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Some theorems in special Finsler spaces and its generalizations in a bivector connected space

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Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. In this paper we would like to prove some reduction theorems for R-quadratic, Ricci-quadratic, W-quadratic, Douglas, and $S(n)$ spaces (in which the stretch tensor vanishes). Moreover, a new, special type of Finsler spaces, the so-called GRH (generalized Ricci-quadratic) space is defined. Finally, in the second part of this paper we develop the theory of quasi-autoparallel mappings.

0. Introduction

Let $F^n(M^n, L)$ be an n-dimensional Finsler space, where M^n is a connected differentiable manifold of dimension n and $L(x, y)$, where $y^i = \dot{x}^i$, is the fundamental function defined on the manifold $TM \setminus \{0\}$ of nonzero tangent vectors. In the following we assume that L is positive and that the fundamental metric tensor $g_{ij} = \frac{1}{2} L^2_{(i)(j)}$ $((i) = \partial/\partial y^i)$ is positive definite. In the first part of the present paper we shall use the terminology and definitions described in MATSUMOTO's monograph [1][1].

The system of differential equations for geodesic curves $x^i(t)$ of F^n with

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[1]Numbers in brackets refer to the references at the end of the paper.

respect to the canonical parameter t is given by $\frac{d^2x^i}{dt^2} = -2G^i(x,y)$, where

$$G^i = \frac{1}{4}g^{i\alpha}(y^\beta(\partial L^2(\alpha)/\partial x^\beta) - \partial^2 L^2/\partial x^\alpha).^2$$

The Berwald connection coefficients $G^i_j(x,y)$, $G^i_{jk}(x,y)$ can be derived from the function G^i, namely $G^i_j = G^i_{(j)}$, $G^i_{jk} = G^i_{j(k)}$. The Berwald covariant derivative with respect to the Berwald connection can be written as

$$T^i_{j\|k} = \partial T^i_j/\partial x^k - T^i_{j(\alpha)}G^\alpha_k - T^i_\alpha G^\alpha_{jk}. \tag{0.1}$$

We denote by $H = H^h_{ijk}$ the h-curvature tensor, where H is defined by

$$H^h_{ijk} = \partial_k G^h_{ij} - G^h_{ij(\alpha)}G^\alpha_k + G^\alpha_{ij}G^h_{\alpha k} - \partial_j G^h_{ik} + G^h_{ik(\alpha)}G^\alpha_j - G^h_{ik}G^h_{\alpha j}, \tag{0.2}$$

where $\partial_k = \partial/\partial x^k$. From (0.2) we obtain the $v(h)$ torsion tensor

$$H^h_{\alpha jk}y^\alpha = H^h_{ojk} = R^h_{jk} \tag{0.3}$$

and

$$H^h_{\alpha\beta k}y^\alpha y^\beta = H^h_{ook} = R^h_k. \tag{0.4}$$

Let us consider two Finsler spaces $F^n(M^n, L)$ and $\bar{F}^n(M^n, \bar{L})$ on a common underlying manifold M^n. A Finsler space is said to be projectively equivalent to $\bar{F}^n$ if it has the same geodesics as F^n as point sets. In this case the change $L \to \bar{L}$ of the metrics is called projective. The Douglas tensor

$$D^h_{ijk} = \left(G^h - \frac{1}{n+1}G^\alpha_{(\alpha)}y^h\right)_{(i)(j)(k)}, \tag{0.5}$$

and the Weyl tensor

$$W^h_{ijk} = W^h_{jk(i)}, \tag{0.6}$$

where

$$\begin{aligned} W^h_{jk} = R^h_{jk} &+ \frac{1}{n+1}\left\{y^h H_{jk} + \delta^h_j(nH_{\alpha k} + H_{k\alpha})y^\alpha/(n-1)\right\} \\ &- \frac{1}{n+1}\left\{y^h H_{kj} + \delta^h_k(nH_{\alpha j} + H_{j\alpha})y^\alpha/(n-1)\right\}, \end{aligned}$$

are invariant under projective changes.

We introduce some sets of special kinds of n-dimensional Finsler spaces $(n > 2)$:

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[2]The Roman and the Greek indices run over the range $1, 2, \ldots, n$; the Roman indices are free but the Greek indices denote summation.

$B(n)\dots$ Berwald spaces ($G^h_{ijk}=0$),

$L(n)\dots$ Landsberg spaces ($y_\alpha G^\alpha_{ijk}=0$),

$D(n)\dots$ Douglas spaces ($D^h_{ijk}=0$) [3],

$S(n)\dots$ spaces with vanishing stretch tensor ($\Sigma_{hijk}=y_\alpha H^\alpha_{hjk(i)}=0$) [1],

$Hx(n)\dots$ spaces with H depending on the position alone, R-quadratic spaces ($H^h_{ijk(l)}=0$) [2],

$Wx(n)\dots$ spaces with W depending on the position alone, W-quadratic spaces ($W^h_{ijk(l)}=0$) [3],

$RHx(n)\dots$ spaces with RH depending on the position alone, Ricci-quadratic spaces ($H^\alpha_{ij\alpha(k)}=H_{ij(k)}=0$) [4], [5],

$GRHx(n)\dots$ generalized Ricci-quadratic spaces ($H^\alpha_{\alpha ij(k)}=0$).

We have some well-known inclusion relations of the above mentioned notions:

$$D(n)\cap L(n)=B(n)\dots\ [6]$$

$$B(n)\subset Hx(n)\subset S(n)$$

$$B(n)\subset L(n)\subset S(n)$$

$$B(n)\subset D(n)\subset Wx(n)\dots\ [3]$$

$$B(n)\subset Hx(n)\subset Wx(n)$$

$$B(n)\subset Hx(n)\subset RHx(n).$$

1. Some reduction theorems among special Finsler spaces

Proposition 1. $Wx(n)\cap RHx(n)=Hx(n)$.

Proof. From (0.6) we have

$$\begin{aligned}W^h_{ijk(l)}&=H^h_{ijk(l)}+\frac{1}{n+1}\Big\{\delta^h_i H^h_{jk(l)}+\delta^h_e H_{jk(i)}+y^h H_{jk(i)(l)}+\frac{1}{n-1}\delta^h_j\\&\quad\times\Big[\big(nH_{\alpha k(i)(l)}+H_{k\alpha(i)(l)}\big)y^\alpha+\big(nH_{lk(i)}+H_{kl(i)}\big)+\big(nH_{ik(l)}+H_{ki(l)}\big)\Big]\Big\}\\&\quad-\frac{1}{n+1}\Big\{\delta^h_i H_{kj(l)}+\delta^h_l H_{kj(i)}+y^h H_{kj(i)(l)}+\frac{1}{n-1}\delta^h_k\\&\quad\times\Big[\big(nH_{\alpha j(i)(l)}+H_{j\alpha(i)(l)}\big)y^\alpha+\big(nH_{lj(i)}+H_{jl(i)}\big)\\&\quad+\big(nH_{ij(l)}+H_{ji(l)}\big)\Big]\Big\}.\end{aligned}\tag{1.7}$$

From (1.7) we easily get Proposition 1. Consequence of (1.7): $Hx(n) \subset Wx(n)$. $\square$

Proposition 2. *$Wx(n) \cap RHx(n) \subset S(n)$.*

Proof. If an F^n is $Wx(n)$ and $RHx(n)$ at the same time, then we have

$$H^h_{ijk(l)} = 0. \tag{1.8}$$

The stretch curvature tensor Σ_{hijk} is written in the form [1]:

$$\Sigma_{hijk} = -y_\alpha H^\alpha_{hjk(i)}. \tag{1.9}$$

(1.9) gives Proposition 2. $\square$

The notion of Douglas spaces, arising from the problem of the equations of the geodesics, yields interesting topics in Finsler geometry. A Finsler space is a Douglas space if and only if the Douglas tensor D^h_{ijk} vanishes identically.

It means that

$$G^i_{hjk} = G_{hjk}y^i/(n+1) + (G_{hj}\delta^i_k + G_{jk}\delta^i_h + G_{kh}\delta^i_j)/(n+1), \tag{1.10}$$

where $G_{hjk} = G^\alpha_{\alpha hj(k)}$ are completely symmetric.

One of the Bianchi identities of $B\Gamma$ (Berwald connection) is the following:

$$H^h_{mij(k)} + G^h_{mjk\|i} - G^h_{mij\|k} = 0.$$

Consequently, we have the expression

$$\Sigma_{hijk} = \big(y_\alpha G^\alpha_{hki}\big)_{\|j} - \big(y_\alpha G^\alpha_{hji}\big)_{\|k}.$$

Therefore F^n is without stretch, iff

$$(y_\alpha G^\alpha_{hij})_{\|k} - (y_\alpha G^\alpha_{hik})_{\|j} = 0. \tag{1.11}$$

(In the above equation, "$\|$" is the h-covariant differentiation in $B\Gamma$). Transvecting (1.10) by h^l_i we easily get

$$G^l_{hjk} = l^l(l_\alpha G^\alpha_{hjk}) + (h^l_h G_{jk} + h^l_j G_{kh} + h^l_k G_{hj})/(n+1). \tag{1.12}$$

Proposition 3. *F^n is a Douglas space without stretch if and only if* (1.11) *and* (1.12) *are satisfied.*

Differentiate (1.12) h-covariantly and we get

$$G^l_{hjk\|i} = l^l\big(l_\alpha G^\alpha_{hjk}\big)_{\|i} + \big(h^l_h G_{jk\|i} + h^l_j G_{kh\|i} + h^l_k G_{hj\|i}\big)/(n+1). \tag{1.13}$$

By changing the indices i and j in (1.13) we have

$$G^l_{hik\|j} = l^l\big(l_\alpha G^\alpha_{hik}\big)_{\|j} + \big(h^l_h G_{ik\|j} + h^l_i G_{kh\|j} + h^l_k G_{hi\|j}\big)/(n+1). \tag{1.13'}$$

From (1.13), (1.13') and $\Sigma = 0$ we obtain

$$\begin{aligned} H^l_{hij(k)} = \big(&h^l_h G_{ik\|j} + h^l_i G_{kh\|j} + h^l_k G_{hi\|j} \\ &- h^l_j G_{kh\|i} - h^l_h G_{jk\|i} - h^l_h G_{hj\|i}\big)/(n+1). \end{aligned} \tag{1.14}$$

So we have:

Corollary 1. *$Hx(n) \subset S(n)$ with $G_{ij\|k} = 0$.*

Transvecting (1.14) by the indices l and h, we get

Corollary 2. *$GRHx(n) \subset S(n)$ with $G_{ij\|k} = 0$.*

2. Quasi-autoparallel mappings

In the last section we want to study the quasi-autoparallel curves and quasi-autoparallel mappings in bivector preserving tensorially connected spaces.

Let M^n be an n-dimensional manifold. A nonlinear (that is, a not necessarily linear) connection of the tensors t^{ij} of type $(2,0)$ is a mapping between the n^2-dimensional vector spaces of the tensors of the given type at two points of M. For two neighbouring points x and $x+dx$ this is given by the vanishing of the absolute differential

$$\mathcal{V}t^{ij} = dt^{ij} + B^{ij}{}_\alpha(x,t)dx, \tag{2.15}$$

where $B^{ij}{}_\alpha(x,t)$ are the coefficients of the connection [7].

$\mathcal{V}t^{ij}$ must be a tensor of type $(2,0)$. This requirement determines the transformation law of the geometric object $B^{ij}{}_\alpha$ [7]. $B^{ij}{}_\alpha(x,t)$ is supposed to be homogeneous of degree one in t.

A tensor connection is linear if

$$B^{ij}{}_r(x,t) = \gamma_{\kappa\lambda}{}^{ij}{}_r(x)t^{\kappa\lambda},$$

and it reduces to a linear vector connection (to an affine connection) with coefficients $\Gamma^i_{j\alpha}$ if and only if

$$\gamma_{\kappa\lambda}{}^{ij}{}_\alpha = \Gamma^i_{\kappa\alpha}\delta^j_\lambda + \Gamma^j_{\lambda\alpha}\delta^i_\kappa.$$

N. S. Sinyukov gave the following notion ([8]):

Definition 1 ([8]). A curve is *almost autoparallel* if there exists a plane $\tau(t)$ in every tangent space of the curve $x^i(t)$ such that:

(a) $\tau(t)$ are parallel translated along $x^i(t)$, and

(b) the tangent dx^i/dt of the curve lies in $\tau(t)$.

This definition can be used in any context in which the parallelism of planes is defined. We showed that the parallel translation of plane positions can be defined in any tensorially connected space in which bivectors, as tensors of type $(2,0)$, are translated again into bivectors. In the following we use the fact that a bivector determines a plane position, and vice versa.

The following theorem characterizes a subclass of tensorial connections $B^{ij}{}_{\alpha}$ which carries bivectors into bivectors.

Theorem 1 ([9]). *The nonlinear tensor connection $B^{ij}{}_{\alpha}(x,t)$ carries bivectors p^{ij} $(p^{12} \neq 0)$ into bivectors if and only if the relations*

$$B^{(ij)}{}_{\alpha}(x,p) = 0$$

$$p^{1\,[\,2}B^{\kappa\lambda]}{}_{\alpha}(x,p) + B^{1\,[\,2}{}_{\alpha}p^{\kappa\lambda\,]} = 0 \quad (\kappa,\lambda = 3,4,\dots,n;\ \kappa \neq \lambda)$$

hold good, where (ij) denotes the cyclic permutation of indices i, j and summation, and $[ij]$ means the interchange of indices i, j and subtraction.

Definition 2 ([9]). A curve $x^i(t)$ is *quasi-autoparallel* if it satisfies Definition 1 with respect to a bivector connection.

Theorem 2 ([9]). *In a canonical parameter the differential equation of a quasi-autoparallel curve $x^i(t)$ is*

$$\frac{dp^{ij}}{dt} + B^{ij}{}_{\alpha}(x,p)\frac{dx^{\alpha}}{dt} = 0 \tag{2.16}$$

$$p^{ij} \wedge \frac{dx^k}{dt} = 0. \tag{2.17}$$

Let two bivector be given, which preserve the tensorially connected spaces T_n and $\widetilde{T}_n$.

Definition 3 ([10]). A mapping $L : T_n \to \widetilde{T}_n$ is called *quasi-autoparallel* if any quasi-autoparallel curve of T_n coincides with a quasi-autoparallel curve $\widetilde{T}_n$ as a set of points and vice versa.

Theorem 3 ([10]). *A mapping $L : T_n \to \widetilde{T}_n$ is quasi-autoparallel if and only if there exists a vector field $\varphi^k(x,p)$ satisfying*

$$B^{kl}{}_r(x,p) = B^{kl}{}_r(x,p) + c\varphi^k(x,p)\delta^l_r - c\cdot\varphi^\alpha(x,p)\delta^k_r, \quad p^{ij}\wedge\varphi^k(x,p) = 0, \tag{2.18}$$

where c is a constant.

From the parameters of a bivector preserving tensorial connection, one can construct a tensor which is invariant under quasi-autoparallel mappings.

Theorem 4 ([10]). *The tensor*

$$T_{kl}{}^{ij}{}_{mnr} = A_{kl}{}^{ij}{}_{mnr} - \frac{1}{n-1}\big(A_{kl}{}^{i}{}_{mn}\delta^j_r - A_{kl}{}^{j}{}_{mn}\delta^i_r\big), \tag{2.19}$$

where $A_{kl}{}^{ij}{}_{mnr} = \frac{\partial}{\partial p^{mn}}\big(\frac{\partial}{\partial p^{kl}}B^{ij}_r\big)$ and $A_{kl}{}^{i}{}_{mn} = A_{kl}{}^{i\alpha}{}_{mn\alpha}$, is invariant under quasi-autoparallel mappings.

From (2.19) it follows that if the bivector preserving tensorial connection is linear with respect to p^{ij}, then $T_{kl}{}^{ij}{}_{mnr} = 0$. Conversely, if the invariant tensor $T_{kl}{}^{ij}{}_{mnr}$ vanishes, then from (2.18) we obtain the following equation:

$$\frac{\partial}{\partial p^{mn}}\left[B_{kl}{}^{ij}{}_r(x,p) - \frac{1}{n-1}\left(B_{kl}{}^{i\alpha}{}_\alpha(x,p)\delta^j_r - B_{kl}{}^{j\alpha}{}_\alpha(x,p)\delta^i_r\right)\right] = 0.$$

Consequently, we have a new geometrical object

$$E_{kl}{}^{ij}{}_r(x) = B_{kl}{}^{ij}{}_r(x,p) - \frac{1}{(n-1)}\left(B_{kl}{}^{i\alpha}{}_\alpha(x,p)\delta^j_r - B_{kl}{}^{j\alpha}{}_\alpha(x,p)\delta^i_r\right). \tag{2.20}$$

Using the contracted object $E_{kl}{}^{ij}{}_r(x)p^{kl} = E^{ij}{}_r(x,p)$ and the contracted connection parameters $B_{kl}{}^{ij}{}_r(x,p)p^{kl} = B^{ij}{}_r(x,p)$, we obtain the following

Proposition 4. *The object*

$$E^{ij}{}_r(x,p) = B^{ij}{}_r(x,p) - \frac{1}{n-1}\left(B^{i\alpha}{}_\alpha(x,p)\delta^j_r - B^{j\alpha}{}_\alpha(x,p)\delta^i_r\right) \tag{2.21}$$

is invariant under quasi-autoparallel mappings.

We say that $E^{ij}{}_r$ are the quasi-projective parameters.

Consequently, we obtain the following

Theorem 5. *In a space where a linear (bivector preserving) connection is provided, the tensor $T_{kl}{}^{ij}{}_{mnr}(x,p)$ vanishes. Conversely, if $T_{kl}{}^{ij}{}_{mnr}(x,p)$ is equal to zero, then the quasi-projective parameters are linear with respect to p^{kl}.*

Let (U, x^i) and $(\bar{U}, \bar{x}^i)$ be two coordinate systems of M^n having the overlapping region $U \cap \bar{U}$, where we have n smooth transition functions $\bar{x}^r = \bar{x}^r(x^i)$. If we put $\partial_k \bar{x}^i = \partial \bar{x}^i / \partial x^k$, $\partial^2_{kl} \bar{x}^i = \partial^2 \bar{x}^i / \partial x^k \partial x^l$ and $\Delta = \|\partial \bar{x}^i\|$, then we get the following transformation law:

$$\bar{E}_{kl}{}^{\alpha\beta}{}_r(\bar{x})\partial_\alpha \bar{x}^i \partial_\beta \bar{x}^j = E_{\alpha\beta}{}^{ij}{}_\gamma(x)\partial_k \bar{x}^\alpha \partial_l \bar{x}^\alpha \partial_l \bar{x}^\beta \partial_r \bar{x}^r + \partial^2_{kr} \bar{x}^i \partial_l \bar{x}^j + \partial^2_{lr} \bar{x}^j \bar{\partial}_k \bar{x}^i - \frac{1}{n-1}$$
$$\times \Big[\Big(\partial^2_{kl} \bar{x}^i \partial_r \bar{x}^j - \partial_l \ln \Delta \partial_k \bar{x}^i \partial_r \bar{x}^j \Big) - \Big(\partial^2_{kl} \bar{x}^j \partial_r \bar{x}^i - \partial_k \ln \Delta \partial_l \bar{x}^j \partial_r \bar{x}^i \Big) \Big]. \quad (2.22)$$

Therefore, summarizing all of the above, we have

Theorem 6. *A tensor invariant with respect to quasi-autoparallel mappings is equal to zero if and only if the bivector connection has the following form:*

$$B_{kl}{}^{ij}{}_r(x,p) = E_{kl}{}^{ij}{}_r(x) + \frac{1}{n-1}(B_{kl}{}^{i\alpha}{}_\alpha(x,p)\delta^j_r - B_{kl}{}^{j\alpha}{}_\alpha(x,p)\delta^i_r).$$

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