditions from Theorem 9, then M is compact.*

Remark. For Berwald manifolds our result follows directly from [10]. Moreover, for Berwald manifolds some results from [11] holds. Indeed, in the case of Berwald manifolds the connection of the Berwald metric lives on the tangent level (the reference vector is irrelevant). Szabó's structure theorem (see [17]) implies that there exists a non-unique Riemannian metric g on M such that the Berwald connection is the connection of the Riemannian metric. Taking into account that the flag curvature of a Berwald metric is equal to the sectional curvature of the Riemannian metric g and the second fundamental form of a submanifold with respect to the Berwald metric will be the analogue counterpart of the Riemannian metric, the results from [10], [11] apply.

Acknowledgements. The first author was supported by a grant of the Romanian National Authority for Scientific Research, CNSS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0256 as well as by the bilateral Romanian–Hungarian grant RO-HU 672/2013.

The work of the second author has been co-funded by the bilateral Romanian–Hungarian grant RO-HU 672/2013 .

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(Received April 2, 2013; revised February 6, 2014)