Publ. Math. Debrecen
84/1-2 (2014), 123–137
DOI: 10.5486/PMD.2014.5844

On contact CR-warped product submanifolds of a quasi-Sasakian manifold

By TRAN QUOC BINH (Debrecen) and AVIK DE (Kolkata)

Dedicated to Professor Lajos Tamássy on his 90th birthday

Abstract. In the present paper we study contact CR-warped product submanifolds of a quasi-Sasakian manifold. We obtain a necessary and sufficient condition for a contact CR-submanifold of a quasi-Sasakian manifold to be a contact CR-product or a contact CR-warped product submanifold. We estimate the squared norm of the second fundamental form in terms of the warping function. Equality cases are also investigated. As a particular case, we obtain some further results for Sasakian manifolds.

1. Introduction

The notion of warped product manifold was introduced by Bishop and O'Neill in 1969 [5] for studying manifolds of negative curvature. Given two Riemannian manifolds (M_1, g_1) and (M_2, g_2) and a positive function f on M_1, on the product manifold $M_1 \times M_2$ the metric tensor $g := g_1 + f^2 g_2$ is said to be a warped metric, and we call $(M_1 \times M_2, g)$ a warped product Riemannian manifold with warping function f. We also denote $(M_1 \times M_2, g)$ by $M_1 \times_f M_2$.

Bejancu introduced the notion of CR-submanifolds of a Kaehler manifold [3]. Let M be a submanifold of a complex manifold $\bar{M}$, and suppose TM denotes the tangent bundle, and $T^{\perp}M$ denotes the normal bundle of M. M is said to be

Mathematics Subject Classification: 53C40, 53C42, 53C15.
Key words and phrases: warped product, submanifold, quasi-Sasakian manifold, inequality, totally geodesic.
The work was supported by the Japanese–Hungarian Bilateral Project No. TÉT_10-1-2011-0062.

a CR-submanifold of $\bar{M}$ if and only if there exist two distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$, $J\mathcal{D} \subset TM$ and $J\mathcal{D}^{\perp} \subset T^{\perp}M$, where J is the complex structure of $\bar{M}$. Chen defined and studied the geometry of warped product CR-submanifolds in Kaehler manifolds ([14], [15], [16] [17]). Gaining inspiration from his results, many mathematicians extended their studies to different special cases of almost complex manifolds, as to nearly Kaehler manifolds ([25]), locally conformal Kaehler manifolds ([10], [26]), etc. We also mention here that the para-Kaehler version of CR-warped products in para-Kaehler manifolds (PR-warped products) was introduced and studied very recently by Chen and Munteanu in [19].

In contact geometry the concept of a contact CR-submanifold was introduced by Bejancu and Papaghiuc [4]. A submanifold M of a contact manifold $(\bar{M}, \phi, \xi, \eta)$ is said to be a contact CR-submanifold of type $(\mathcal{D} \oplus \langle\xi\rangle, \mathcal{D}^{\perp})$ if there exist distributions $\mathcal{D}$, $\mathcal{D}^{\perp}$ and $\langle\xi\rangle$, such that $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle\xi\rangle$, $\phi D \subset TM$ and $\phi\mathcal{D}^{\perp} \subset T^{\perp}M$. Later the studies of warped product CR-submanifolds in Kaehler manifolds were also extended to the case of contact geometry. Contact CR-warped product submanifolds of Sasakian manifolds were studied by Hasegawa and Mihai [22], Mihai [27] and Munteanu [28], etc. Contact CR-warped product submanifolds of Kenmotsu space forms were studied by Arslan, Ezentas, Mihai and Murathan [2], and recently Ozgur and Sular [29] studied contact CR-warped product submanifolds of a generalized Sasakian space form, and obtained many good results.

On the other hand, the notion of quasi-Sasakian structure was introduced by D. E. Blair [7] to unify Sasakian and cosymplectic structures. Also Tanno [30] obtained some results on quasi-Sasakian structures. A necessary and sufficient condition for an almost contact metric manifold to be quasi-Sasakian was given by Kanemaki [24]. Contact CR-submanifolds of quasi-Sasakian manifolds were studied intensively and successfully by Calin ([11], [12], [13]). Recently quasi-Sasakian manifolds became the subject of growing interest, and gained significant applications to physics, in particular, to super gravity and magnetic theory [1]. Quasi-Sasakian structures have a wide range of applications in the mathematical analysis of string theory [21]. Motivated by these applications, in the present paper we study contact CR-warped product submanifolds of quasi-Sasakian manifolds.

The paper is organized as follows:
After Preliminaries, in Section 3 we study warped product submanifolds of a quasi-Sasakian manifold. Among other results, we prove that under certain conditions a contact CR-submanifold of a quasi-Sasakian manifold reduces to a contact

CR-warped product. Finally, in Section 4, we establish an inequality between the squared norm of the second fundamental form and the warping function. As a corollary, we obtain some results for the Sasakian case.

2. Preliminaries

An n-dimensional manifold M^n is said to admit an almost contact structure ([6], [8], [31]) if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0. \tag{2.2}$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $M^n \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable, where X is tangent to M^n, t is the coordinate of $\mathbb{R}$, and f is a smooth function on $M^n \times \mathbb{R}$. Let g be the compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.3}$$

Then M^n becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.3) it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.4}$$

for any vector fields X, Y on the manifold. In an almost contact metric structure we define the fundamental 2-form by $\Phi(X, Y) := g(X, \phi Y)$. An almost contact metric structure becomes a contact metric structure if $\Phi(X, Y) = d\eta(X, Y)$, for all vector fields X, Y.

An almost contact metric structure is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η) is normal, and the fundamental 2-form Φ is closed, that is $d\Phi = 0$. This was first introduced by Blair [7]. Kanemaki proved ([24]) that a necessary and sufficient condition for an almost contact metric manifold (M, ϕ, ξ, η, g) to be quasi-Sasakian is that there exists a symmetric linear transformation field F, such that

$$(\nabla_X \phi)Y = \eta(Y)FX - g(FX, Y)\xi, \quad F\phi X = \phi FX,$$

for any vector fields X and Y of M with respect to the Riemannian connection ∇ of the metric g. It can be easily checked that for all vector fields X in a quasi-Sasakian manifold M

$$\nabla_X \xi = \phi F X, \quad F\xi = \eta(F\xi)\xi.$$

Let $i : (M, g) \longrightarrow (\bar{M}, g)$ be an isometric immersion. We denote by ∇ and $\bar{\nabla}$ the Levi–Civita connections of M and $\bar{M}$ respectively, and by $T^{\perp}M$ the normal bundle of M. Then for any vector fields $X, Y \in TM$ and normal vector field $N \in T^{\perp}M$ the second fundamental form h and the Weingarten map A_N are given by the Gauss and Weingarten formulas:

$$h(X, Y) = \bar{\nabla}_X Y - \nabla_X Y, \tag{2.5}$$

$$A_N X = \nabla^{\perp}_X N - \bar{\nabla}_X N, \tag{2.6}$$

where $\nabla^{\perp}$ denotes the normal connection of M. The second fundamental form h and A_N are related by $g(h(X, Y), N) = g(A_N X, Y)$. We say that M is totally umbilical if $h(X, Y) = g(X, Y)H$, where H is the mean curvature defined by $H = \sum_{i=1}^{n} h(e_i, e_i)$ for some basis $\{e_1, e_2, \ldots, e_n\}$ of TM. M is said to be totally geodesic if $h(X, Y) = 0$, and minimal if $H = 0$.

Now let $M = M_1 \times_f M_2$ be a submanifold of $\bar{M}$. We say that M is a CR-warped product submanifold of $\bar{M}$ if and only if either M_1 is invariant and M_2 is anti-invariant, or M_2 is invariant and M_1 is anti-invariant.

3. Warped product submanifolds

In this section we investigate warped products $M = M_1 \times_f M_2$, which are contact CR-submanifolds of a quasi-Sasakian manifold $\bar{M}$. By definition such submanifolds are always tangent to ξ. Similarly to Hasegawa and Mihai, here we also distinguish only two cases:

(a) ξ is tangent to M_1;

(b) ξ is tangent to M_2.

For a warped product Riemannian manifold $M_1 \times_f M_2 = (M_1 \times M_2, g = g_1 + f^2 g_2)$ we recall the following well-known identity [5]:

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z, \tag{3.1}$$

for any $X \in TM_1$, $Z \in TM_2$.

Lemma 3.1. *If $M = M_1 \times_f M_2$ is a warped product submanifold of a quasi-Sasakian manifold $\bar{M}$ such that ξ is tangent to M_2, then it becomes a Riemannian product submanifold.*

Proof. Supppose $\xi \in TM_2$. In (3.1), putting $Z = \xi$, we obtain

$$\nabla_X \xi = (X \ln f)\xi. \tag{3.2}$$

Now,

$$\phi FX = \bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi) = (X \ln f)\xi + h(X, \xi). \tag{3.3}$$

Since, $g(\phi FX, \xi) = -g(FX, \phi\xi) = 0$, and $g(h(X, \xi), \xi) = 0$, from (3.3) we obtain $X \ln f = 0$, $\forall X \in TM_1$.

Hence, f is constant, and the warped product is nothing, but simply a Riemannian product. □

So, for studying a proper contact CR-warped product submanifold we only need to consider the case a) where ξ is tangent to M_1. We have two subcases:

(i) M_1 is invariant, ξ is tangent to M_1, and M_2 is anti-invariant,

(ii) M_1 is anti-invariant, ξ is tangent to M_1, and M_2 is invariant.

For the case (ii) we have the following theorem in a more general setting:

Theorem 3.1. *If $M = M_1 \times_f M_2$ is a warped product contact CR-submanifold of a quasi-Sasakian manifold $\bar{M}$, such that $\xi \in TM_1$, and M_2 is invariant, then f is constant, that is, M is a CR-product.*

Proof. For $X \in TM_2$, $Z \in TM_1$ we have,

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X. \tag{3.4}$$

Putting $Z = \xi$ in (3.4), we obtain

$$\nabla_X \xi = (\xi \ln f)X. \tag{3.5}$$

Since $\bar{M}$ is quasi-Sasakian, we have

$$\phi FX = \bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi) = (\xi \ln f)X + h(X, \xi), \quad \text{[from (3.5)]}, \tag{3.6}$$

which implies

$$g(\phi FX, X) = (\xi \ln f)g(X, X), \quad \text{for all } X \in TM_2. \tag{3.7}$$

But $g(\phi FX, X) = 0$, since F is symmetric and $F\phi = \phi F$, which, together with (3.7), gives

$$\xi \ln f = 0. \tag{3.8}$$

Let h^T be the second fundamental form of M_2 in M. Then for $X, Y \in TM_2$ and $Z \in TM_1$, we have

$$\begin{aligned} g(h^T(X,Y),Z) &= g(\nabla_X Y, Z) = -g(Y, \nabla_X Z) \\ &= -g(Y,(Z \ln f)X) = -(Z \ln f) g(X,Y). \end{aligned} \tag{3.9}$$

Let $\hat{h}$ be the second fundamental form of the immersion of M_2 in $\bar{M}$, and let ∇^T be the Levi–Civita connection in M_2 induced from $\bar{\nabla}$. Then,

$$\hat{h}(X,Y) = h(X,Y) + h^T(X,Y). \tag{3.10}$$

So,

$$g(\hat{h}(X,Y),Z) = g(h^T(X,Y),Z) = -(Z \ln f) g(X,Y). \tag{3.11}$$

Since M_2 is an invariant submanifold of $\bar{M}$, we have

$$\bar{\nabla}_X \phi Y = \nabla^T_X \phi Y + \hat{h}(X, \phi Y). \tag{3.12}$$

Hence,

$$\begin{aligned} \nabla^T_X \phi Y + \hat{h}(X,\phi Y) &= (\bar{\nabla}_X \phi) Y + \phi \bar{\nabla}_X Y \\ &= \eta(Y) FX - g(FX,Y)\xi + \phi(\nabla^T_X Y) + \phi \hat{h}(X,Y) \\ &= -g(FX,Y)\xi + \phi(\nabla^T_X Y) + \phi \hat{h}(X,Y). \end{aligned} \tag{3.13}$$

Since M_2 is invariant, from (3.13) we obtain,

$$\hat{h}(X,\phi Y) = \phi \hat{h}(X,Y) - g(FX,Y)\xi. \tag{3.14}$$

Now, for $Z \perp \langle \xi \rangle$ we have from (3.11)

$$\begin{aligned} -(Z \ln f) g(\phi X, \phi X) &= g(\hat{h}(\phi X, \phi X), Z) = g(\phi \hat{h}(\phi X, X) - g(F\phi X, X)\xi, Z) \\ &= g(\phi \hat{h}(\phi X, X), Z) = g(\phi(\phi \hat{h}(X,X) - g(FX,X)\xi), Z) \\ &= g(\phi^2 \hat{h}(X,X), Z) = g(-\hat{h}(X,X), Z) = (Z \ln f) g(X,X), \end{aligned}$$

which implies

$$(Z \ln f) g(X,X) = 0, \quad \text{for all } X \in TM_2. \tag{3.15}$$

Hence,

$$Z \ln f = 0, \quad \text{for any vector field } Z \perp \langle \xi \rangle \text{ in } TM_1. \tag{3.16}$$

This, together with (3.8), gives us

$$V \ln f = 0, \quad \text{for all } V \in TM_1. \tag{3.17}$$

Hence f is constant. □

If TM is invariant under F, then from the above theorem we obtain

$$\phi FX = F\phi X \in TM \tag{3.18}$$

for all $X \in TM_2$. Now,

$$\phi FX = \bar{\nabla}_X \xi = \nabla_X \xi + h(X,\xi) = (\xi \ln f)X + h(X,\xi) = h(X,\xi), \tag{3.19}$$

which implies

$$\phi FX = 0 = h(X,\xi), \quad \text{for all } X \in TM_2. \tag{3.20}$$

So, for all $X \in TM_2$ we get

$$\begin{aligned} 0 = \phi^2 FX &= -FX + g(FX,\xi)\xi = -FX + g(X,F\xi)\xi \\ &= -FX + g(X,\eta(F\xi)\xi)\xi = -FX. \end{aligned} \tag{3.21}$$

So, if $F|_{TM_2}$ is injective, then we obtain:

Corollary 3.1. *There does not exist any warped product submanifold $M = M_1 \times_f M_2$ of a quasi-Sasakian manifold $\bar{M}$, such that $\xi \in TM_1$, and M_2 is an invariant submanifold, provided, TM is invariant under F, and $F|_{TM_2}$ is injective.*

In the Sasakian case $F = Id$, and then from Corollary 3.1 we obtain the theorem of Hasegawa and Mihai [22]:

Theorem 3.2. *Let $\bar{M}$ be a $2m+1$-dimensional Sasakian manifold. Then there do not exist warped product submanifolds $M_1 \times_f M_2$ such that M_1 is an anti-invariant submanifold tangent to ξ, and M_2 is an invariant submanifold of $\bar{M}$.*

The warped product submanifolds $M_1 \times_f M_2$ are called of type $(\mathcal{D}^\perp \oplus \langle\xi\rangle, \mathcal{D})$ if M_1 is an anti-invariant submanifold tangent to ξ, and M_2 is an invariant submanifold of $\bar{M}$. From Theorem 3.1 we also have that a contact $(\mathcal{D}^\perp \oplus \langle\xi\rangle, \mathcal{D})$ CR-warped product must be a CR-product. But, when is a contact CR-submanifold, even locally, a contact CR-product of type $(\mathcal{D}^\perp \oplus \langle\xi\rangle, \mathcal{D})$? From the Theorems 1.1 and 1.4 in [13] of Calin, and from the well-known de Rham's decomposition theorem we obtain the following theorem:

Theorem 3.3. *A contact CR-submanifold M of a quasi-Sasakian manifold $\bar{M}$ is locally a contact CR-product of type $(\mathcal{D}^\perp \oplus \langle\xi\rangle, \mathcal{D})$ if and only if*

$$A_{\phi Z} X = 0,$$

for all $X \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$, and

$$F\mathcal{D} \perp \mathcal{D}.$$

Proof. If a contact CR-submanifold M of a quasi-Sasakian manifold $\bar{M}$ is locally a contact $(\mathcal{D}^\perp \oplus \langle\xi\rangle, \mathcal{D})$ CR-product, then the distributions $\mathcal{D}^\perp \oplus \langle\xi\rangle$ and $\mathcal{D}$ are integrable, and their leaves are totally geodesic in M. From Theorems 1.1

and 1.4 of [13] we have $h(X,U) \in \nu$, and $F\mathcal{D} \perp \mathcal{D}$ for all $X \in \mathcal{D}$, $U \in TM$, where ν is the orthogonal complementer of $\phi(\mathcal{D})$ in $T^\perp M$. From this it follows that $A_{\phi Z}X = 0$ and $F\mathcal{D} \perp \mathcal{D}$.

Conversely, if M is a contact CR-submanifold of a quasi-Sasakian manifold $\bar{M}$, and $A_{\phi Z}X = 0$, for all $X \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$, and $F\mathcal{D} \perp \mathcal{D}$, then from $A_{\phi Z}X = 0$ we have $g(h(X,Y), \phi Z) = 0$ for any $X, Y \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$. Therefore $h(X,Y) \in \nu$ for all $X, Y \in \mathcal{D}$. From Theorem 1.1 in [13] it follows that the distribution $\mathcal{D}$ is integrable, and its leaves are totally geodesic. On the other hand, from $A_{\phi Z}X = 0$ we also have $h(X,V) \in \nu$ for all $X \in \mathcal{D}, V \in \mathcal{D}^\perp \oplus \langle \xi \rangle$. From Theorem 1.4 in [13] we have that the distribution $\mathcal{D}^\perp \oplus \langle \xi \rangle$ is integrable, and its leaves are totally geodesic. Thus, M is locally a contact $(\mathcal{D}^\perp \oplus \langle \xi \rangle, \mathcal{D})$ CR-product according to the de Rham's decomposition theorem. □

In the case of a Sasakian manifold $F = Id$. Then the condition $F\mathcal{D} \perp \mathcal{D}$ is never satisfied. From the above theorem we have the weaker form of Theorem 3.2.

Theorem 3.4. *In a Sasakian manifold, there exists no contact CR-product submanifold of type* $(\mathcal{D}^\perp \oplus \langle \xi \rangle, \mathcal{D})$

Now we consider the case of a contact CR-submanifold of type $(\mathcal{D} \oplus \langle \xi \rangle, \mathcal{D}^\perp)$ in a quasi-Sasakian manifold.

Theorem 3.5. *Let M be a contact CR-submanifold of type $(\mathcal{D} \oplus \langle \xi \rangle, \mathcal{D}^\perp)$ in a quasi-Sasakian manifold $\bar{M}$. Then M is locally a contact CR-warped product of type $(\mathcal{D} \oplus \langle \xi \rangle, \mathcal{D}^\perp)$ if and only if*

$$F\mathcal{D}^\perp \perp \phi\mathcal{D}^\perp$$

and

$$A_{\phi Z}X = (\phi X \mu)Z, \quad \text{for } X \in \mathcal{D}, Z \in \mathcal{D}^\perp,$$

for some C^∞ function μ on M satisfying $W\mu = 0$, for all $W \in \mathcal{D}^\perp \oplus \langle \xi \rangle$.

Proof. Suppose M is a contact $(\mathcal{D} \oplus \langle \xi \rangle, \mathcal{D}^\perp)$ CR-warped product of the form $M = N_\top \times_f N_\perp$. Then $\mathcal{D} \oplus \langle \xi \rangle$ is integrable, and its leaves are totally geodesic in M. Thus we have from Theorem 1.2 of [13] that

$$g(h(X,Y), \phi Z) = 0, \quad \text{for all } X \in \mathcal{D},\ Y \in \mathcal{D} \oplus \langle \xi \rangle,\ Z \in \mathcal{D}^\perp,$$

which implies

$$g(A_{\phi Z}X, Y) = 0. \tag{3.22}$$

Let $X = \phi Y$, $X, Y \in \mathcal{D}$. Then for all $V \in \mathcal{D}^\perp$ we have,

$$g(A_{\phi Z}X, V) = g(h(X,V), \phi Z) = g(h(\phi Y, V), \phi Z) = g(\bar{\nabla}_V \phi Y, \phi Z)$$

$$= g((\bar{\nabla}_V\phi)Y + \phi(\bar{\nabla}_V Y), \phi Z) = g(\phi(\bar{\nabla}_V Y), \phi Z) = g(\bar{\nabla}_V Y, Z)$$
$$= g(\nabla_V Y, Z) = g((Y \ln f)V, Z) = -(\phi X \ln f)g(Z, V). \qquad (3.23)$$

Here we have used $g((\bar{\nabla}_V\phi)Y, \phi Z) = g(\eta(Y)FX - g(FX, Y)\xi, \phi Z) = 0$ and $g(h(V, Y), Z) = 0$.

From (3.22) and (3.23) we get

$$A_{\phi Z}X = -(\phi X \ln f)Z, \quad \text{for all } X \in \mathcal{D},\ Z \in \mathcal{D}^{\perp}.$$

On the other hand $\mathcal{D}^{\perp}$ is also integrable, so from Theorem 1.1 of [12] we obtain

$$F\mathcal{D}^{\perp} \perp \phi\mathcal{D}^{\perp}.$$

Let $\mu = -\ln f$. Then, $W\mu = -\frac{Wf}{f} = 0$, for all $W \in \mathcal{D}^{\perp} \oplus \langle\xi\rangle$ since f is a function on $N_{\top}$, and from the proof of Theorem 3.1 it is easy to see that $\xi(\ln f) = 0$.

Conversely, let M be a contact $(\mathcal{D} \oplus \langle\xi\rangle, \mathcal{D}^{\perp})$ CR-submanifold of a quasi-Sasakian manifold $\bar{M}$, such that

$$F\mathcal{D}^{\perp} \perp \phi\mathcal{D}^{\perp}$$

and

$$A_{\phi Z}X = (\phi X\mu)Z, \quad \text{for } X \in \mathcal{D},\ Z \in \mathcal{D}^{\perp},$$

for some C^{∞} function μ on M satisfying $W\mu = 0$ for all $W \in \mathcal{D}^{\perp} \oplus \langle\xi\rangle$.

We have to prove that

$$M = N_{\top} \times_f N_{\perp},\ \xi \in \mathcal{X}(N_{\top}).$$

From $A_{\phi Z}X = \phi X(\mu)Z$, for $X \in \mathcal{D}$, $Z \in \mathcal{D}^{\perp}$ we have $g(A_{\phi Z}X, Y) = 0$ for all $Y \in D \oplus \langle\xi\rangle$, which implies

$$g(h(X, Z), \phi Z) = 0. \qquad (3.24)$$

Hence

$$h(X, Y) \in \nu. \qquad (3.25)$$

Thus from Theorem 2.2 of [12] we have that $\mathcal{D} \oplus \langle\xi\rangle$ is integrable, and its leaf are totally geodesic in M. From $F\mathcal{D}^{\perp} \perp \phi\mathcal{D}^{\perp}$ and from Theorem 1.1 of [12] we have that $\mathcal{D}^{\perp}$ is integrable. Now, let $N_{\top}$ be a leaf of $\mathcal{D} \oplus \langle\xi\rangle$, and $N_{\perp}$ be a leave of $\mathcal{D}^{\perp}$. Then $N_{\top}$ is a totally geodesic submanifold of M.

We now prove that $N_{\perp}$ is an extrinsic sphere in M, that is, $N_{\perp}$ is a totally umbilical submanifold of M, and its mean curvature is parallel according to the normal connection of $N_{\perp}$. Let $h^{\perp}$ and $A^{\perp}$ be the second fundamental form and the shape operator of the submanifold $N_{\perp}$ in M. First we prove the following:

Lemma 3.2.

$$g(\phi A_{\phi Z}U, X) = g(\nabla_U Z, X), \quad \textit{for all } X \in \mathcal{D},\ Z \in \mathcal{D}^{\perp}, U \in TM.$$

Proof. For any $Y \in \mathcal{D}$ and $U \in TM$ we have,

$$\begin{aligned} g(\nabla_U Z, \phi Y) &= g(\bar{\nabla}_U Z, \phi Y) = -g(Z, \bar{\nabla}_U \phi Y) = -g(Z, (\bar{\nabla}_U \phi)Y + \phi(\bar{\nabla}_U Y)) \\ &= -g(Z, \phi(\bar{\nabla}_U Y)) = g(\phi Z, \bar{\nabla}_U Y) \\ &= -g(Y, \bar{\nabla}_U \phi Z) = g(Y, A_{\phi Z}U). \end{aligned} \tag{3.26}$$

Putting $X = \phi Y$ in the above equation we get

$$-g(\nabla_U Z, X) = g(A_{\phi Z}U, \phi X) = -g(\phi A_{\phi Z}U, X).$$

This yields the lemma. □

Using this lemma for any $V, Z \in TN_{\perp}$ where X is a normal vector field on $N_{\perp}$, we obtain:

$$\begin{aligned} g(\nabla_Z X, V) &= -g(X, \nabla_Z V) = -g(\phi A_{\phi V} Z, X) = g(A_{\phi V} Z, \phi X) \\ &= g(A_{\phi V}\phi X, Z) = -g((X\mu)V, Z) - (X\mu)g(V, Z). \end{aligned} \tag{3.27}$$

But

$$g(\nabla_Z X, V) = g(A^{\perp}_X Z, V) = g(h^{\perp}(Z, V), X). \tag{3.28}$$

$W(\mu) = 0$ for all $W \in \mathcal{D}^{\perp} \oplus \langle\xi\rangle$ implies that $\nabla\mu \in \mathcal{D}$. From this and from (3.28) it follows that

$$g(h^{\perp}(Z, V), X) = -(X\mu)g(Z, V) = -g(X, \nabla\mu)g(Z, V). \tag{3.29}$$

On the other hand, from $\phi F = F\phi$ and from $\phi FZ = \nabla_Z \xi$ it follows that

$$\begin{aligned} g(h^{\perp}(Z, V), \xi) &= g(\nabla_Z V, \xi) = -g(V, \nabla_Z \xi) = -g(V, \bar{\nabla}_Z \xi) \\ &= -g(V, \phi FZ) = -g(V, F\phi Z) = -g(FV, \phi V). \end{aligned} \tag{3.30}$$

From the conditions $F\mathcal{D}^{\perp} \perp \phi\mathcal{D}^{\perp}$ and $g(\xi, \nabla\mu) = \xi(\mu) = 0$, we have

$$g(h^{\perp}(Z, V), \xi) = 0 = -g(\xi, \nabla\mu)g(Z, V).$$

From this and from (3.30) we have

$$h^{\perp}(Z, V) = -g(Z, V)\nabla\mu. \tag{3.31}$$

This means that $N_{\perp}$ is totally umbilical with mean curvature vector $\nabla\mu$.

Now we prove that $\nabla\mu$ is parallel according to the normal connection of $N_{\perp}$ in M. Since the leaves of $\mathcal{D} \oplus \langle\xi\rangle$ are totally geodesic, and $\mathcal{D}^{\perp}$ is integrable, from Theorem A of Blumenthal and Hebda [9] we know that M is locally diffeomorphic to a product $N_\top \times N_\perp$, where $N_\top$ is a leaf of $\mathcal{D} \oplus \langle\xi\rangle$, and $N_\perp$ is a leaf of $\mathcal{D}^\perp$. So we can introduce a local coordinate system $\{x^i, z^\alpha\}$ on M, such that $\{\frac{\partial}{\partial x^i}\}$ and $\{\frac{\partial}{\partial z^\alpha}\}$ are bases of $\mathcal{D} \oplus \langle\xi\rangle$ and $\mathcal{D}^\perp$ respectively.

Thus, for any $X \in \mathcal{D} \oplus \langle\xi\rangle$ and $Z \in \mathcal{D}^\perp$, we have, $[X, Z] = 0$, which implies

$$\nabla_X Z = \nabla_Z X. \tag{3.32}$$

Let $\nabla^\perp$ be the normal connection of $N_\perp$ in M. Then, for $Y \in \mathcal{D} \oplus \langle\xi\rangle$ and $Z \in N_\perp$ we obtain,

$$\begin{aligned} g(\nabla^\perp_Z \nabla\mu, Y) &= g(\nabla_Z \nabla\mu, Y) = Zg(\nabla\mu, Y) - g(\nabla\mu, \nabla_Z Y) \\ &= Z(Y(\mu)) - g(\nabla\mu, \nabla_Y Z) \\ &= Y(Z(\mu)) - \{Yg(\nabla\mu, Z) - g(\nabla_Y \nabla\mu, Z)\} = 0, \end{aligned} \tag{3.33}$$

since $Z(\mu) = 0$ for all $Z \in \mathcal{D}^\perp$ and $\nabla_Y \nabla\mu \in \mathcal{D} \oplus \langle\xi\rangle$. This means that the mean curvature of $N_\perp$ is parallel. So, we have proved that the leaves of $\mathcal{D} \oplus \langle\xi\rangle$ are totally geodesic, implying that $\mathcal{D} \oplus \langle\xi\rangle$ is autoparallel. Also the leaves of $\mathcal{D}^\perp$ are totally umbilical, and their mean curvatures are parallel, consequently they are extrinsic spheres. Therefore, by using the result of [23] (see also [20], Remark 2.1 and [14], [25]), M is a warped product of type $M = N_\top \times_f N_\perp$, ξ is tangent to $N_\perp$, for some function f on $N_\top$.

From the first part of the proof we can easily see that,

$$\nabla \ln f = -\nabla\mu,$$

from which we obtain $f = ce^{-\mu}$ for some constant c. □

In the Sasakian case we have $F = Id$. Thus the condition $F\mathcal{D}^\perp = \mathcal{D}\perp\phi\mathcal{D}^\perp$ is always satisfied. From Theorem 3.4 we have:

Corollary 3.2. *A contact* $(\mathcal{D} \oplus \langle\xi\rangle, \mathcal{D}^\perp)$ *CR-submanifold* M *of a Sasakian manifold* $\bar{M}$ *is locally a contact CR-warped product if and only if*

$$A_{\phi Z} X = (\phi X\mu) Z, \quad \text{for } X \in \mathcal{D}, Z \in \mathcal{D}^\perp,$$

for some C^∞ *function* μ *on* M *satisfying* $W\mu = 0$ *for all* $W \in \mathcal{D}^\perp \oplus \langle\xi\rangle$.

4. Inequality between the warping function and the squared norm of the second fundamental form

Theorem 4.1. *Let $M = N_\top \times_f N_\perp$ be a contact CR-warped product submanifold of a quasi-Sasakian manifold $\bar{M}$, such that $N_\top$ is an invariant submanifold tangent to ξ, and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$. Suppose that dimension $N_\top = 2n+1$, dimension $N_\perp = \beta$. Then,*

(i) $\|h\|^2 \geq 2\beta\|\nabla \ln f\|^2 + 2Tr_\perp F^2$, *where $Tr_\perp F^2 := \sum g(Fe_\alpha, e_\alpha)$, which is independent of the choice of the orthonormal basis e_α ($\alpha = 2n+2, \dots, 2n+1+\beta$) on $N_\perp$.*

(ii) *If the equality holds, then $N_\top$ is totally geodesic in $\bar{M}$, $N_\perp$ is totally umbilical in $\bar{M}$, and M is a minimal submanifold of $\bar{M}$.*

Proof. Let $X \in TN_\top$, $Z \in TN_\perp$ be two unit vector fields. We have

$$\begin{aligned} g(h(\phi X, Z), \phi Z) &= g(\bar{\nabla}_Z \phi X, \phi Z) \\ &= g(\eta(X)FZ - g(FZ, X)\xi + \phi\bar{\nabla}_Z X, \phi Z) \\ &= g(\eta(X)FZ + \phi\bar{\nabla}_Z X, \phi Z) + g(FZ, X)g(\phi\xi, Z). \end{aligned} \tag{4.1}$$

Since $g(FZ, \phi Z) = -g(\phi FZ, Z) = -g(F\phi Z, Z) = -g(\phi Z, FZ)$ implies $g(FZ, \phi Z) = 0$, from (4.1) we obtain:

$$\begin{aligned} g(h(\phi X, Z), \phi Z) &= g(\phi\bar{\nabla}_Z X, \phi Z) = g(\bar{\nabla}_Z X, Z) \\ &= g(\nabla_Z X, Z) = X \ln f g(Z, Z) = X \ln f. \end{aligned} \tag{4.2}$$

Since $TN_\perp$ is anti-invariant

$$\begin{aligned} g(h(Z, \xi), \phi Z) &= g(\bar{\nabla}_Z \xi - \nabla_Z \xi, \phi Z) = g(\bar{\nabla}_Z \xi, \phi Z) \\ &= g(\phi FZ, \phi Z) = g(FZ, Z). \end{aligned} \tag{4.3}$$

Suppose that h^* is the second fundamental form from $N_\perp$ to M. Then we have

$$\begin{aligned} g(h^*(Z, W), X) &= g(\nabla_Z W, X) = -g(W, \nabla_Z X) \\ &= -X \ln f g(Z, W) = -g(g(Z, W)\nabla \ln f, X). \end{aligned} \tag{4.4}$$

Hence,

$$h^*(Z, W) = g(Z, W)\nabla \ln f. \tag{4.5}$$

Let $e_1, e_2, \dots, e_{2n}, e_{2n+1} = \xi$ be an orthonormal basis of $Tn_\top$, while $\{e_\alpha,\ \alpha = 2n+2, \dots, 2n+1+\beta\}$ is a basis of $TN_\perp$. Then $\{\phi e_1, \phi e_2, \dots, \phi e_{2n}, e_{2n+1} = \xi\}$ is also an orthonormal basis of $TN_\top$. Let $E_a, a = 1, \dots, 2n+1+\beta$be a basis of

TM such that $E_i = \phi e_i, i = 1, \ldots, 2n, E_{2n+1=\xi}$ and $E_\alpha = e_\alpha$. Then we obtain

$$\begin{aligned}\|h\|^2 &= \sum \|h(E_a, E_b)\|^2 \geq \sum 2\|h(\phi e_i, e_\alpha)\|^2 + 2\|h(\xi, e_\alpha)\|^2) \\ &\geq 2\sum\{\|g(h(\phi e_i, e_\alpha), \phi e_\alpha)\|^2 + \|g(h(\xi, e_\alpha), \phi e_\alpha)\|^2\}) \\ &\geq 2\beta\|\nabla \ln f\|^2 + 2\sum g(Fe_\alpha, e_\alpha) \geq 2\beta\|\nabla \ln f\|^2 + 2Tr_\perp F^2, \end{aligned} \tag{4.6}$$

where we have used $\xi \ln f = 0$.

If the equality holds, then

$$h(TN_\top, TN_\top) = 0, \tag{4.7}$$

$$h(TN_\perp, TN_\perp) = 0, \tag{4.8}$$

and

$$h(TN_\top, TN_\perp) \subset \phi TN_\perp. \tag{4.9}$$

Since $N_\top$ is always totally geodesic in M, from (4.7) we can conclude that $N_\top$ is also totally geodesic in $\bar{M}$. From (4.5) we have that $N_\perp$ is totally umbilical in M. Combining this with (4.8), we conclude that $N_\perp$ is totally umbilical in $\bar{M}$. From (4.6) and (4.7) we also obtain that M is a minimal submanifold of $\bar{M}$. □

In the case of Sasakian manifolds $F = Id$. Then $Tr_\perp F^2 = \beta$, and we obtain the result of Hasegawa and Mihai [22].

Corollary 4.1. *Let $M = N_\top \times_f N_\perp$ be a contact CR-warped product submanifold of a Sasakian manifold $\bar{M}$, such that $N_\top$ is an invariant submanifold tangent to ξ, and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$. Suppose that dimension $N_\top = 2n + 1$, dimension $N_\perp = \beta$. Then*

(i) $\|h\|^2(\geq 2\beta\|\nabla \ln f\|^2 + 1)$,

(ii) *If the equality holds, then $N_\top$ is totally geodesic in $\bar{M}$, $N_\perp$ is totally umbilical in $\bar{M}$, and M is a minimal submanifold of $\bar{M}$.*

References

[1] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull and B. Spence, Branes at conical singularities and holography, *Adv. Theor. Math. Phys.* **2** (1998), 1249–1286.

[2] K. Arslan, R. Ezentas , I. Mihai and C. Murathan, Contact CR-warped product submanifolds in Kenmotsu space forms, *J. Korean Math. Soc.* **42** (2005), 1101–1110.

[3] A. Bejancu, CR submanifolds of a Kaehler manifold, *Proc. Amer. Math. Soc.* **69** (1978), 135–142.

[4] A. Bejancu and N. Papaghiuc, Semi-invariant submanifolds of a Sasakian manifold, *Ann. St. Univ. "Al. I. Cuza" Iasi* **27** (1981), 163–170.

[5] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.* (1969), 1–49.

[6] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Math., Vol. 203, *Berlin*.

[7] D. E. Blair, The theory of quasi-Sasakian structure, *J. Differential Geom.* **1** (1967), 331–345.

[8] D. E. Blair, Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics, 509, *Springer-Verlag, Berlin*, 2003.

[9] R. A. Blumenthal and J. J. Hebda, De Rham decomposition theorems for foliated manifolds, *Annales de l'institut Fourier* **33** (1983), 183–198.

[10] V. Bonanzinga and K. Matsumoto, Warped product CR-submanifolds in locally conformal Kaehler manifolds, *Period. Math. Hungar.* **48** (2004), 207–221.

[11] C. Calin, Contact CR-submanifolds of quasi-Sasakian manifolds, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **36** (1992), 217–226.

[12] C. Calin, Geometry of leaves on a CR-submanifold of a quasi-Sasakian manifold, *Bul. Inst. Politeh. Iasi. Sec. I. Mat. Mec. Teor. Fiz.* **39** (1993), 37–43.

[13] C. Calin, CR-product of quasi-Sasakian manifold, *Bul. Inst. Politeh. Iasi. Sec. I. Mat. Mec. Teor. Fiz.* **40** (1994), 37–44.

[14] B. Y. Chen, Geometry of Submanifolds, Pure and Applied Mathematics, No. 22, *Marcel Dekker, Inc. New York*, 1973.

[15] B. Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds, *Monatsh. Math.* **133** (2001), 177–195.

[16] B. Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds. II, *Monatsh. Math.* **134** (2001), 103–119.

[17] B. Y. Chen, On warped product immersions, *J. Geom.* **82** (2005), 36–49.

[18] B. Y. Chen, Pseudo-Riemannian Geometry, δ-Invariants and Applications, *World Scientific*, 2011.

[19] B. Y. Chen and M. I. Munteanu, Geometry of PR-Warped products in Para–Kahler manifolds, *Taiwanese J. Math.* **16** (2012), 1293–1327.

[20] F. Dillen and S. Nölker, Semi-parallelity, muti-rotation surfaces and the helix-property, *J. Reine Angew. Math.* **435** (1993), 33–63.

[21] T. Friedrich and S. Ivanov, Almost contact manifolds, connections with torsion and parallel spinors, *J. Reine Angew. Math.* **559** (2003), 217–236.

[22] I. Hasegawa and I. Mihai, Contact CR-warped product submanifolds in Sasakian manifolds, *Geom. Dedicata* **102** (2003), 143–150.

[23] S. Hiepko, Eine inner Kennzeichung der verzerrten Produkte, *Math. Ann.* **241** (1979), 209–215.

[24] S. Kanemaki, Quasi-Sasakian manifolds, *Tohoku Math. J.* **29** (1977), 227–233.

[25] V. A. Khan, K. A. Khan and Siraj Uddin, CR-warped product submanifolds in a Kaehler manifold, *South. Asian Bull. Math.* **33** (2009), 865–874.

[26] V. A. Khan and N. Jamal, CR-warped product submanifolds of locally conformal Kaehler manifolds, *Toyama Math. J.* **33** (2010), 1–19.

[27] I. Mihai, Contact CR-warped product submanifolds in Sasakian space forms, *Geom. Dedicata* **109** (2004), 165–173.

[28] M. I. Munteanu, Warped product contact CR-submanifolds of Sasakian space forms, *Publi. Math. Debrecen* **66** (2005), 75–120.

[29] S. Sular and C. Ozgur, Contact CR-warped product submanifolds in generalized Sasakian space forms, *Turk. J. Math.* **36** (2012), 485–497.

[30] S. Tanno, Quasi-Sasakian structure of rank $2p+1$, *J. Differential Geom.* **5** (1971), 317–324.

[31] K. Yano and M. Kon, Structures on manifolds, series in Pure Mathematics (*World scientific, Singapore*, 1984).

TRAN QUOC BINH
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: binh@science.unideb.hu

AVIK DE
DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA
35, B.C.ROAD, KOLKATA-700019
WEST BENGAL
INDIA

E-mail: de.math@gmail.com

(Received April 5, 2013)