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Lepage forms in Kawaguchi spaces and the Hilbert form

By DEMETER KRUPKA (Slatinice)

In honor of the 90th Birthday of Professor Lajos Tamássy

Abstract. A well-known construction in geometric mechanics and Riemann - Finsler geometry assigns to a (first order) homogeneous Lagrangian the Hilbert form, serving as an integrand in the corresponding variational functional. Analogous constructions, needed for higher-order mechanics and Finsler–Kawaguchi geometry, have not been found yet. In this paper we construct Lepage equivalents of Lagrangians, satisfying higher-order homogeneity (Zermelo) condition. We show that the homogeneity determines uniquely higher-order momenta and annihilates local Hamiltonians. The resulting Lepage equivalents then represent higher-order generalizations of the Hilbert form. This result extends geometric foundations of variational theory to higher-order parameter-invariant variational functionals.

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1. Introduction

This paper contains the lecture, presented by the author at the IRSES Meeting on Differential Geometry and Mechanics 14–16, January 2013, Ghent University, Belgium. It is devoted to variational structures that could be called *Kawaguchi structures* (KAWAGUCHI [3], [4]). Many references to the concepts, such as *Kawaguchi space* and *Kawaguchi fundamental function*, can be found in the literature; some of them are not always clearly established. We consider in this paper the Kawaguchi structure as a special case of a higher-order, simple integral variational structure whose Lagrangian satisfies certain additional conditions (higher-order Zermelo conditions).

Underlying structures for this class of variational problems are *higher-order velocity spaces*, the manifolds of r-jets of curves in a given smooth manifold (see URBAN and KRUPKA [14]). The special case $r = 1$ corresponds with tangent bundles, the underlying spaces for *Finsler structures* (see e.g. TAMÁSSY [11]).

Main objective of the research in this field is to extend the notions and results of Finsler geometry to higher order Kawaguchi variational structures. It is well known, in particular, that one can naturally assign to a Finsler fundamental function the so called *Hilbert form* (CRAMPIN and SAUNDERS [1], [2], TANAKA and KRUPKA [10]); it turns out that the Hilbert form is a special case of the Lepage forms, and as such, it can be used to characterize in a geometric way all fundamental properties of the underlying variational functional, such as extremals and symmetries (see e.g. KRUPKA [5], KRUPKA, KRUPKOVA and SAUNDERS [6], KRUPKA and SAUNDERS [7]). The aim of this paper is to find an analogue of the Hilbert form for Kawaguchi structures.

A basic general concept of the geometric variational theory is the Lepage form. In higher order fibred mechanics, to each Lagrangian λ one can assign in a unique way its Lepage equivalent Θ_λ. Our main idea in this paper is to determine Θ_λ for Lagrangians, satisfying the *Zermelo condition*, describing independence of variational functionals on parametrizations (see ZERMELO [15]). For more discussion of homogeneity problems and Zermelo conditions we refer to MATSYUK [8], SAUNDERS [9], and URBAN and KRUPKA [12], [13]). We call the Lagrangians satisfying the Zermelo condition, the *Kawaguchi Lagrangians*.

Our result consists in explicit expression for the Lepage form Θ_λ in terms of the Kawaguchi Lagrangian λ. We know that the Lepage form Θ_λ describes all basic general properties of variational functional, defined by λ (see Section 2); in this way we get an immediate descriptions of extremals and symmetries of the Kawaguchi variational functionals. It should also be pointed out that the Zermelo condition, leading to singularity of λ, has a specific influence on local Hamiltonians, related with Θ_λ; namely, these local Hamiltonians vanish identically.

Note that for the 1st order Lagrangians λ, the form Θ_λ reduces to the well-known *Hilbert form*.

2. Lepage forms in higher-order fibred mechanics

In this section we summarize basic concepts and theorems of the global variational theory in fibred manifolds over 1-dimensional bases (*fibred mechanics*); complete references can be found in Krupka and Saunders [7].

Throughout, standard symbols of the differential calculus on smooth manifolds are used: d (exterior derivative), T (tangent functor), ∂_ξ (Lie derivative by a vector field ξ), i_ξ (contraction by ξ), and * stands for the pull-back operation. Y is a fibred manifold with 1-dimensional base X and projection π, and we denote $\dim Y = m+1$. J^rY, where $r \geq 0$, is the r-jet prolongation of Y, and $\pi^{r,s} : J^rY \to J^sY$, $\pi^r : J^rY \to X$ are the canonical *jet projections*. The points of J^rY are r-jets $J^r_x\gamma$ of sections γ of Y at $x \in X$; the *r-jet prolongation* of γ is the mapping $x \to J^r\gamma(x) = J^r_x\gamma$. Any *fibred chart* (V, ψ), $\psi = (t, y^\sigma)$, on Y induces the *associated charts* (V^r, ψ^r), $\psi^r = (t, y^\sigma_{(0)}, y^\sigma_{(1)}, y^\sigma_{(2)}, \dots, y^\sigma_{(r)})$, on J^rY, and (U, φ), $\varphi = (t)$, on X; here $V^r = (\pi^{r,0})^{-1}(V)$, $U = \pi(V)$, and $1 \leq \sigma \leq m$. For lower dimensions we usually use a modified notation; if e.g. $r = 3$ we write $\psi^3 = (t, y^\sigma, \dot{y}^\sigma, \ddot{y}^\sigma, \dddot{y}^\sigma)$. A vector Ξ at a point $y \in Y$ is said to be *π-vertical*, if $T_y\pi \cdot \Xi = 0$; a differential form ρ on Y is *π-horizontal*, if it vanishes whenever one of its arguments is a π-vertical vector. Clearly, these concepts apply to the canonical jet projections.

For any open set $W \subset Y$, we denote by $\Omega^r_0 W$ (resp. $\Omega^r_k W$) the ring of smooth functions (resp. the $\Omega^r_0 W$-module of smooth k-forms) on $W^r = (\pi^{r,0})^{-1}(W)$. We also use some submodules of these modules; $\Omega^r_{k,X} W \subset \Omega^r_k W$ (resp. $\Omega^r_{k,Y} W \subset \Omega^r_k W$) are submodules of π^r-horizontal (resp. $\pi^{r,0}$-horizontal) forms. We have a morphism of exterior algebras $h : \Omega^r_k W \to \Omega^{r+1}_{k,X} W$ defined by

$$hf = f\pi^{r+1,r}, \quad hdt = dt, \quad hdy^\sigma_{(l)} = y^\sigma_{(l+1)}dt,$$

where $f : V^r \to \mathbb{R}$ is a function; obviously, $J^r\gamma^*\rho = J^{r+1}\gamma^* h\rho$ for every section γ of Y. We call h the *π-horizontalization*. We say that a form $\rho \in \Omega^r_k W$ is *contact*, if $h\rho = 0$. For any fibered chart (V, ψ), $\psi = (t, y^\sigma)$, the 1-forms

$$\omega^\sigma_{(l)} = dy^\sigma_{(l)} - y^\sigma_{(l+1)}dt, \quad 0 \leq l \leq r-1,$$

are examples of contact forms. The system of forms $dt, \omega^\sigma_{(l)}, dy^\sigma_{(r)}$ is a basis of linear forms on V^r. A form $\rho \in \Omega^r_k W$ has a unique decomposition

$$(\pi^{r+1,r})^*\rho = h\rho + p_1\rho + p_2\rho + \ldots + p_k\rho,$$

in which $p_k\rho$ contains, in any fibred chart, exactly k exterior factors $\omega^\sigma_{(l)}$; transformation properties of these forms ensure invariance of the decomposition. If $k = 1$, this formula reads $(\pi^{r+1,r})^*\rho = h\rho + p_1\rho$; if $k \geq 2$, then $(\pi^{r+1,r})^*\rho = p_{k-1}\rho + p_k\rho$. $p_k\rho$ is the *k-contact component* of ρ. We say ρ is of *order of contactness* k, if $(\pi^{r+1,r})^*\rho = p_k\rho$.

By a *Lagrangian* (of order r) for Y we mean a 1-form $\lambda \in \Omega^r_{1,X} W$. In a fibred chart,

$$\lambda = \mathscr{L}\omega_0,$$

where

$$\omega_0 = dt.$$

The component $\mathscr{L} : V^r \to \mathbb{R}$ of the form λ is the *Lagrange function.* For any piece Ω of X with boundary $\partial\Omega$, λ gives rise to the variational functional

$$\Gamma_\Omega Y \ni \gamma \to \lambda_\Omega(\gamma) = \int_\Omega J^r\gamma^*\lambda \in \mathbb{R}, \tag{1}$$

where $\Gamma_\Omega Y$ is the set of sections of Y, defined on Ω.

Let $V \subset Y$ be an open set, $\alpha : V \to Y$ a diffeomorphism, commuting with π, $U = \pi(V)$, and let $\alpha_0 : U \to X$ be the π-projection of α. Setting

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}$$

for every $J^r_x\gamma \in V^r$, we obtain the *r-jet prolongation* $J^r\alpha : V^r \to J^rY$. Applying this concept to the flow of a π-projectable vector field Ξ on Y, and differentiating, we obtain the *r-jet prolongation* of Ξ, denoted by $J^r\Xi$.

Let $U \subset X$ be an open set, let $\gamma : U \to Y$ be a section. Let Ξ be a π-projectable vector field on an open set $W \subset Y$ such that $\gamma(U) \subset W$. If α_t is the flow of Ξ, and $\alpha_{(0)t}$ is its π-projection, then since $\pi\alpha_t = \alpha_{(0)t}\pi$ for all t, Ξ defines a 1-parameter family of sections of Y, $\gamma_t = \alpha_t\gamma\alpha^{-1}_{(0)t}$, depending smoothly on the parameter t; γ_t is the *variation* of γ, *induced* by Ξ.

Choose an element $\gamma \in \Gamma_\Omega Y$ and a π-projectable vector field Ξ on Y, and consider the variation γ_t of γ, induced by Ξ. Since the domain of γ_t contains Ω for all sufficiently small t, we get a function on a neighbourhood $(-\varepsilon, \varepsilon)$ of the origin $0 \in \mathbb{R}$,

$$(-\varepsilon, \varepsilon) \ni t \to \lambda_{\alpha_{(0)t}(\Omega)}(\alpha_t\gamma\alpha^{-1}_{(0)t}) = \int_{\alpha_{(0)t}(\Omega)} J^r(\alpha_t\gamma\alpha^{-1}_{(0)t})^*\lambda = \int_\Omega J^r\gamma^* J^r\alpha_t^*\lambda \in \mathbb{R},$$

where we used the identity $J^r(\alpha_t\gamma\alpha_{(0)t}^{-1})^*\lambda = (\alpha_{(0)t}^{-1})^* J^r\gamma^* J^r\alpha_t^*\lambda$. Differentiating this real-valued function at $t = 0$ we obtain

$$(\partial_{J^r\Xi}\lambda)_\Omega(\gamma) = \int_\Omega J^r\gamma^*\partial_{J^r\Xi}\lambda. \tag{2}$$

The number (2) is the *variation* of the variational functional λ_Ω at γ, *induced* by the vector field Ξ. The function $\gamma \to (\partial_{J^r\Xi}\lambda)_\Omega(\gamma)$ is the *variational derivative*, or the *first variation* of λ_Ω by Ξ.

Extremals of λ_Ω are defined in a standard way. To describe them formally by means of invariant differential-geometric operations, we proceed as follows.

We say that a form $\Theta \in \Omega_1^s W$ is a *Lepage equivalent* of λ, if (a) $h\Theta = \lambda$ (up to a canonical jet projection), and (b) $hi_\zeta d\Theta = 0$ for every $\pi^{s,0}$-vertical vector field ζ on W^s; condition (b) is equivalent to saying that $p_1 d\Theta$ is $\pi^{s+1,0}$-horizontal.

We now state existence and describe basic properties of Lepage equivalents, namely their meaning for extremals and invariance properties of the variational functional (1).

Let $\Theta \in \Omega_1^s W$ be a Lepage equivalent of $\lambda \in \Omega_{1,X}^r W$. Condition (a) implies that for every section γ defined on Ω

$$\int_\Omega J^s\gamma^*\Theta = \int_\Omega J^{s+1}\gamma^* h\Theta = \int_\Omega J^r\gamma^*\lambda,$$

which means that the variational functional on the left-hand side, associated with Θ, coincides with λ_Ω. For any π-projectable vector field Ξ on W, we have

$$J^s\gamma^*\partial_{J^s\Xi}\Theta = J^s\gamma^* i_{J^s\Xi} d\Theta + dJ^s\gamma^* i_{J^s\Xi}\Theta. \tag{3}$$

Since by (b) the term $J^s\gamma^* i_{J^s\Xi} d\Theta$ in (3) depends linearly on $J^s\Xi$ via the vector field Ξ only, and is independent of derivatives of the components of Ξ. Moreover, since $J^s\gamma^*\partial_{J^s\Xi}\Theta = J^{s+1}\gamma^* h\partial_{J^s\Xi}\Theta = J^{s+1}\gamma^*\partial_{J^{s+1}\Xi} h\Theta = J^r\gamma^*\partial_{J^r\Xi}\lambda$, (3) can also be written as $\partial_{J^r\Xi}\lambda = hi_{J^s\Xi} d\Theta + hdi_{J^s\Xi}\Theta$. Therefore, it is *a priori* clear that for a Lepage form Θ, the decomposition (3) has the properties which are required in the integrand expressions of the classical first variation formula: If $\Omega \subset \pi(W)$ is a piece, then integrating (3) one gets

$$\int_\Omega J^r\gamma^*\partial_{J^r\Xi}\lambda = \int_\Omega J^s\gamma^* i_{J^s\Xi} d\Theta + \int_{\partial\Omega} J^s\gamma^* i_{J^s\Xi}\Theta.$$

The expression on the right-hand side of (3) can be further simplified using the 1-contact component of $d\Theta$. Since $J^s\gamma^* i_{J^s\Xi} d\Theta = J^{s+1}\gamma^* i_{J^{s+1}\Xi} p_1 d\Theta$, we have the following result (the *infinitesimal first variation formula*). For the proofs of Theorems 1–3 we refer to [5], [7].

Theorem 1. *For any Lepage equivalent* $\Theta \in \Omega_1^s W$ *of a Lagrangian* $\lambda \in \Omega_{1,X}^r W$,

$$J^r\gamma^*\partial_{J^r\Xi}\lambda = J^s\gamma^* i_{J^s\Xi} d\Theta + dJ^s\gamma^* i_{J^s\Xi}\Theta.$$

The following theorem ensures existence of (global) Lepage equivalents.

Theorem 2. *Every Lagrangian* $\lambda \in \Omega_{1,X}^r W$ *has a unique Lepage equivalent* Θ_λ. *If in a fibred chart* (V,ψ), $\psi = (t, y^\sigma)$, λ *is expressed as* $\lambda = \mathscr{L}\omega_0$, *then* Θ_λ *has an expression*

$$\Theta_\lambda = \mathscr{L}\omega_0 + \sum_{k=0}^{r-1}\left(\sum_{l=0}^{r-1-k}(-1)^l \frac{d^l}{dt^l}\frac{\partial \mathscr{L}}{\partial y^\sigma_{(k+l+1)}}\right)\omega^\sigma_{(k)}. \tag{4}$$

Theorem 3. *Let* $W \subset Y$ *be an open set, let* $\lambda \in \Omega_{1,X}^r W$ *be a Lagrangian. Then the form* $d\Theta_\lambda$ *has an expression*

$$(\pi^{s+1,s})^* d\Theta_\lambda = E_\lambda + F_\lambda,$$

where E_λ *is a 2-form*

$$E_\lambda = E_\sigma(\mathscr{L})\omega^\sigma \wedge \omega_0,$$

with components

$$E_\sigma(\mathscr{L}) = \sum_{l=0}^{r}(-1)^l \frac{d^l}{dt^l}\frac{\partial \mathscr{L}}{\partial y^\sigma_{(l)}}, \tag{5}$$

and F_λ *is of order of contactness 2.*

The form E_λ is called the *Euler–Lagrange form* associated with λ. The components $E_\sigma(\mathscr{L})$ (5) are the *Euler–Lagrange expressions.* Obviously, E_λ belongs to $\Omega_{2,Y}^{2r} W$. The mapping $\Omega_{1,X}^r W \ni \lambda \to E_\lambda \in \Omega_{2,Y}^{2r} W$, assigning to a Lagrangian its Euler–Lagrange form, is called the *Euler–Lagrange mapping.*

A section $\gamma \in \Gamma_\Omega Y$ is an extremal of λ if and only if $E_\lambda \circ J^{2r}\gamma = 0$, or, which is the same, it is a solution of the *Euler–Lagrange equations* $E_\sigma(\mathscr{L}) = 0$. The following standard theorem is mentioned for the record.

Theorem 4. *Let* $\lambda \in \Omega_{1,X}^r W$ *be a Lagrangian,* $\Theta_\lambda \in \Omega_1^s W$ *the Lepage equivalent of* λ, *and* γ *a section of* Y. *The following conditions are equivalent:*

(a) γ *is an extremal of* λ.

(b) *For every* π*-vertical vector field* Ξ,

$$J^{2r}\gamma^* i_{J^{2r}\Xi} d\Theta_\lambda = 0.$$

(c) *For any fibred chart* (V, ψ), $\psi = (t, y^\sigma)$, γ *satisfies the system of partial differential equations*

$$E_\sigma(\lambda) \circ J^{s+1}\gamma = 0, \quad 1 \leq \sigma \leq m. \tag{6}$$

Equations (6) are the *Euler–Lagrange equations.*

Theorem 4 explains the meaning of the Lepage equivalent Θ_λ for a chart-independent description of extremals. Θ_λ is also a basic quantity for the study of *invariance properties* of the Lagrangian (*Noether's theory*). We say that λ is *invariant* with respect to a π-projectable vector field Ξ, if

$$\partial_{J^r\Xi}\lambda = 0.$$

The following theorem, establishing a relation between extremals and conservation law equations, follows from the first variation formula.

Theorem 5. *Let* $\lambda \in \Omega^r_{1,X}W$ *be a Lagrangian,* $\Theta_\lambda \in \Omega^s_1 W$ *its Lepage equivalent. The following two conditions are equivalent:*

(a) λ *is invariant with respect to* Ξ.

(b) *For every section* γ *of* Y,

$$J^s\gamma^* i_{J^s\Xi} d\Theta_\lambda + dJ^s\gamma^* i_{J^s\Xi}\Theta_\lambda = 0.$$

Clearly, also other kinds of invariance can be considered (e.g. invariance of the Euler–Lagrange form).

3. Higher-order positive homogeneous functions

In this section we recall basic definitions and notations on the theory of higher-order velocity spaces and present a theorem on homogeneous functions.

Let Q be a smooth manifold of dimension m. We denote by T^rQ the set of r-jets $J^r_0\zeta$ with source $0 \in \mathbb{R}$ and target $\zeta(0)$, an arbitrary point in Q. Elements of the set T^rQ are called *r-velocities.* We consider the set T^rQ with its canonical smooth structure: if (W, χ), $\chi = (y^\sigma)$, is a chart on Q then the associated chart on T^rQ is denoted by (W^r, χ^r), $\chi^r = (y^\sigma_{(0)}, y^\sigma_{(1)}, y^\sigma_{(2)}, \ldots, y^\sigma_{(r)})$. For lower orders r we also use a more convenient notation; if for instance $r = 3$, then we write $\chi^3 = (y^\sigma, \dot{y}^\sigma, \ddot{y}^\sigma, \dddot{y}^\sigma)$. T^rQ is referred to as the *manifold of r-velocities.*

We denote by $\mathrm{Imm}T^rQ$ the open subset of the manifold of r-velocities, consisting of r-jets whose representatives are *immersions* at the origin $0 \in \mathbb{R}$. Elements of the set $\mathrm{Imm}T^rQ$ are called *regular r-velocities.* Note that for $r = 1$, $\mathrm{Imm}T^rQ$ coincides with the *slit tangent bundle* of Q.

Suppose we have a function $F : \mathrm{Imm}T^rQ \to \mathbb{R}$. Let $S \subset \mathbb{R}$ be a compact interval, and $\gamma : S \to Q$ an immersion. These data define the integral

$$F_S(\gamma) = \int_S (F \circ T^r\gamma)(t)dt, \tag{7}$$

where $T^r\gamma(t) = J^r_0(\gamma \circ \mathrm{tr}_{-t})$ is the canonical lift of the curve γ to $\mathrm{Imm}T^rQ$, and tr_{-t} is the translation $s \to s + t$ of $\mathbb{R}$ sending the origin 0 into the point t. We shall say that the integral (7) is *parameter-invariant*, if for any curve $\gamma : S \to Q$, any open interval $I \subset S$ and any diffeomorphism $\mu : J \to I$, such that $\mu(J) = I$ and $D\mu > 0$,

$$F_I(\gamma) = F_J(\gamma \circ \mu).$$

The following is a criterion of parameter-invariance.

Theorem 6. *Suppose that the function $F : \mathrm{Imm}T^rQ \to \mathbb{R}$ is differentiable. Then the following two conditions are equivalent:*

(a) *Integral (7) is parameter-invariant.*

(b) *For any chart (W, χ), $\chi = (y^\sigma)$, on Q*

$$\frac{\partial F}{\partial y^\sigma_{(1)}} y^\sigma_{(1)} + 2\frac{\partial F}{\partial y^\sigma_{(2)}} y^\sigma_{(2)} + 3\frac{\partial F}{\partial y^\sigma_{(3)}} y^\sigma_{(3)} + \ldots + r\frac{\partial F}{\partial y^\sigma_{(r)}} y^\sigma_{(r)} = F,$$

$$\frac{\partial F}{\partial y^\sigma_{(r-k+1)}} y^\sigma_{(1)} + \binom{r-k+2}{1} \frac{\partial F}{\partial y^\sigma_{(r-k+2)}} y^\sigma_{(2)} + \binom{r-k+3}{2} \frac{\partial F}{\partial y^\sigma_{(r-k+3)}} y^\sigma_{(3)}$$
$$+ \ldots + \binom{r}{k-1} \frac{\partial F}{\partial y^\sigma_{(r)}} y^\sigma_{(k)} = 0, \quad k = 1, 2, \ldots, r-1. \tag{8}$$

Condition (8) is called the *Zermelo condition.*

Remark 1. The Zermelo condition is equivalent with *positive homogeneity condition*, which can be formulated as *equivariance* of F with respect to the canonical right action of the differential group $L^r_{(+)}$ on $\mathrm{Imm}T^rQ$, and the canonical multiplicative action of $L^1_{(+)}$ on $\mathbb{R}$. The group $L^r_{(+)}$ consists of r-jets $J^r_0\alpha$ of diffeomorphisms of $\mathbb{R}$ with source and target at the origin $0 \in \mathbb{R}$, such that $D\alpha > 0$, and the group action is the mapping

$$\mathrm{Imm}T^rQ \times L^r_{(+)} \ni (J^r_0\zeta, J^r_0\alpha) \to J^r_0(\zeta \circ \alpha) \in \mathrm{Imm}T^rQ.$$

This action represents parameter changes in the manifold $\mathrm{Imm}T^rQ$ induced by parameter changes on $\mathbb{R}$.

One can consider variational problems on the manifolds of regular velocities $\mathrm{Imm}T^rQ$ as a special case of variational problems on fibred manifolds. We set

$$Y = \mathbb{R} \times Q.$$

Then Y becomes a fibration over $\mathbb{R}$ with type fibre Q. The Cartesian product Y has an atlas, consisting of fibred charts (V, ψ), $\psi = (t, y^\sigma)$, where $V = \mathbb{R} \times W$, t is the canonical coordinate on $\mathbb{R}$, and (W, χ), $\chi = (y^\sigma)$, is a chart on Q. Sections $t \to \gamma(t)$ of Y are canonically identified with curves $t \to \zeta(t)$ in Q, where $\gamma(t) = (t, \zeta(t))$. Then the r-jets $J^r_t\gamma$ are canonically identified with the pairs $(t, J^r_t\zeta) = (t, J^r_0(\zeta \circ \mathrm{tr}_{-t})) \in \mathbb{R} \times T^rQ$. Thus, using this identification, we can write

$$J^rY = \mathbb{R} \times T^rQ.$$

Then the variational theory as expressed in Section 2 can also be applied to Lagrangians, defined on the open set $\mathbb{R} \times \mathrm{Imm}T^rQ \subset J^rY$.

4. Lepage forms and homogeneous Lagrangians

Our objective in this section is to find Θ_λ for Lagrangians λ, satisfying the Zermelo homogeneity conditions. Once Θ_λ is found we can say, in view of Section 2, that the basic (global) variational theory for higher-order homogeneous Lagrangians on smooth manifolds (the *Kawaguchi Lagrangians*) is established.

To this purpose we set

$$P^{(k+1)}_\sigma = \sum_{l=0}^{r-1-k} (-1)^l \frac{d^l}{dt^l} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(k+l+1)}},$$

and write the Lepage form Θ_λ as

$$\Theta_\lambda = \mathscr{L} dt + \sum_{k=0}^{r-1} P^{(k+1)}_\sigma \omega^\sigma_{(k)} = -\mathscr{H} dt + \sum_{k=0}^{r-1} P^{(k+1)}_\sigma dy^\sigma_{(k)},$$

where

$$-\mathscr{H} = \mathscr{L} - P^{(1)}_\sigma y^\sigma_{(1)} - P^{(2)}_\sigma y^\sigma_{(2)} - \ldots - P^{(r)}_\sigma y^\sigma_{(r)}.$$

The functions $P_\sigma^{(k+1)}$ and $\mathscr{H}$ can be called, on analogy with the first order mechanics, the *momenta* and the *Hamiltonian*, associated with $\mathscr{L}$.

To study homogeneous Lagrangians, we need an explicit expression for the Hamiltonian $\mathscr{H}$. Our idea is to collect together all terms containing total derivatives of a given order. The following theorem on the structure of $\mathscr{H}$ is new.

Theorem 7. *For any Lagrangian λ on J^rY, the function $\mathscr{H}$ reads*

$$\begin{aligned}-\mathscr{H} = \mathscr{L} &- \sum_{1\le i\le r}\binom{i}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i)}}y^\sigma_{(i)}\\ &+ \sum_{1\le l\le r-1}(-1)^{l-1}\frac{d^l}{dt^l}\sum_{1\le i\le r-l}\binom{i+l}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i+l)}}y^\sigma_{(i)}.\end{aligned}\tag{9}$$

Proof. 1. Consider the sum

$$\sum_{1\le i\le r}P_\sigma^{(i)}y^\sigma_{(i)} = P_\sigma^{(1)}y^\sigma_{(1)} + P_\sigma^{(2)}y^\sigma_{(2)} + \ldots + P_\sigma^{(r)}y^\sigma_{(r)}.$$

Substituting for $1\le k\le r$

$$P_\sigma^{(i)} = \frac{\partial\mathscr{L}}{\partial y^\sigma_{(i)}} - \frac{dP_\sigma^{(i+1)}}{dt},$$

and $P_\sigma^{(r+1)} = 0$, we have

$$P_\sigma^{(i)}y^\sigma_{(i)} = \frac{\partial\mathscr{L}}{\partial y^\sigma_{(i)}}y^\sigma_{(i)} - \frac{d}{dt}(P_\sigma^{(i+1)}y^\sigma_{(i)}) + P_\sigma^{(i+1)}y^\sigma_{(i+1)}.$$

Note that

$$P_\sigma^{(r)}y^\sigma_{(r)} = \frac{\partial\mathscr{L}}{\partial y^\sigma_{(r)}}y^\sigma_{(r)}.$$

Consequently, after some calculation,

$$\begin{aligned}\sum_{1\le i\le r}P_\sigma^{(i)}y^\sigma_{(i)} &= \frac{\partial\mathscr{L}}{\partial y^\sigma_{(1)}}y^\sigma_{(1)} + \frac{\partial\mathscr{L}}{\partial y^\sigma_{(2)}}y^\sigma_{(2)} + \ldots + \frac{\partial\mathscr{L}}{\partial y^\sigma_{(r)}}y^\sigma_{(r)} + P_\sigma^{(2)}y^\sigma_{(2)} + P_\sigma^{(3)}y^\sigma_{(3)}\\ &\quad + \ldots + P_\sigma^{(r)}y^\sigma_{(r)} - \frac{d}{dt}(P_\sigma^{(2)}y^\sigma_{(1)} + P_\sigma^{(3)}y^\sigma_{(2)} + \ldots + P_\sigma^{(r)}y^\sigma_{(r-1)})\\ &= \ldots = \sum_{1\le i\le r}\binom{i}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i)}}y^\sigma_{(i)} - \frac{d}{dt}\sum_{1\le i\le r-1}\binom{i}{i-1}P_\sigma^{(i+1)}y^\sigma_{(i)}.\end{aligned}$$

2. Now consider the sum

$$\sum_{1\le i\le r-1}\binom{i}{i-1}P_\sigma^{(i+1)}y^\sigma_{(i)} = P_\sigma^{(2)}y^\sigma_{(1)} + 2P_\sigma^{(3)}y^\sigma_{(2)} + \ldots + (r-1)P_\sigma^{(r)}y^\sigma_{(r-1)}.\tag{10}$$

We have for $1 \le i \le r-1$

$$\begin{aligned}
P_\sigma^{(i+1)} y^\sigma_{(i)} &= \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i+1)}} y^\sigma_{(i)} - \frac{d}{dt}(P_\sigma^{(i+2)} y^\sigma_{(i)}) + P_\sigma^{(i+2)} y^\sigma_{(i+1)} \\
&= \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i+1)}} y^\sigma_{(i)} - \frac{d}{dt}(P_\sigma^{(i+2)} y^\sigma_{(i)}) + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i+2)}} y^\sigma_{(i+1)} - \frac{dP_\sigma^{(i+3)}}{dt} y^\sigma_{(i+1)} \\
&= \ldots = \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i+1)}} y^\sigma_{(i)} + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i+2)}} y^\sigma_{(i+1)} + \ldots + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-1)} \\
&\quad - \frac{d}{dt}(P_\sigma^{(i+2)} y^\sigma_{(i)} + P_\sigma^{(i+3)} y^\sigma_{(i+1)} + \ldots + P_\sigma^{(r)} y^\sigma_{(r-2)}). \qquad (11)
\end{aligned}$$

The term in (10) (after the substitution from (11)) not containing formal derivatives is

$$\begin{aligned}
&\frac{\partial \mathscr{L}}{\partial y^\sigma_{(2)}} y^\sigma_{(1)} + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(3)}} y^\sigma_{(2)} + \ldots + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-1)} + 2\frac{\partial \mathscr{L}}{\partial y^\sigma_{(3)}} y^\sigma_{(2)} \\
&\quad + 2\frac{\partial \mathscr{L}}{\partial y^\sigma_{(4)}} y^\sigma_{(3)} + \ldots + 2\frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-1)} + \ldots + (r-2)\frac{\partial \mathscr{L}}{\partial y^\sigma_{(r-1)}} y^\sigma_{(r-2)} \\
&\quad + (r-2)\frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-1)} + (r-1)\frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-1)} = \sum_{1 \le i \le r-1} \binom{i+1}{i-1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i+1)}} y^\sigma_{(i)}.
\end{aligned}$$

The remaining summands in the formal derivative term are, analogously,

$$\begin{aligned}
&P_\sigma^{(3)} y^\sigma_{(1)} + P_\sigma^{(4)} y^\sigma_{(2)} + \ldots + P_\sigma^{(r)} y^\sigma_{(r-2)} + 2(P_\sigma^{(4)} y^\sigma_{(2)} + P_\sigma^{(5)} y^\sigma_{(3)} + \ldots + P_\sigma^{(r)} y^\sigma_{(r-2)}) \\
&\quad + \ldots + (r-3)(P_\sigma^{(r-1)} y^\sigma_{(r-3)} + P_\sigma^{(r)} y^\sigma_{(r-2)}) + (r-2) P_\sigma^{(r)} y^\sigma_{(r-2)} \\
&= \sum_{1 \le i \le r-2} \binom{i+1}{i-1} P_\sigma^{(i+2)} y^\sigma_{(i)}.
\end{aligned}$$

Summarizing,

$$\begin{aligned}
\sum_{1 \le i \le r-1} \binom{i}{i-1} P_\sigma^{(i+1)} y^\sigma_{(i)} &= \sum_{1 \le i \le r-1} \binom{i+1}{i-1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i+1)}} y^\sigma_{(i)} \\
&\quad - \frac{d}{dt} \sum_{1 \le i \le r-2} \binom{i+1}{i-1} P_\sigma^{(i+2)} y^\sigma_{(i)}.
\end{aligned}$$

3. For induction, note that the sums we already obtained

$$\sum_{1 \le i \le r-1} \binom{i}{i-1} P_\sigma^{(i+1)} y^\sigma_{(i)}, \qquad \sum_{1 \le i \le r-2} \binom{i+1}{i-1} P_\sigma^{(i+2)} y^\sigma_{(i)},$$

can be expressed as

$$\sum_{1\le i\le r-j}\binom{i+j-1}{i-1}P_\sigma^{(i+j)}y_{(i)}^\sigma,$$

where $j=1,2$. Now consider for some integer j the expression

$$\sum_{1\le i\le r-j}\binom{i+j-1}{i-1}P_\sigma^{(i+j)}y_{(i)}^\sigma=\binom{j}{0}P_\sigma^{(j+1)}y_{(1)}^\sigma \\ +\binom{j+1}{1}P_\sigma^{(j+2)}y_{(2)}^\sigma+\ldots+\binom{r-1}{r-j-1}P_\sigma^{(r)}y_{(r-j)}^\sigma. \quad (12)$$

In this formula

$$P_\sigma^{(i+j)}y_{(i)}^\sigma=\frac{\partial\mathscr{L}}{\partial y_{(i+j)}^\sigma}y_{(i)}^\sigma-\frac{dP_\sigma^{(i+j+1)}}{dt}y_{(i)}^\sigma \\ =\ldots=\frac{\partial\mathscr{L}}{\partial y_{(i+j)}^\sigma}y_{(i)}^\sigma+\frac{\partial\mathscr{L}}{\partial y_{(i+1+j)}^\sigma}y_{(i+1)}^\sigma+\frac{\partial\mathscr{L}}{\partial y_{(i+2+j)}^\sigma}y_{(i+2)}^\sigma \\ -\frac{d}{dt}(P_\sigma^{(i+j+1)}y_{(i)}^\sigma+P_\sigma^{(i+j+2)}y_{(i+1)}^\sigma+P_\sigma^{(i+j+3)}y_{(i+2)}^\sigma)+P_\sigma^{(i+j+3)}y_{(i+3)}^\sigma.$$

We can proceed this way further on. After l steps

$$P_\sigma^{(i+j)}y_{(i)}^\sigma=\frac{\partial\mathscr{L}}{\partial y_{(i+j)}^\sigma}y_{(i)}^\sigma+\frac{\partial\mathscr{L}}{\partial y_{(i+1+j)}^\sigma}y_{(i+1)}^\sigma+\ldots+\frac{\partial\mathscr{L}}{\partial y_{(i+l+j)}^\sigma}y_{(i+l)}^\sigma \\ -\frac{d}{dt}(P_\sigma^{(i+j+1)}y_{(i)}^\sigma+P_\sigma^{(i+j+2)}y_{(i+1)}^\sigma+\ldots+P_\sigma^{(i+j+l+1)}y_{(i+l)}^\sigma) \\ +P_\sigma^{(i+j+l+1)}y_{(i+l+1)}^\sigma.$$

However, if l is such that $i+j+l+1=r+1$, then $P_\sigma^{(i+j+l+1)}=0$. Thus, setting $l=r-i-j$ we get

$$P_\sigma^{(i+j)}y_{(i)}^\sigma=\frac{\partial\mathscr{L}}{\partial y_{(i+j)}^\sigma}y_{(i)}^\sigma+\frac{\partial\mathscr{L}}{\partial y_{(i+1+j)}^\sigma}y_{(i+1)}^\sigma+\ldots+\frac{\partial\mathscr{L}}{\partial y_{(r)}^\sigma}y_{(r-j)}^\sigma \\ -\frac{d}{dt}(P_\sigma^{(i+j+1)}y_{(i)}^\sigma+P_\sigma^{(i+j+2)}y_{(i+1)}^\sigma+\ldots+P_\sigma^{(r)}y_{(r-j-1)}^\sigma).$$

Consequently

$$\binom{i+j-1}{i-1}P_\sigma^{(i+j)}y_{(i)}^\sigma$$

$$= \binom{i+j-1}{i-1} \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_{(i+j)}} y^\sigma_{(i)} + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i+1+j)}} y^\sigma_{(i+1)} + \ldots + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-j)} \right)$$
$$- \binom{i+j-1}{i-1} \frac{d}{dt} (P^{(i+j+1)}_\sigma y^\sigma_{(i)} + P^{(i+j+2)}_\sigma y^\sigma_{(i+1)} + \ldots + P^{(r)}_\sigma y^\sigma_{(r-j-1)}),$$

and to obtain (12), we sum these expressions through $i = 1, 2, \ldots, r-j$. We have

$$\sum_{1 \le i \le r-j} \binom{i+j-1}{i-1} P^{(i+j)}_\sigma y^\sigma_{(i)}$$
$$= \binom{j}{0} \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_{(j+1)}} y^\sigma_{(1)} + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(j+2)}} y^\sigma_{(2)} + \ldots + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-j)} \right)$$
$$- \binom{j}{0} \frac{d}{dt} (P^{(j+2)}_\sigma y^\sigma_{(1)} + P^{(j+3)}_\sigma y^\sigma_{(2)} + \ldots + P^{(r)}_\sigma y^\sigma_{(r-j-1)})$$
$$+ \binom{j+1}{1} \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_{(j+2)}} y^\sigma_{(2)} + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(j+3)}} y^\sigma_{(3)} + \ldots + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-j)} \right)$$
$$- \binom{j+1}{1} \frac{d}{dt} (P^{(j+3)}_\sigma y^\sigma_{(2)} + P^{(j+4)}_\sigma y^\sigma_{(3)} + \ldots + P^{(r)}_\sigma y^\sigma_{(r-j-1)})$$
$$+ \ldots + \binom{r-2}{r-j-2} \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_{(r-1)}} y^\sigma_{(r-j-1)} + \frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-j)} \right)$$
$$- \binom{r-2}{r-j-2} \frac{d}{dt} P^{(r)}_\sigma y^\sigma_{(r-j-1)} + \binom{r-1}{r-j-1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-j)}.$$

We can also write this expression as

$$\sum_{1 \le i \le r-j} \binom{i+j-1}{i-1} P^{(i+j)}_\sigma y^\sigma_{(i)}$$
$$= \binom{j}{0} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(j+1)}} y^\sigma_{(1)} + \left(\binom{j}{0} + \binom{j+1}{1} \right) \frac{\partial \mathscr{L}}{\partial y^\sigma_{(j+2)}} y^\sigma_{(2)}$$
$$+ \left(\binom{j}{0} + \binom{j+1}{1} + \binom{j+2}{2} \right) \frac{\partial \mathscr{L}}{\partial y^\sigma_{(j+3)}} y^\sigma_{(3)}$$
$$+ \ldots + \left(\binom{j}{0} + \binom{j+1}{1} + \ldots + \binom{r-1}{r-j-1} \right) \frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-j)}$$
$$- \binom{j}{0} \frac{d}{dt} P^{(j+2)}_\sigma y^\sigma_{(1)} - \frac{d}{dt} \left(\binom{j}{0} + \binom{j+1}{1} \right) P^{(j+3)}_\sigma y^\sigma_{(2)}$$
$$- \frac{d}{dt} \left(\binom{j}{0} + \binom{j+1}{1} + \binom{j+2}{2} \right) P^{(j+4)}_\sigma y^\sigma_{(3)}$$

$$-\ldots-\frac{d}{dt}\left(\binom{j}{0}+\binom{j+1}{1}+\ldots+\binom{r-2}{r-j-2}\right)P_\sigma^{(r)}y^\sigma_{(r-j-1)}.$$

Finally, the formula

$$\sum_{0\le i\le k}\binom{j+i}{i}=\binom{j+k+1}{k}$$

yields

$$\begin{aligned}
&\sum_{1\le i\le r-j}\binom{i+j-1}{i-1}P_\sigma^{(i+j)}y^\sigma_{(i)}\\
&\quad=\binom{j+1}{0}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(j+1)}}y^\sigma_{(1)}+\binom{j+2}{1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(j+2)}}y^\sigma_{(2)}\\
&\qquad+\ldots+\binom{r}{r-j-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(r)}}y^\sigma_{(r-j)}-\binom{j+1}{0}\frac{d}{dt}P_\sigma^{(j+2)}y^\sigma_{(1)}\\
&\qquad-\binom{j+2}{1}\frac{d}{dt}P_\sigma^{(j+3)}y^\sigma_{(2)}-\ldots-\binom{r-1}{r-j-2}\frac{d}{dt}P_\sigma^{(r)}y^\sigma_{(r-j-1)}\\
&\quad=\sum_{1\le i\le r-j}\binom{j+i}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(j+i)}}y^\sigma_{(i)}-\frac{d}{dt}\sum_{1\le i\le r-j-1}\binom{j+i}{i-1}P_\sigma^{(j+i+1)}y^\sigma_{(i)}. \qquad (13)
\end{aligned}$$

4. Using the recurrence relation (13) we get

$$\begin{aligned}
\sum_{1\le i\le r}P_\sigma^{(i)}y^\sigma_{(i)}&=\sum_{1\le i\le r}\binom{i}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i)}}y^\sigma_{(i)}-\frac{d}{dt}\sum_{1\le i\le r-1}\binom{i}{i-1}P_\sigma^{(i+1)}y^\sigma_{(i)}\\
&=\sum_{1\le i\le r}\binom{i}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i)}}y^\sigma_{(i)}-\frac{d}{dt}\sum_{1\le i\le r-1}\binom{i+1}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i+1)}}y^\sigma_{(i)}\\
&\quad+\frac{d^2}{dt^2}\sum_{1\le i\le r-2}\binom{i+2}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i+2)}}y^\sigma_{(i)}-\frac{d^3}{dt^3}\sum_{1\le i\le r-3}\binom{i+3}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i+3)}}y^\sigma_{(i)}\\
&\quad+\ldots+(-1)^{r-2}\frac{d^{r-2}}{dt^{r-2}}\left(\binom{r-1}{0}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(r-1)}}y^\sigma_{(1)}+\binom{r}{1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(r)}}y^\sigma_{(2)}\right)\\
&\quad+(-1)^{r-1}\frac{d^{r-1}}{dt^{r-1}}\binom{r}{0}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(r)}}y^\sigma_{(1)},
\end{aligned}$$

and

$$\begin{aligned}
-\mathscr{H}=\mathscr{L}&-\sum_{1\le i\le r}\binom{i}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i)}}y^\sigma_{(i)}+\frac{d}{dt}\sum_{1\le i\le r-1}\binom{i+1}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i+1)}}y^\sigma_{(i)}\\
&-\frac{d^2}{dt^2}\sum_{1\le i\le r-2}\binom{i+2}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i+2)}}y^\sigma_{(i)}+\frac{d^3}{dt^3}\sum_{1\le i\le r-3}\binom{i+3}{i-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{(i+3)}}y^\sigma_{(i)}
\end{aligned}$$

$$- \ldots - (-1)^{r-2} \frac{d^{r-2}}{dt^{r-2}} \left(\binom{r-1}{0} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{(r-1)}} y^{\sigma}_{(1)} + \binom{r}{1} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{(r)}} y^{\sigma}_{(2)} \right)$$
$$- (-1)^{r-1} \frac{d^{r-1}}{dt^{r-1}} \binom{r}{0} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{(r)}} y^{\sigma}_{(1)},$$

which proves formula (9). □

Consider a smooth manifold Q, the Cartesian product $Y = \mathbb{R} \times Q$, manifolds of r-velocities $T^r Q$ and the r-jet prolongation $J^r Y = \mathbb{R} \times T^r Q$ (cf. Section 3). Recall that a *Lagrangian* (of order r) for Y is a 1-form $\lambda \in \Omega^r_{1,\mathbb{R}} W$, where W is an open set in Y; we shall consider open sets of the form $\mathbb{R} \times W_0$. Due to the Cartesian product structure of Y and the existence of the (global) coordinate t and (global) volume element dt on the base $\mathbb{R}$, λ can be written as

$$\lambda = \mathscr{L} dt,$$

where $\mathscr{L} : \mathbb{R} \times T^r Q \to \mathbb{R}$ is a (globally defined) *Lagrange function.* Recall that the variational integral, related with λ, satisfies

$$\int_{\Omega} J^r \gamma^* \lambda = \int_{\Omega} J^s \gamma^* \Theta_{\lambda},$$

where Θ_{λ} is the Lepage equivalent of λ (Section 2, (4)).

In the following main theorem we consider Lagrangians λ on $\mathbb{R} \times \mathrm{Imm} T^r Q$, satisfying the *Zermelo condition*

$$\frac{\partial \mathscr{L}}{\partial y^{\sigma}_{(1)}} y^{\sigma}_{(1)} + 2 \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{(2)}} y^{\sigma}_{(2)} + 3 \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{(3)}} y^{\sigma}_{(3)} + \ldots + r \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{(r)}} y^{\sigma}_{(r)} = \mathscr{L},$$
$$\frac{\partial \mathscr{L}}{\partial y^{\sigma}_{(r-k+1)}} y^{\sigma}_{(1)} + \binom{r-k+2}{1} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{(r-k+2)}} y^{\sigma}_{(2)} + \binom{r-k+3}{2} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{(r-k+3)}} y^{\sigma}_{(3)}$$
$$+ \ldots + \binom{r}{k-1} \frac{\partial \mathscr{L}}{\partial y^{\sigma}_{(r)}} y^{\sigma}_{(k)} = 0, \quad k = 1, 2, \ldots, r-1. \tag{14}$$

We wish to determine the corresponding Lepage equivalent Θ_{λ}.

Theorem 8. *Let λ be a Lagrangian of order r on $Y = \mathbb{R} \times Q$, expressed as $\lambda = \mathscr{L} dt$, with the Lagrange function $\mathscr{L} : \mathbb{R} \times \mathrm{Imm} T^r Q \to \mathbb{R}$. If $\mathscr{L}$ satisfies the Zermelo condition, then the fundamental Lepage equivalent Θ_{λ} is given by*

$$\Theta_{\lambda} = \sum_{k=0}^{r-1} P^{(k+1)}_{\sigma} dy^{\sigma}_{(k)}, \tag{15}$$

where

$$P_\sigma^{(k+1)} = \sum_{l=0}^{r-1-k} (-1)^l \frac{d^l}{dt^l} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(k+l+1)}}.$$

PROOF. 1. To understand our general method, we first prove formula (15) independently for $r \leq 3$. By definition, we have in this case

$$\Theta_\lambda = \mathscr{L} dt + \left(\frac{\partial \mathscr{L}}{\partial \dot{y}^\sigma} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} + \frac{d^2}{dt^2} \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \right) \omega^\sigma + \left(\frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \right) \dot{\omega}^\sigma + \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \ddot{\omega}^\sigma.$$

Θ_λ can also be written as

$$\begin{aligned} \Theta_\lambda = -\mathscr{H} dt &+ \left(\frac{\partial \mathscr{L}}{\partial \dot{y}^\sigma} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} + \frac{d^2}{dt^2} \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \right) dy^\sigma \\ &+ \left(\frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \right) d\dot{y}^\sigma + \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} d\ddot{y}^\sigma, \end{aligned}$$

where

$$\begin{aligned} -\mathscr{H} = \mathscr{L} &- \left(\frac{\partial \mathscr{L}}{\partial \dot{y}^\sigma} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} + \frac{d^2}{dt^2} \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \right) \dot{y}^\sigma \\ &- \left(\frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \right) \ddot{y}^\sigma - \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \dddot{y}^\sigma. \end{aligned} \tag{16}$$

The Zermelo condition reduces in this case to the equations

$$\begin{aligned} &\frac{\partial \mathscr{L}}{\partial \dot{y}^\sigma} \dot{y}^\sigma + 2 \frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} \ddot{y}^\sigma + 3 \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \dddot{y}^\sigma = \mathscr{L}, \\ &\frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \dot{y}^\sigma = 0, \quad \frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} \dot{y}^\sigma + 3 \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \ddot{y}^\sigma = 0. \end{aligned} \tag{17}$$

To substitute from (4) to (16) we express the function $-\mathscr{H}$ in terms of formal derivatives:

$$\begin{aligned} -\mathscr{H} = \mathscr{L} &- \frac{\partial \mathscr{L}}{\partial \dot{y}^\sigma} \dot{y}^\sigma - 2 \frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} \ddot{y}^\sigma - 3 \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \dddot{y}^\sigma \\ &+ \frac{d}{dt} \left(\frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} \dot{y}^\sigma + 3 \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \ddot{y}^\sigma \right) - \frac{d^2}{dt^2} \left(\frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \dot{y}^\sigma \right). \end{aligned}$$

Now we see that (4) implies

$$\mathscr{H} = 0$$

and

$$\begin{aligned} \Theta_\lambda = &\left(\frac{\partial \mathscr{L}}{\partial \dot{y}^\sigma} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} + \frac{d^2}{dt^2} \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \right) dy^\sigma \\ &+ \left(\frac{\partial \mathscr{L}}{\partial \ddot{y}^\sigma} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} \right) d\dot{y}^\sigma + \frac{\partial \mathscr{L}}{\partial \dddot{y}^\sigma} d\ddot{y}^\sigma, \end{aligned} \tag{18}$$

as required.

2. Now consider the general case. In view of Theorem 7, it is sufficient to show that the Zermelo condition (14) implies $\mathscr{H} = 0$. Thus, we want to show that the function

$$\mathscr{L} - \sum_{1\le i\le r} \binom{i}{i-1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i)}} y^\sigma_{(i)} + \sum_{1\le l\le r-1} (-1)^{l-1} \frac{d^l}{dt^l} \sum_{1\le i\le r-l} \binom{i+l}{i-1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i+l)}} y^\sigma_{(i)} \tag{19}$$

vanishes. The first one of equations (14) can be written as

$$\mathscr{L} - \sum_{1\le i\le r} \binom{i}{i-1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i)}} y^\sigma_{(i)} = 0.$$

Substituting $l = r - k$, where $l = 1, 2, \dots, r-2, r-1$, into the next equations (14), we get the conditions

$$\begin{aligned} &\frac{\partial \mathscr{L}}{\partial y^\sigma_{(r-k+1)}} y^\sigma_{(1)} + \binom{r-k+2}{1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(r-k+2)}} y^\sigma_{(2)} + \dots + \binom{r}{k-1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(k)} \\ &\quad = \frac{\partial \mathscr{L}}{\partial y^\sigma_{(l+1)}} y^\sigma_{(1)} + \binom{l+2}{1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(l+2)}} y^\sigma_{(2)} + \dots + \binom{r}{r-l-1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(r)}} y^\sigma_{(r-l)} \\ &\quad = \sum_{1\le i\le r-l} \binom{i+l}{i-1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{(i+l)}} y^\sigma_{(i)} = 0. \end{aligned}$$

This implies, however, that expression (19) vanishes. □

Remark 2. If $\mathscr{L}$ does not depend on $\ddot{y}^\sigma$ and $\dddot{y}^\sigma$, then Θ_λ (18) reduces to the well-known *Hilbert form*

$$\Theta_\lambda = \frac{\partial \mathscr{L}}{\partial \dot{y}^\sigma} dy^\sigma.$$

Formula (18) also describes the second- and third-order generalisations of the Hilbert form; (15) can be considered as the higher-order generalisation. All these forms inherit properties of a Lepage form, providing this way a geometric tool for further analysis of local and global structure of variational theory (see e.g. Krupka and Saunders [7]).

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DEMETER KRUPKA
LEPAGE RESEARCH INSTITUTE
783 42 SLATINICE
CZECH REPUBLIC

AND

SCHOOL OF MATHEMATICS
BEIJING INSTITUTE OF TECHNOLOGY
5 SOUTH ZHONGGUANCUN STREET
HAIDIAN ZONE, BEIJING 100081
CHINA

AND

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF OSTRAVA
70103 OSTRAVA
CZECH REPUBLIC

E-mail: demeter.krupka@lepageri.eu

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