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Homogeneous differential equations and the inverse problem of the calculus of variations

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Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. We study second order differential equations considering positive homogeneity of a general degree of the equations and of functions connected with them (like, for example, metrics or Lagrangians). Special attention is payed to semi-variational equations and to relationships between homogeneity properties and variationality (existence of local Lagrangians.

1. Introduction

In this paper we shall be concerned with systems of second order ordinary differential equations

$$B_{jk}(x^i, \dot{x}^i)\ddot{x}^k + A_j(x^i, \dot{x}^i) = 0, \quad 1 \leq j \leq n, \tag{1.1}$$

for curves $\gamma : I \to U$, $\gamma(t) = (x^i(t))$, $1 \leq i \leq n$, where I is an open interval in $\mathbb{R}$ and U is an open subset of an n-dimensional smooth manifold M (here and in what follows summation over repeated indices applies). In a geometric setting, equations of this kind can be modelled by a differential two-form, so-called dynamical form, E, on the second jet bundle $J^2(\mathbb{R} \times M) \to \mathbb{R}$ of the fibered manifold $\mathbb{R} \times M \to \mathbb{R}$. We remind the reader the identification of $J^1(\mathbb{R} \times M)$

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with $\mathbb{R} \times TM$ and of $J^2(\mathbb{R} \times M)$ with $\mathbb{R} \times T^2M$. We denote by T^oM the TM with the zero section excluded. Next we denote by t the global coordinate on $\mathbb{R}$, by (x^i), $1 \leq i \leq n$, local coordinates on M, and by $(t, x^i, \dot{x}^i)$ and $(t, x^i, \dot{x}^i, \ddot{x}^i)$ the associated coordinates on $\mathbb{R} \times TM$ and $\mathbb{R} \times T^2M$, respectively. In such coordinates,

$$E = E_j dx^j \wedge dt, \quad \text{where } E_j = B_{jk}\ddot{x}^k + A_j \tag{1.2}$$

are functions on an open subset of $\mathbb{R} \times T^2M$. Then equations (1.1) can be expressed in an intrinsic form $E \circ J^2\hat{\gamma} = 0$, where $\hat{\gamma} : I \to \mathbb{R} \times M$ is a local section of the bundle $\mathbb{R} \times M \to \mathbb{R}$ (the graph of γ) and $J^2\hat{\gamma}$ is its second jet prolongation. We shall be interested in autonomous (time independent) equations, such that the components B and A do not depend explicitly on t. On the other hand, we put no a priori regularity assumption on the matrix B, so that our study concern both regular equations (representable by a semispray) and equations in implicit form.

Throughout the paper we assume that all mappings are locally defined (domains are open sets), and *smooth with a possible exception of points P where $\dot{x}^k(P) = 0$ for all $k = 1, \ldots, n$.*

In the theory of ordinary differential equations, in the calculus of variations, in differential geometry and in mechanics an important role is played by equations with certain (different) homogeneity properties. The most familiar examples of such equations appear in Riemannian and Finsler geometry, where the corresponding equations of interest are positively homogenous of degree 2, or 1. The aim of this paper is to study second order differential equations from a more general point of view, considering positive homogeneity of a general degree of equations and of functions connected with the equations (like, for example, metrics or Lagrangians). Attention is payed to relationships between homogeneity properties and variationality (existence of local Lagrangians). In this sense our results contribute to the recent investigations of geometric and variational properties of differential equations on Finsler manifolds and on manifolds with variational metrics, and to studies of the structure of variational and semi-variational equations (see eg. [1], [2], [3], [4], [7], [8], [9], [10], [11], [12], [14], [15], [16], [18], [19]).

As mentioned above, we shall deal with second order functions with homogeneity properties concerning the first and second derivatives. In the existing literature one can find different concepts of positive homogeneity for higher order functions, appearing as a generalization of the (common) first order case. For second order functions one has to distinguish two levels of positive homogeneity. To avoid confusion, we shall use the following terminology:

Definition 1.1. Let $F(t, x^i, \dot{x}^i, \ddot{x}^i)$ be a function such that $\dot{x}^k \neq 0$, for at least one $k = 1, \dots, n$. F is called *first level positively homogeneous of degree c in velocities and accelerations* if

$$F(t, x^i, a\dot{x}^i, a^2\ddot{x}^i) = a^c\, F(t, x^i, \dot{x}^i, \ddot{x}^i) \tag{1.3}$$

for all $a > 0$. F is called *second level positively homogeneous of degree c in velocities and accelerations* if

$$F(t, x^i, a\dot{x}^i, a^2\ddot{x}^i + b\dot{x}^i) = a^c\, F(t, x^i, \dot{x}^i, \ddot{x}^i) \tag{1.4}$$

for all $a > 0$ and all $b \in \mathbb{R}$.

The first level positive homogeneity of F is equivalent with differential conditions

$$\frac{\partial F}{\partial \dot{x}^i}\dot{x}^i + 2\frac{\partial F}{\partial \ddot{x}^i}\ddot{x}^i = c\,F\,, \tag{1.5}$$

while the second level positive homogeneity is equivalent with the conditions

$$\frac{\partial F}{\partial \dot{x}^i}\dot{x}^i + 2\frac{\partial F}{\partial \ddot{x}^i}\ddot{x}^i = c\,F, \quad \frac{\partial F}{\partial \ddot{x}^i}\dot{x}^i = 0. \tag{1.6}$$

In the case $c = 1$ the latter conditions are called *Zermelo conditions*. As it is known, Zemelo conditions have a deep geometric meaning: solutions of differential equations whose left-hand sides satisfy the Zermelo conditions are invariant under orientation preserving reparametizations [18]. Remarkably, differential equations of this kind appear for example in Riemannian and Finsler geometry as equations for geodesics, or in physics as equations of motion for relativistic particles.

The plan of the paper is as follows: In Section 2 we introduce semi-variational equations. In Section 3 we study properties of differential equations connected with different homogeneity assumptions. Main results are as follows: We find the structure of semi-variational equations which are positively homogeneous of degree $c \neq 0, 1$ (Theorem 3.3). Next, we give a proof that positive homogeneity of degree $c \neq 0, 1$ of the functions A_i partially substitutes variationality in the sense that a part of the Helmholtz conditions [6] for such equations is redundant (Theorem 3.4, Corollary 3.5). This is a generalization of a similar result known for the case $c = 2$ ([1], [12], [14]). We also disprove the conjecture [14] that this property holds for any c. Further we find an explicit structure of variational equations which are first level positively homogeneous of degree c for different values of c (Theorems 3.6, 3.10 and 3.11). We also find all c-homogeneous first order Lagrangians for c-homogeneous variational equations and show that for $c \neq 0, 1$

such Lagrangian is unique. We stress that when speaking about Lagrangians we have in mind local Lagrangians (unless otherwise stated). The last section is devoted to second level positively homogeneous of degree 1 second order differential equations, which are a special case of positively homogeneous equations studied in the previous section. As already mentioned, solutions of such equations are invariant under orientation preserving reparametrizations. We show that for semi-variational equations and variational equations the Zermelo conditions simplify (Theorem 4.3). For the case of variational equations (Finsler geometry) we show that the concepts of first-level and second-level positive 1-homogeneity coincide, and that the class of the corresponding first order positively 1-homogeneous Lagrangians contains the Engels Lagrangian (Theorem 4.7). Finally we give necessary and sufficient conditions for ODEs to be variational and positively 1-homogeneous (Helmholtz conditions in the homogeneous background) and clarify the structure of these equations (Theorems 4.9 and 4.10).

2. Semi-variational equations

Definition 2.1. Equations (1.1) are called *semi-variational* if their components B_{ij} at the second derivatives satisfy the following symmetry and integrability conditions respectively:

$$B_{ik} = B_{ki}, \quad \frac{\partial B_{ik}}{\partial \dot{x}^j} = \frac{\partial B_{ij}}{\partial \dot{x}^k}. \tag{2.1}$$

We note that, as shown in [11], the property of being semi-variational intrinsically means that the Lepage equivalent of the corresponding dynamical form E is projectable onto J^1Y.

Theorem 2.2. *Equations* (1.1) *are semi-variational if and only if there exist functions L (Lagrangian) and $\Phi = (\Phi_j)$ (force), depending on $(x^i, \dot{x}^i)$, such that*

$$E_j = \frac{\partial L}{\partial x^j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^j} - \Phi_j. \tag{2.2}$$

A solution (L, Φ) is non-unique; namely, L is determined up to a function affine in velocities $(\dot{x}^j)$, and Φ is determined up to a Lorentz-like force.

Proof. One way is obvious, because if the equations take the form

$$\frac{\partial L}{\partial x^j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^j} = \Phi_j, \tag{2.3}$$

then B is the negative Hessian matrix of L.

Conversely, the integrability conditions of (2.1) guarantee the existence of functions $p_i(x^j, \dot{x}^j)$ such that

$$B_{ij} = -\frac{\partial p_i}{\partial \dot{x}^j} \tag{2.4}$$

(the negative sign is chosen to keep relationship with conventions in classical mechanics). The symmetry conditions of (2.1) then give

$$\frac{\partial p_i}{\partial \dot{x}^k} = \frac{\partial p_k}{\partial \dot{x}^i}, \tag{2.5}$$

which again is an integrability condition, ensuring the existence of a function $L(x^j, \dot{x}^j)$ such that

$$p_i = \frac{\partial L}{\partial \dot{x}^i}, \quad B_{ij} = -\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}. \tag{2.6}$$

Functions Φ_i, $1 \leq i \leq n$, are then obtained by putting $\Phi_i = \mathcal{E}_i(L) - E_i$, where $\mathcal{E}_i(L)$ are the Euler–Lagrange expressions of L.

The nonuniqueness of L follows immediately from (2.6). If L, L' are two Lagrangians giving the same matrix B then $L' = L + V_i \dot{x}^i + U$, where V_i and U do not depend upon velocities. Since $\Phi'_i = \mathcal{E}_i(L') - E_i$, we have

$$\Phi'_i - \Phi_i = \mathcal{E}_i(L') - \mathcal{E}_i(L) = \mathcal{E}_i(L' - L) = \left(\frac{\partial V_k}{\partial x^i} - \frac{\partial V_i}{\partial x^k} \right) \dot{x}^k + \frac{\partial U}{\partial x^i}, \tag{2.7}$$

(i.e. the difference is a Lorentz-type force), proving our assertion. □

Remarkably, every system of semi-variational equations has a *canonical Lagrangian*: In the class of all admissible pairs (L, Φ) there is a distinguished one, represented by a Lagrangian determined by the matrix $B = (B_{ij})$ [10]. It is given locally by the formula

$$L = -\dot{x}^i \dot{x}^j \int_0^1 \left(\int_0^1 (B_{ij} \circ \bar{\chi}) dv \right) \circ \bar{\chi} \, v \, dv \tag{2.8}$$

where (for a proper open set $W \subset M$),

$$\bar{\chi} : [0,1] \times TW \ni (v, (x^i, \dot{x}^i)) \to (x^i, v\dot{x}^i) \in TW. \tag{2.9}$$

The above coordinate formula takes a nice geometric form in terms of the Poincaré homotopy operator $\bar{\mathcal{P}}$ associated with the map $\bar{\chi}$ as follows:

$$L = -\bar{\mathcal{P}}^2(B). \tag{2.10}$$

The canonical Lagrangian is global if E is global (see [10]).

It is worth notice that if B is defined everywhere on TW with the exception of the zero section, $\bar{\mathcal{P}}^2(B)$ still can be defined by extending B to the zero section (the extension even need not be continuous), and the value of the integral does not depend on the extension.

Apparently, if $-B = g$ is a Riemannian metric on M then L (2.8) is the kinetic energy, $T = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j$, and the same assertion can be proved also for the case when g is a Finsler metric [10].

Theorem 2.3. *Given semi-variational equations as above, assume that the coefficients A_j, B_{jk} satisfy the identities*

$$\frac{\partial A_i}{\partial \dot{x}^k} + \frac{\partial A_k}{\partial \dot{x}^i} = 2\frac{\partial B_{ik}}{\partial x^j}\dot{x}^j. \tag{2.11}$$

Then the Hessian matrix of A_j is completely determined by the B_{jk}'s as follows:

$$\frac{\partial^2 A_i}{\partial \dot{x}^j \partial \dot{x}^k} = G_{ijk} = 2\Gamma_{ijk} + \frac{\partial^2 B_{jk}}{\partial x^p \partial \dot{x}^i}\dot{x}^p, \tag{2.12}$$

where Γ_{ijk} are the formal Christoffel symbols of B, i.e. functions defined by

$$\Gamma_{ijk} = \Gamma_{ikj} = \frac{1}{2}\left(\frac{\partial B_{ij}}{\partial x^k} + \frac{\partial B_{ik}}{\partial x^j} - \frac{\partial B_{jk}}{\partial x^i}\right). \tag{2.13}$$

Proof. The proof is obtained easily by differentiating relation (2.11) with respect to $\dot{x}$, cycling the indices and summing up, accounting the properties of B. □

With help of the Poincaré homotopy operator $\bar{\mathcal{P}}$ defined above a solution of equation (2.12) takes the form

$$A_i = \bar{\mathcal{P}}^2(G_i) \tag{2.14}$$

where G_i, $1 \le i \le n$, are symmetric matrices with components G_{ijk} defined by the right-hand sides of (2.12). Again, the solution is determined up to a function affine in velocities. Summarizing, we have:

Corollary 2.4. *Semi-variational equations satisfying additional condition* (2.11) *have the following form:*

$$B_{ik}\ddot{x}^k + \bar{\mathcal{P}}^2(G_i) = \Phi_i \quad \text{where } \Phi_i \text{ are affine in velocities.} \tag{2.15}$$

Theorem 2.5. *The left-hand sides* $B_{ik}\ddot{x}^k + \bar{\mathcal{P}}^2(G_i)$ *of equations* (2.15) *are Euler–Lagrange expressions of the Lagrangian* $L = -\bar{\mathcal{P}}^2(B)$.

Proof. Computing the Euler–Lagrange expressions $\mathcal{E}_i(L)$ of $L = -\bar{\mathcal{P}}^2(B)$ we obtain the corresponding functions $A_i(L) = \mathcal{E}_i(L) - B_{ik}\ddot{x}^k$ in the following form (see [10], Theorem 6.7 and Appendix therein)

$$\begin{aligned}
A_i(L) &= \left[\frac{1}{2}\int_0^1 \left(\frac{\partial B_{ij}}{\partial x^k} + \frac{\partial B_{ik}}{\partial x^j} - 2\frac{\partial B_{jk}}{\partial x^i}\right)\circ\bar{\chi}\,dv + \int_0^1\left(\frac{\partial B_{jk}}{\partial x^i}\circ\bar{\chi}\right)v\,dv\right]\dot{x}^j\dot{x}^k\\
&= \left[\int_0^1 (2\Gamma_{ijk}\circ\bar{\chi})\,dv - \int_0^1\left(\frac{\partial B_{ij}}{\partial x^k}\circ\bar{\chi}\right)dv + \int_0^1\left(\frac{\partial B_{jk}}{\partial x^i}\circ\bar{\chi}\right)v\,dv\right]\dot{x}^j\dot{x}^k\\
&= \left[\int_0^1 (2\Gamma_{ijk}\circ\bar{\chi})\,dv - \int_0^1 (2\Gamma_{ijk}\circ\bar{\chi})\,v\,dv - \int_0^1\left(\frac{\partial B_{ij}}{\partial x^k}\circ\bar{\chi}\right)dv\right.\\
&\quad \left. + \int_0^1\left(\frac{\partial B_{ij}}{\partial x^k} + \frac{\partial B_{ik}}{\partial x^j}\right)\circ\bar{\chi}\,v\,dv\right]\dot{x}^j\dot{x}^k\\
&= \left[\int_0^1\left(\int_0^1 (2\Gamma_{ijk}\circ\bar{\chi})dv\right)\circ\bar{\chi}\,v\,dv - \int_0^1\left(\int_0^1\left(\frac{\partial B_{ij}}{\partial x^k}\circ\bar{\chi}\right)dv\right)\circ\bar{\chi}\,v\,dv\right.\\
&\quad \left. + \int_0^1\left(\frac{\partial B_{ij}}{\partial x^k}\circ\bar{\chi}\right)v\,dv\right]\dot{x}^j\dot{x}^k = \dot{x}^j\dot{x}^k\int_0^1\left(\int_0^1 (G_{ijk}\circ\bar{\chi})dv\right)\circ\bar{\chi}\,v\,dv\\
&= \bar{\mathcal{P}}^2(G_i),
\end{aligned} \tag{2.16}$$

since with the use of the properties of B, and after some computations we get

$$\begin{aligned}
\dot{x}^j\dot{x}^k &\left[\int_0^1\left(\frac{\partial B_{ij}}{\partial x^k}\circ\bar{\chi}\right)v\,dv\right.\\
&\left. - \int_0^1\left(\int_0^1\left(\left(\frac{\partial B_{ij}}{\partial x^k} + \frac{\partial^2 B_{jk}}{\partial x^p\partial\dot{x}^i}\dot{x}^p\right)\circ\bar{\chi}\right)dv\right)\circ\bar{\chi}\,v\,dv\right] = 0.
\end{aligned} \tag{2.17}$$

□

Remark 2.6. Recall that in case of *regular equations*, i.e. such that the matrix B is regular, and, consequently, the equations are represented by a semispray Γ on $J^1(\mathbb{R}\times M)$, the condition (2.11) has the intrinsic form

$$\mathcal{L}_\Gamma B = 0, \tag{2.18}$$

i.e. the Lie derivative along Γ of the morphism (generalized metric) B vanishes. This condition is a generalization to semispray connections of the classical condition on metrizability of a linear connection (see [10]).

Remark 2.7. Note that semi-variational equations are variational if and only if they satisfy conditions (2.11) in the above theorem plus one additional set of conditions as follows:

$$\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} = \frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial A_i}{\partial \dot{x}^k} - \frac{\partial A_k}{\partial \dot{x}^i}\right)\dot{x}^j. \tag{2.19}$$

However, then (2.19) reduce to conditions concerning only Φ_i (which, as we already know, is affine in $\dot{x}$), and mean that Φ is a Lorenz-like force (see [7]).

We remind the reader that (2.1), (2.11) and (2.19) are called *Hemlholtz conditions.*

3. Semi-variational equations with homogeneous coefficients

Starting from this section we shall consider all functions defined and smooth on open subsets such that, at each point P, $\dot{x}^k(P) \neq 0$, for at least one $k = 1, \ldots, n$.

As above, we shall consider time-independent second-order ODE's of the form (1.1).

Recall that a first-order function $F(x^i, \dot{x}^i)$ is called *positively homogeneous of degree c in velocities* if

$$\frac{\partial F}{\partial \dot{x}^k}\dot{x}^k = cF. \tag{3.1}$$

Differentiating this relation we can see that

$$\frac{\partial^2 F}{\partial \dot{x}^j \partial \dot{x}^k}\dot{x}^k = (c-1)\frac{\partial F}{\partial \dot{x}^j}, \quad \text{and} \quad \frac{\partial^2 F}{\partial \dot{x}^j \partial \dot{x}^k}\dot{x}^j \dot{x}^k = c(c-1)F. \tag{3.2}$$

For second order functions the concept of positive homogeneity is generalized as follows (cf. Definition 1.1 and the comments around): $F(x^i, \dot{x}^i, \ddot{x}^i)$ is called *first level positively homogeneous of degree c in the first and second derivatives* if

$$\frac{\partial F}{\partial \dot{x}^k}\dot{x}^k + 2\frac{\partial F}{\partial \ddot{x}^k}\ddot{x}^k = cF. \tag{3.3}$$

Differential equations $E_i(x^k, \dot{x}^k, \ddot{x}^k) = 0$ are called *first level positively homogeneous of degree c* if their left-hand sides E_i are first level positively homogeneous functions of degree c in the variables $\dot{x}^k$ and $\ddot{x}^k$, $1 \leq k \leq n$.

For brevity, having in mind the above definitions we shall speak simply about “homogeneous functions of degree c”, or “c-homogeneous functions”.

First, let us prove the following important consequence of homogeneity of the morphism B:

Theorem 3.1. *If B is homogeneous of degree $c-2$ then the canonical Lagrangian $L=-\bar{\mathcal{P}}^2(B)$ is homogeneous of degree c.*

Proof. By a direct computation we have

$$\frac{\partial L}{\partial \dot{x}^k}\dot{x}^k = -2\dot{x}^i\dot{x}^k\int_0^1\left(\int_0^1 (B_{ik}\circ\bar{\chi})dv\right)\circ\bar{\chi}\,v\,dv$$
$$-\dot{x}^i\dot{x}^j\int_0^1\left(\int_0^1\left(\frac{\partial B_{ij}}{\partial \dot{x}^k}\dot{x}^k\right)\circ\bar{\chi}\,dv\right)\circ\bar{\chi}\,v\,dv = 2L+(c-2)L=cL \tag{3.4}$$

□

Now, let us discuss homogeneity in the context of differential equations. From the definition we easily obtain:

Theorem 3.2. *Equations* (1.1) *are homogeneous of degree c if and only if A_i are homogeneous of degree c and B_{ij} are homogeneous of degree $c-2$.*

Proof. Substituting $E_i = A_i + B_{ij}\ddot{x}^j$ into (3.3) gives us

$$\frac{\partial A_i}{\partial \dot{x}^k}\dot{x}^k + \frac{\partial B_{ij}}{\partial \dot{x}^k}\dot{x}^k\ddot{x}^j = cA_i + (c-2)B_{ij}\ddot{x}^j. \tag{3.5}$$

Since this is a polynomial in $\ddot{x}$, we can see that A_i are homogeneous of degree c and B_{ij} are homogeneous of degree $c-2$.

Conversely, if A_i are homogeneous of degree c and B_{ij} are homogeneous of degree $c-2$ then

$$\frac{\partial E_i}{\partial \dot{x}^k}\dot{x}^k + 2\frac{\partial E_i}{\partial \ddot{x}^k}\ddot{x}^k = \frac{\partial A_i}{\partial \dot{x}^k}\dot{x}^k + \frac{\partial B_{ij}}{\partial \dot{x}^k}\dot{x}^k\ddot{x}^j + 2B_{ik}\ddot{x}^k = cE_i, \tag{3.6}$$

as desired. □

The relationship between coefficients A_i and B_{ij} of semi-variational equations given by Theorem 2.3 becomes of a particular importance if the coefficients are homogeneous functions:

Theorem 3.3. *Let $A_i + B_{ij}\ddot{x}^j = 0$, $1\le i\le n$, be a system of semi-variational equations satisfying condition* (2.11). *Assume that the functions A_i are homogeneous of degree $c\neq 0,1$. Then*

$$A_i = \frac{1}{c(c-1)}G_{ijk}\dot{x}^j\dot{x}^k = \frac{1}{c(c-1)}\left(\frac{\partial B_{ij}}{\partial x^k} + \frac{\partial B_{ik}}{\partial x^j} - \frac{\partial B_{jk}}{\partial x^i} + \frac{\partial^2 B_{jk}}{\partial x^p\partial \dot{x}^i}\dot{x}^p\right)\dot{x}^j\dot{x}^k. \tag{3.7}$$

Moreover, the functions G_{ijk} (2.12) *satisfy the following identity:*

$$\frac{\partial G_{ipr}}{\partial \dot{x}^k}\dot{x}^p\dot{x}^r = (c-2)G_{ikr}\dot{x}^r. \tag{3.8}$$

Proof. Formula (3.2) and Theorem 2.3 immediately give us (3.7), so that it remains to prove (3.8). Differentiating (3.7) we get

$$\frac{\partial A_i}{\partial \dot{x}^k} = \frac{1}{c(c-1)} \frac{\partial G_{ipr}}{\partial \dot{x}^k} \dot{x}^p \dot{x}^r + \frac{2}{c(c-1)} G_{ikr} \dot{x}^r. \tag{3.9}$$

On the other hand, the first formula of (3.2), applied to A_i, and (2.12) yield

$$\frac{\partial A_i}{\partial \dot{x}^k} = \frac{1}{c-1} \frac{\partial^2 A_i}{\partial \dot{x}^k \partial \dot{x}^r} \dot{x}^r = \frac{1}{c-1} G_{ikr} \dot{x}^r. \tag{3.10}$$

Now, formula (3.8) easily follows. □

Surprisingly, semi-variational equations as above have the following much stronger property, so far known only for the case $c = 2$ (see [1], [12], [14]); as we shall see later, for $c = 0$ and $c = 1$ a similar result no longer holds true.

Theorem 3.4. *Every system of semi-variational equations satisfying condition* (2.11), *and such that the functions A_i are homogeneous of degree $c \neq 0, 1$, is variational (meaning that it satisfies all Helmholtz conditions).*

Proof. One has to check that the last Helmholtz condition (2.19) is redundant. It is worth note here that for regular equations (i.e. such that $\det B \neq 0$) this condition expresses properties of the Jacobi endomorphism introduced in [13].

This, of course, can be done by substituting the A_i in the form (3.7) into (2.19); the result is then obtained after quite long and boring calculations. Here we shall present another proof based on the structure of considered semi-variational equations (Corollary 2.4 and Theorem 2.5).

By the corollary,

$$A_i = \bar{\mathcal{P}}^2(G_i) - \Phi_i = \frac{1}{c(c-1)} G_{ijk} \dot{x}^j \dot{x}^k, \tag{3.11}$$

where Φ_i are affine in velocities. Differentiating the A_i, we get on one hand using (2.12)

$$\begin{aligned} \frac{\partial^2 A_i}{\partial \dot{x}^p \partial \dot{x}^r} &= \frac{1}{c(c-1)} \frac{\partial^2 G_{ijk}}{\partial \dot{x}^p \partial \dot{x}^r} \dot{x}^j \dot{x}^k + \frac{2}{c(c-1)} \Big(\frac{\partial G_{irk}}{\partial \dot{x}^p} + \frac{\partial G_{ipk}}{\partial \dot{x}^r} \Big) \dot{x}^k + \frac{2}{c(c-1)} G_{ipr} \\ &= G_{ipr} \end{aligned} \tag{3.12}$$

so that

$$\frac{\partial^2 G_{ijk}}{\partial \dot{x}^p \partial \dot{x}^r} \dot{x}^j \dot{x}^k + 2 \left(\frac{\partial G_{irk}}{\partial \dot{x}^p} + \frac{\partial G_{ipk}}{\partial \dot{x}^r} \right) \dot{x}^k = (c(c-1) - 2) G_{ipr}, \tag{3.13}$$

and on the other hand, accounting the above identity for the G's,

$$
\begin{aligned}
\frac{\partial^2 A_i}{\partial \dot{x}^p \partial \dot{x}^r} &= 2 \int_0^1 \left(\int_0^1 (G_{ipr} \circ \bar{\chi}) dv \right) \circ \bar{\chi}\, v\, dv \\
&\quad + 2\dot{x}^k \int_0^1 \left(\int_0^1 \Big(\frac{\partial G_{ipk}}{\partial \dot{x}^r} \circ \bar{\chi} \Big) v\, dv \right) \circ \bar{\chi}\, v^2\, dv \\
&\quad + 2\dot{x}^k \int_0^1 \left(\int_0^1 \Big(\frac{\partial G_{irk}}{\partial \dot{x}^p} \circ \bar{\chi} \Big) v\, dv \right) \circ \bar{\chi}\, v^2\, dv \\
&\quad + \dot{x}^j \dot{x}^k \int_0^1 \left(\int_0^1 \Big(\frac{\partial^2 G_{ijk}}{\partial \dot{x}^p \partial \dot{x}^r} \circ \bar{\chi} \Big) v^2\, dv \right) \circ \bar{\chi}\, v^3\, dv \\
&= c(c-1) \int_0^1 \left(\int_0^1 (G_{ipr} \circ \bar{\chi}) dv \right) \circ \bar{\chi}\, v\, dv, \qquad (3.14)
\end{aligned}
$$

since Φ is affine in velocities. So, we have obtained

$$
G_{ipr} = c(c-1) \int_0^1 \Big(\int_0^1 (G_{ipr} \circ \bar{\chi}) dv \Big) \circ \bar{\chi}\, v\, dv, \qquad (3.15)
$$

meaning that

$$
\begin{aligned}
\Phi_i &= \bar{\mathcal{P}}^2(G_i) - \frac{1}{c(c-1)} G_{ijk} \dot{x}^j \dot{x}^k \\
&= \left(\int_0^1 \left(\int_0^1 (G_{ijk} \circ \bar{\chi}) dv \right) \circ \bar{\chi}\, v\, dv - \frac{1}{c(c-1)} G_{ijk} \right) \dot{x}^j \dot{x}^k = 0. \qquad (3.16)
\end{aligned}
$$

Hence A_i are equal to $\bar{\mathcal{P}}^2(G_i)$, which by Theorem 2.5 means that $B_{ij}\ddot{x}^j + A_i$ are Euler–Lagrange expressions of the Lagrangian $-\bar{\mathcal{P}}^2(B)$; we are done. □

Summarizing (and reformulating), we have obtained

Corollary 3.5. *Let E be a dynamical form with components affine in the second derivatives, $E_i = A_i + B_{ij}\ddot{x}^j$, and with A_i homogeneous of degree $c \neq 0, 1$. E_i are variational if and only if*

$$
B_{ij} = B_{ji}, \quad \frac{\partial B_{ij}}{\partial \dot{x}^k} = \frac{\partial B_{ik}}{\partial \dot{x}^j}, \quad \frac{\partial A_i}{\partial \dot{x}^j} + \frac{\partial A_j}{\partial \dot{x}^i} = 2 \frac{\partial B_{ij}}{\partial x^k} \dot{x}^k \qquad (3.17)
$$

(i.e. one of the Helmholtz conditions – that one for the Jacobi endomorphism – is redundant).

A corresponding Lagrangian is $L = -\bar{\mathcal{P}}^2(B)$, and the structure of the equations is as follows:

$$
B_{ij}\ddot{x}^j + \frac{1}{c(c-1)} G_{ijk} \dot{x}^j \dot{x}^k = 0, \qquad (3.18)
$$

where

$$G_{ijk} = \frac{\partial B_{ij}}{\partial x^k} + \frac{\partial B_{ik}}{\partial x^j} - \frac{\partial B_{jk}}{\partial x^i} + \frac{\partial^2 B_{jk}}{\partial x^p \partial \dot{x}^i} \dot{x}^p. \tag{3.19}$$

The G_{ijk} *satisfy the identity* (3.8).

Combining Corollary 3.5, Theorem 3.2 and Theorem 3.1 we immediately get the following strong result:

Theorem 3.6. *Let* E *be a dynamical form with components affine in the second derivatives,* $E_i = A_i + B_{ij}\ddot{x}^j$*, and homogeneous of degree* $c \neq 0, 1$*.* E_i *are variational if and only if*

$$B_{ij} = B_{ji}, \quad \frac{\partial B_{ij}}{\partial \dot{x}^k} = \frac{\partial B_{ik}}{\partial \dot{x}^j}, \quad \frac{\partial A_i}{\partial \dot{x}^j} + \frac{\partial A_j}{\partial \dot{x}^i} = 2\frac{\partial B_{ij}}{\partial x^k}\dot{x}^k. \tag{3.20}$$

If the variationality conditions are satisfied then the structure of the equations is as follows

$$B_{ij}\ddot{x}^j + \frac{1}{c-1}\left(\frac{1}{2}\left(\frac{\partial B_{ij}}{\partial x^k} + \frac{\partial B_{ik}}{\partial x^j}\right) - \frac{1}{c}\frac{\partial B_{jk}}{\partial x^i}\right)\dot{x}^j\dot{x}^k = 0. \tag{3.21}$$

A corresponding Lagrangian for E *is the canonical Lagrangian* $L = -\bar{\mathcal{P}}^2(B)$*. Moreover, the canonical Lagrangian is homogeneous of degree* c*, and it is a unique first order Lagrangian for* E *possessing this homogeneity property.*

Proof. It only remains to prove that the canonical Lagrangian is the unique first-order Lagrangian for E which is homogeneous of degree c. Hence, let L' be a Lagrangian equivalent with the canonical Lagrangian L (i.e. giving the same Euler–Lagrange expressions). Then $L' = L + df/dt$ for a function $f(t, x^i)$, so that it holds

$$\frac{\partial L'}{\partial \dot{x}^k}\dot{x}^k - cL' = (1-c)\frac{\partial f}{\partial x^k}\dot{x}^k - c\frac{\partial f}{\partial t}. \tag{3.22}$$

The right-hand side is a function affine in $\dot{x}$. Since $c \neq 0, 1$ by assumption, the homogeneity condition for L' gives $f = \text{const.}$ Hence $df/dt = 0$ and $L' = L$, proving the uniqueness. □

It is worth mention that identity (3.8) is of particular importance if equations (3.21) are equations for geodesics of a semispray in Finsler geometry. In this case $c = 2$, and $B = -g$ is a Finsler metric. Then

$$G_{ijk} = -\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right), \tag{3.23}$$

and (3.8) take the form

$$\frac{\partial G_{ipr}}{\partial \dot{x}^k}\dot{x}^p\dot{x}^r = 0. \tag{3.24}$$

An interesting situation arises when B does not depend upon velocities (as e.g. in Riemannian geometry). Then the homogeneity condition

$$\frac{\partial B_{ij}}{\partial \dot{x}^k}\dot{x}^k = (c-2)B_{ij} \tag{3.25}$$

gives us

$$(c-2)B = 0, \tag{3.26}$$

and we obtain:

Corollary 3.7. *If $E_i = A_i + B_{ij}\ddot{x}^j$, $1 \leq j \leq n$, are homogeneous of degree $c \neq 2$, and*

$$\frac{\partial B_{ij}}{\partial \dot{x}^k} = 0, \tag{3.27}$$

then $B = 0$, meaning that the equations are implicit first order differential equations, $A_i(x^k(t), \dot{x}^k(t)) = 0$. In other words, (nontrivially) second order differential equations with B independent upon velocities admit the only homogeneity property of being homogeneous of degree 2:

$$\frac{\partial E_i}{\partial \dot{x}^k}\dot{x}^k + 2\frac{\partial E_i}{\partial \ddot{x}^k}\ddot{x}^k = 2\,E_i, \quad \text{i.e.} \quad \frac{\partial A_i}{\partial \dot{x}^k}\dot{x}^k = 2A_i. \tag{3.28}$$

If, moreover, B is symmetric and condition (2.11) is satisfied, then the equations are variational with the unique homogeneous of degree 2 first order Lagrangian $L = -\frac{1}{2}B_{ij}\dot{x}^i\dot{x}^j$, and take the form

$$B_{ij}\ddot{x}^j + \Gamma_{ijk}\dot{x}^j\dot{x}^k = 0 \tag{3.29}$$

where Γ_{ijk} are formal Christoffel symbols of the (not necessarily regular) morphism B.

Remark 3.8. The above results can be applied to the case of equations in normal form

$$\ddot{x}^i + \Gamma^i = 0 \tag{3.30}$$

describing integral sections of a second order vector field Γ on TM. Such equations are called variational if there exists a matrix (B_{ij}) such that the "covariant equations"

$$B_{ij}(\ddot{x}^j + \Gamma^j) = 0 \tag{3.31}$$

satisfy the Helmholtz conditions. Usually in addition also $\det B \neq 0$ is required in order to guarantee that equations (3.30) and (3.31) are equivalent (having the same set of solutions). If we set $A_i = B_{ij}\Gamma^j$, we immediately obtain:

- If Γ^i are c-homogeneous, A_i are a-homogeneous and B_{ij} are b-homogeneous then
$$a = b + c. \tag{3.32}$$
- The degree of homogeneity of the contravariant force Γ^i and the covariant force A_i is the same if and only if the multiplier (B_{ij}) is 0-homogeneous.
- If the covariant equations (3.31) are required be homogeneous and of the same homogeneity degree, c, as the contravariant forces Γ^i then $a = c = 2$ (since $b = a - 2$ by Theorem 3.2), and $b = 0$.
- If the contravariant forces Γ^i are homogeneous of degree c and we require the multiplier (B_{ij}) be b-homogeneous where $b + c \neq 0, 1$, then by Corollary 3.5 equations (3.31) are variational if and only if conditions (3.17) hold (recall that for B regular the (2.11) take the form (2.18)). That is, the last Helmholtz condition (2.19) is redundant. This was proved in [12] for the case $c = 2$ and $b = 0$ (the Finsler metric inverse problem).

In the sequel, let us discuss relationships between homogeneity of Lagrangians and homogeneity of equations in more detail. Recall that, as above, "c-homogeneity" of equations here means first-level positive homogeneity of degree c in $\dot{x}^i$ and $\ddot{x}^i$ in the sense of Definition 1.1., of the functions on the left-hand sides of the equations.

Theorem 3.9. *Euler–Lagrange expressions of a time independent first order Lagrangian which is homogenous of degree c are homogeneous of degree c.*

Proof. We have
$$E_i = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} \tag{3.33}$$
so that with help of (3.1) and (3.2)
$$\begin{aligned}\frac{\partial E_i}{\partial \dot{x}^k}\dot{x}^k + 2\frac{\partial E_i}{\partial \ddot{x}^k}\ddot{x}^k &= \frac{\partial^2 L}{\partial x^i \partial \dot{x}^k}\dot{x}^k - \Big(\frac{d}{dt}\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^k}\Big)\dot{x}^k - \frac{\partial^2 L}{\partial x^k \partial \dot{x}^i}\dot{x}^k - 2\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^k}\ddot{x}^k \\ &= c\frac{\partial L}{\partial x^i} - (c-1)\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial^2 L}{\partial x^k \partial \dot{x}^i}\dot{x}^k - \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^k}\ddot{x}^k = cE_i.\end{aligned} \tag{3.34}$$
□

Now, using Theorem 2.2 we can conclude:

Corollary 3.10. *Consider semi-variational equations $B_{ij}\ddot{x}^j + A_i = 0$ written in the form*
$$\mathcal{E}_i(L) = \Phi_i, \tag{3.35}$$

where the left-hand sides $\mathcal{E}_i(L)$ are Euler–Lagrange expressions of the canonical Lagrangian related with B. If B is homogeneous of degree $c-2$ then equations (3.35) *are homogeneous of degree c if and only if Φ_i are homogeneous of degree c.*

Using Theorem 2.5, Theorem 3.1 and Theorem 3.9 one can immediately see that for semi-variational equations satisfying condition (2.11) and such that B is homogeneous of degree $c-2$, the left-hand sides $B_{ik}\ddot{x}^k + \bar{\mathcal{P}}^2(G_i)$ are homogeneous of degree c. In this case, however, as we know, the force Φ is affine in velocities; denote

$$\Phi_i(x, \dot{x}) = \alpha_{ij}(x)\dot{x}^j + \beta_i(x). \tag{3.36}$$

Then the "almost variational" equations (2.4) are homogeneous of degree c if and only if

$$(c-1)\alpha_{ik}\dot{x}^k + c\beta_i = 0. \tag{3.37}$$

If $c \neq 0, 1$, the above condition means that $\Phi = 0$, that is, the equations are *variational*, being the Euler–Lagrange equations of the canonical Lagrangian $L = -\bar{\mathcal{P}}^2(B)$. In this way we arrive once again to the assertions of Theorem 3.6. Moreover, joining the results, we can see that in this case

$$\bar{\mathcal{P}}^2(G_i) = \frac{1}{c-1}\left(\frac{1}{2}\Big(\frac{\partial B_{ij}}{\partial x^k} + \frac{\partial B_{ik}}{\partial x^j}\Big) - \frac{1}{c}\frac{\partial B_{jk}}{\partial x^i}\right)\dot{x}^j\dot{x}^k, \tag{3.38}$$

and due to Corollary 3.7 this form of the equations is fully relevant only for B *dependent upon velocities*. If $\partial B_{ij}/\partial \dot{x}^k = 0$ then there is the only possibility $c = 2$, which for regular B's means that the equations are equations for geodesics of a metrizable linear connection.

Now, with help of (3.37) we shall clarify the situation for the remaining cases $c = 1$ and $c = 0$:

Theorem 3.11. *Semi-variational equations satisfying condition* (2.11)*, and homogeneous of degree* 1*, take the form*

$$B_{ik}\ddot{x}^k + \bar{\mathcal{P}}^2(G_i) = \alpha_{ik}\dot{x}^k, \tag{3.39}$$

where B is homogeneous of degree -1, and the left-hand sides are the Euler–Lagrange expressions of the canonical Lagrangian $L = -\bar{\mathcal{P}}^2(B)$ (we remind that they take an explicit form as given in the proof of Theorem 2.3 or an equivalent form as in the proof of Theorem 2.5). Equations (3.39) *are variational if and only if*

$$\alpha_{ij} = -\alpha_{ji} \quad \text{and} \quad (\partial\alpha_{ij}/\partial x^k)_{\mathrm{cycl}(ijk)} = 0. \tag{3.40}$$

If the equations are variational, they come from a first order Lagrangian

$$L = -\bar{\mathcal{P}}^2(B) + V_i \dot{x}^i, \quad \text{where } V_i(x) \text{ are defined by} \quad \alpha_{ij} = \frac{\partial V_i}{\partial x^j} - \frac{\partial V_j}{\partial x^i}, \tag{3.41}$$

which is homogeneous of degree 1*, and non-unique, determined up to* $\frac{\partial f}{\partial x^i}\dot{x}^i$*, where* $f(x)$ *is an arbitrary function.*

Proof. It remains only to prove (3.40) and the assertion concerning the form of Lagrangians. The former is very easy: conditions (3.40) come from the Helmholtz conditions for $\phi_i = \alpha_{ij}\dot{x}^j$. Notice that the differential conditions on α_{ij} come from (2.19) which now cannot be omitted. Next, L (3.41) is obviously a Lagrangian for (3.39), homogeneous of degree 1. Finally, if L is a first order Lagrangian and L' is an equivalent Lagrangian of the same order then $L' = L + df/dt$ where $f(t, x)$ is a function. Assuming homogeneity of both the Lagrangians, we get

$$\frac{\partial L'}{\partial \dot{x}^k}\dot{x}^k - L' = \frac{\partial f}{\partial x^k}\dot{x}^k - \frac{df}{dt} = -\frac{\partial f}{\partial t} = 0 \quad \text{iff } f \text{ does not depend on } t. \tag{3.42}$$

□

Theorem 3.12. *Semi-variational equations satisfying condition* (2.11)*, and homogeneous of degree* 0*, take the form*

$$B_{ik}\ddot{x}^k + \bar{\mathcal{P}}^2(G_i) = \beta_i \tag{3.43}$$

(the force independent on velocities), where B *is homogeneous of degree* -2*, and the left-hand sides are the Euler–Lagrange expressions of the canonical Lagrangian* $L = -\bar{\mathcal{P}}^2(B)$*. The equations are variational if and only if* $\beta_i = \partial U/\partial x^i$ *for some function* $U(x)$*.*

If the equations are variational, they come from a first order Lagrangian

$$L = -\bar{\mathcal{P}}^2(B) - U \tag{3.44}$$

which is homogeneous of degree 0*, and non-unique, determined up to an arbitrary function of* t *(respectively, up to a constant, if we restrict to autonomous Lagrangians).*

Proof. As above, potentiality of the force in (3.43) comes from the Helmholtz conditions, namely from the condition (2.19). Then any first order Lagrangian for the equations has the form $L = -\bar{\mathcal{P}}^2(B) - U + df/dt$ where $f(t, x)$ is an arbitrary function. Homogeneity now means that $\frac{\partial L}{\partial \dot{x}^i}\dot{x}^i = 0$, and since the canonical Lagrangian is homogeneous of degree zero by Theorem 3.1, we get $\partial f/\partial x^i = 0$, proving the assertion. □

We have seen that homogeneous equations of degree c have a first order Lagrangian which is homogeneous of degree c (unique for $c \neq 0, 1$). It is worth note that one has also second order homogeneous Lagrangians:

Theorem 3.13. *If E_i are variational and homogeneous of degree c then the Tonti Lagrangian satisfies the same homogeneity condition.*

PROOF. The Tonti Lagrangian has the form $L_{\mathrm{Ton}} = \mathcal{P}(E)$ where $\mathcal{P}$ is the Poincaré homotopy operator associated with the map

$$\chi : [0,1]\times(\mathbb{R}\times T^2W) \ni (u, (t, x^i, \dot{x}^i, \ddot{x}^i)) \to (t, ux^i, u\dot{x}^i, u\ddot{x}^i) \in \mathbb{R}\times T^2W, \tag{3.45}$$

where $W \subset M$ is a proper subset (see [17]). In coordinates,

$$L_{\mathrm{Ton}} = x^i \int_0^1 (E_i \circ \chi) du. \tag{3.46}$$

Now,

$$\frac{\partial L_{\mathrm{Ton}}}{\partial \dot{x}^k}\dot{x}^k + 2\frac{\partial L_{\mathrm{Ton}}}{\partial \ddot{x}^k}\ddot{x}^k - c\, L_{\mathrm{Ton}} = x^i \int_0^1 \Big(\frac{\partial E_i}{\partial \dot{x}^k}\dot{x}^k + 2\frac{\partial E_i}{\partial \ddot{x}^k}\ddot{x}^k - c\, E_i\Big) \circ \chi\, du = 0. \tag{3.47}$$

We note that as in the case of the operator $\bar{\mathcal{P}}^2$ (see Section 2), $\mathcal{P}(E)$ is obtained similarly, by extending the E_i to the set $(\dot{x}^1, \ldots, \dot{x}^n) = (0, \ldots, 0)$. □

Finally, we notice that homogeneity of a Lagrangian implies interesting properties of its momenta p_i and Hamiltonian H. Recall that

$$p_i = \frac{\partial L}{\partial \dot{x}^i}, \quad H = -L + \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i = -L + p_i\dot{x}^i. \tag{3.48}$$

Then assuming that L is c-homogeneous,

$$\frac{\partial L}{\partial \dot{x}^i}\dot{x}^i = cL, \tag{3.49}$$

immediately yields:

Theorem 3.14. *Let L be a time independent first order Lagrangian, homogeneous of degree c. Then*

$$H = (c-1)L, \tag{3.50}$$

$$\frac{\partial H}{\partial \dot{x}^k}\dot{x}^k = cH, \quad \textit{i.e. } H \textit{ is } c\textit{-homogeneous}, \tag{3.51}$$

$$\frac{\partial p_i}{\partial \dot{x}^k}\dot{x}^k = (c-1)p_i, \quad \textit{i.e. momenta are } (c-1)\textit{-homogeneous}, \tag{3.52}$$

$$\frac{\partial p_i}{\partial \dot{x}^k}\dot{x}^k\dot{x}^i = c(c-1)L = cH. \tag{3.53}$$

If $c \neq 0$ then

$$L = \frac{1}{c} p_k \dot{x}^k, \tag{3.54}$$

and the Euler–Lagrange expressions of L are completely determined by momenta:

$$E_i = \left(\frac{1}{c} \frac{\partial p_k}{\partial x^i} - \frac{\partial p_i}{\partial x^k} \right) \dot{x}^k - \frac{\partial p_i}{\partial \dot{x}^k} \ddot{x}^k. \tag{3.55}$$

Note that if momenta are given, one can find the corresponding Lagrangian L directly (without integration procedure) with help of formula (3.54).

4. Second-level positively homogeneous equations

In the sequel we shall discuss in more detail equations satisfying the second-level positive homogeneity conditions in the sense of Definition 1.1. As above, we assume that the domain of definition of the functions under consideration does not contain points where $(\dot{x}^1, \ldots, \dot{x}^n) = (0, \ldots, 0)$. And, as above, we consider ODEs of the form (1.1). Moreover, we assume that the equations are nontrivially of second order ($B_{ij} \neq 0$ for at least some i, j).

Remarkably, by the following theorem only the case $c = 1$ is of interest. We recall that in this case solutions of the equations are invariant under orientation preserving reparametrizations.

Theorem 4.1. *Let* (1.1) *be second-level positively homogeneous of degree c. If the equations are semi-variational then $c = 1$.*

Proof. Applying homogeneity conditions (1.6) to functions E_i affine in the second derivatives we obtain

$$\frac{\partial A_i}{\partial \dot{x}^k} \dot{x}^k = c A_i, \quad \frac{\partial B_{ij}}{\partial \dot{x}^k} \dot{x}^k = (c-2) B_{ij}\,, \quad B_{ik} \dot{x}^k = 0. \tag{4.1}$$

Differentiating the last condition and using semi-variationality yields

$$\frac{\partial B_{ik}}{\partial \dot{x}^j} \dot{x}^k + B_{ij} = \frac{\partial B_{ij}}{\partial \dot{x}^k} \dot{x}^k + B_{ij} = 0. \tag{4.2}$$

This means, however, that $B_{ij} + (c-2) B_{ij} = 0$, hence $c = 1$. $\square$

Since second-level positively 1-homogeneous equations are first-level positively 1-homogeneous, all results obtained in the previous section for the case $c = 1$ apply.

We can see that for equations (1.1) Zermelo conditions take the form

$$\frac{\partial A_i}{\partial \dot{x}^k}\dot{x}^k = A_i, \quad \frac{\partial B_{ij}}{\partial \dot{x}^k}\dot{x}^k = -B_{ij}, \quad B_{ik}\dot{x}^k = 0. \tag{4.3}$$

Thus the assumption of second-level positive homogeneity adds to the homogeneity conditions studied in the previous section one additional condition, which has an obvious, however very important consequence:

Theorem 4.2. *If equations* (1.1) *are second-level positively homogeneous then the matrix* $B = (B_{ij})$ *is singular, i.e.* $\det B = 0$. *This means that the equations cannot be put into a normal form* $\ddot{x}^i = f^i(x^k(t), \dot{x}^k(t))$, *or, otherwise speaking, are not representable by means of a second-order vector field (semi-spray) on* $T^{\circ}M$.

Proof. Understanding conditions $B_{ik}\dot{x}^k = 0$ as a system of homogeneous linear algebraic equations for unknowns $\dot{x}^k$, $1 \le k \le n$, we obtain the result. □

By Theorem 3.11, first-level positively 1-homogeneous variational equations posses 1-homogeneous Lagrangians, and conversely, by Theorem 3.9, positively 1-homogeneous Lagrangians give first-level positively 1-homogeneous Euler–Lagrange equations. However, differentiating the homogeneity condition for L and using the formula for B_{ij} yields

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^k}\dot{x}^k = -B_{ik}\dot{x}^k = 0. \tag{4.4}$$

In other words, in Finsler geometry the concepts of first and second-level positive 1-homogeneity for differential equations coincide:

Theorem 4.3. *First-level positively* 1*-homogeneous variational equations satisfy Zermelo conditions (i.e. they are second-level positively* 1*-homogeneous).*

In view of the above, when dealing with variational equations we shall just speak about "positive 1-homogeneity".[1]

Furthermore, for semi-variational and variational equations *Zermelo conditions take the following simple form*:

Theorem 4.4. (1) *Semi-variational equations are second-level positively* 1*-homogeneous if and only if*

$$\frac{\partial A_i}{\partial \dot{x}^k}\dot{x}^k = A_i, \quad B_{ik}\dot{x}^k = 0. \tag{4.5}$$

[1]In particular, we can see that variational equations satisfying Zermelo conditions posses local positively homogeneous Lagrangians of degree one, and conversely, such Lagrangians give rise to equations satisfying Zermelo conditions, which is a result earlier proved in [18].

(2) *Semi-variational equations satisfying condition* (2.11) *are second-level positively* 1*-homogeneous if and only if*

$$A_k \dot{x}^k = 0, \quad B_{ik} \dot{x}^k = 0. \tag{4.6}$$

(3) *Variational equations* (1.1) *are positively* 1*-homogeneous if and only if* (4.6) *hold true.*

Proof. (1) Indeed, for semi-variational equations the second set of the Zermelo conditions (4.3) is superfluous, since it is obtained by differentiating the condition $B_{ik}\dot{x}^k = 0$ and using (2.1).

(2) We have to prove that in this case the first set of the Zermelo conditions can be replaced by $A_k \dot{x}^k = 0$. However, differentiating $A_k \dot{x}^k$ and assuming (2.11) and $B_{ik}\dot{x}^k = 0$, we obtain

$$0 = A_i + \frac{\partial A_k}{\partial \dot{x}^i} \dot{x}^k = A_i + 2 \frac{\partial B_{ik}}{\partial x^j} \dot{x}^j \dot{x}^k - \frac{\partial A_i}{\partial \dot{x}^k} \dot{x}^k = A_i - \frac{\partial A_i}{\partial \dot{x}^k} \dot{x}^k. \tag{4.7}$$

Conversely, if the first set of Zermelo conditions (4.3) is satisfied then due to (2.11)

$$A_k \dot{x}^k = \frac{\partial A_k}{\partial \dot{x}^i} \dot{x}^i \dot{x}^k = \frac{1}{2} \Big(\frac{\partial A_k}{\partial \dot{x}^i} + \frac{\partial A_i}{\partial \dot{x}^k} \Big) \dot{x}^i \dot{x}^k = \frac{\partial B_{ik}}{\partial x^j} \dot{x}^j \dot{x}^i \dot{x}^k = 0, \tag{4.8}$$

since $B_{ik}\dot{x}^k = 0$.

Assertion (3) follows from (2). □

Corollary 4.5. *Consider semi-variational equations satisfying condition* (2.11) *and second-level positively* 1*-homogeneous. Then at least one of the equations is linearly dependent (a linear combination of the remaining ones) and can be omitted.*

Of course, the same assertion holds for positively 1-homogeneous variational equations.

Proof. The number of independent equations equals to the rank of the $(n+1) \times n$ matrix (A_i, B_{ik}), with rows labelled by i and columns labelled by k. Taking a vector $(\dot{x}^1, \ldots \dot{x}^n) \in T^o_x M$, and using homogeneity conditions (4.6) we get an equivalent matrix

$$\begin{pmatrix} A_\sigma & B_{\sigma k} \\ 0 & 0 \end{pmatrix}, \tag{4.9}$$

where $1 \leq \sigma \leq n-1$, proving our assertion. □

Remark 4.6. The number of independent equations and their structure in a neighborhood of a point in T^oM depends on the ranks of the matrices (A_i, B_{ik}) and (B_{ik}), hence, can be specified even more precisely. If both the ranks are constant and $\mathrm{rank}(A_i, B_{ik}) = \mathrm{rank}(B_{ik}) = N$ then we have N independent second order ODEs. If $\mathrm{rank}(A_i, B_{ik}) > \mathrm{rank}(B_{ik})$ then we have a system of mixed second order and first order ODEs. Recall that by Theorem 3.11

$$B_{ik} = -\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^k}, \quad A_i = \bar{\mathcal{P}}^2(G_i) - \alpha_{ij}\dot{x}^j \tag{4.10}$$

where $L = -\bar{\mathcal{P}}^2(B)$ is the canonical Lagrangian; in Finsler geometry $L = F$, a Finsler function.

Additionally we get that due to the homogeneity condition $B_{ik}\dot{x}^k = 0$, the class of local positively 1-homogeneous first order Lagrangians contains a distinguished Lagrangian as follows:

Theorem 4.7. *For positively* 1*-homogeneous variational equations, the Engels Lagrangian defined by the formula*

$$\begin{aligned} L_{\mathrm{Eng}} &= L_{\mathrm{Ton}} + \frac{d}{dt}\Big(x^j \int_0^1 (p_j \circ \chi) du\Big) \\ &= x^j \int_0^1 (A_j \circ \chi) du + \dot{x}^j \int_0^1 (p_j \circ \chi) du + x^j \dot{x}^k \int_0^1 \Big(\frac{\partial p_j}{\partial x^k} \circ \chi\Big) u\, du, \end{aligned} \tag{4.11}$$

where L_{Ton} is the Tonti Lagrangian and $(p_1, \dots, p_n)$ is any solution of the equations $B_{jk} = -\partial p_j / \partial \dot{x}^k$, is positively homogeneous of degree 1*.*

The assertion immediately follows from Theorem 3.11, since the Engels Lagrangian is a time-independent Lagrangian equivalent with the Lagrangian (3.41); alternatively the assertion is easily checked by a direct computation.

As expected, the Tonti Lagrangian for positively 1-homogeneous equations (which is a first-level positively 1-homogeneous function, as we know from the previous section) is second-level positively 1-homogeneous, satisfying *all* the Zermelo conditions.

Note that we have the following direct consequence of Theorem 4.4:

Corollary 4.8. *For equations* (1.1) *the following are necessary conditions to be variational and positively* 1*-homogeneous:*

$$B_{ik}\dot{x}^k = 0, \quad B_{ik} = B_{ki}, \quad \frac{\partial B_{ik}}{\partial \dot{x}^j} = \frac{\partial B_{ij}}{\partial \dot{x}^k}, \tag{4.12}$$

$$A_i \dot{x}^i = 0. \tag{4.13}$$

Of course, *necessary and sufficient conditions* are obtained by adding the remaining two sets of Helmholtz conditions: (2.11) and (2.19). However, these conditions can be "solved" to get an explicit form of the functions A_i. We shall finish with two theorems describing the structure of variational positively 1-homogenous equations. First, combining the homogeneity properties discussed above with Theorem 3.11 we get:

Theorem 4.9. *Equations* (1.1) *are variational and positively* 1*-homogeneous if and only if (4.12) hold and*

$$A_i = \bar{\mathcal{P}}^2(G_i) - \alpha_{ij}\dot{x}^j, \tag{4.14}$$

where α_{ij} *satisfy* (3.40).

Finally, we obtain *"positively homogeneous Helmholtz conditions"*:

Theorem 4.10. *Equations* (1.1) *are variational and positively* 1*-homogeneous if and only if* (4.12) *hold and*

$$A_i = a_{ik}\dot{x}^k, \tag{4.15}$$

where

$$a_{ik} = -a_{ki}, \quad \left(\frac{\partial a_{ij}}{\partial x^k} + \frac{\partial a_{ki}}{\partial x^j} + \frac{\partial a_{jk}}{\partial x^i}\right)\dot{x}^k = 0, \quad \left(\frac{\partial a_{ik}}{\partial \dot{x}^j} - \frac{\partial a_{jk}}{\partial \dot{x}^i}\right)\dot{x}^k = 0,$$
$$\left(\frac{\partial a_{ik}}{\partial \dot{x}^j} + \frac{\partial a_{jk}}{\partial \dot{x}^i} - 2\frac{\partial B_{ij}}{\partial x^k}\right)\dot{x}^k = 0. \tag{4.16}$$

Proof. Assume (1.1) be variational and positively 1-homogeneous. Denote

$$a_{ik} = \frac{1}{2}\Big(\frac{\partial A_i}{\partial \dot{x}^k} - \frac{\partial A_k}{\partial \dot{x}^i}\Big) = -a_{ki}. \tag{4.17}$$

Then

$$A_i = \frac{\partial A_i}{\partial \dot{x}^k}\dot{x}^k = a_{ik}\dot{x}^k + \frac{1}{2}\Big(\frac{\partial A_i}{\partial \dot{x}^k} + \frac{\partial A_k}{\partial \dot{x}^j}\Big)\dot{x}^k = a_{ik}\dot{x}^k, \tag{4.18}$$

since differentiating $A_k\dot{x}^k = 0$ we arrive at

$$0 = \frac{\partial A_k}{\partial \dot{x}^i}\dot{x}^k + A_i = \left(\frac{\partial A_k}{\partial \dot{x}^i} + \frac{\partial A_i}{\partial \dot{x}^k}\right)\dot{x}^k. \tag{4.19}$$

Now, using (2.19),

$$-\left(\frac{\partial a_{ij}}{\partial x^k} + \frac{\partial a_{ki}}{\partial x^j} + \frac{\partial a_{jk}}{\partial x^i}\right)\dot{x}^k = \frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial \dot{x}^j} - \frac{\partial A_j}{\partial \dot{x}^i}\right)\dot{x}^k - \frac{\partial A_i}{\partial x^j} + \frac{\partial A_j}{\partial x^i} = 0, \tag{4.20}$$

and from

$$\frac{\partial a_{ik}}{\partial \dot{x}^j}\dot{x}^k = \frac{\partial(a_{ik}\dot{x}^k)}{\partial \dot{x}^j} - a_{ij} = \frac{\partial A_i}{\partial \dot{x}^j} - \frac{1}{2}\left(\frac{\partial A_i}{\partial \dot{x}^j} - \frac{\partial A_j}{\partial \dot{x}^i}\right) = \frac{1}{2}\left(\frac{\partial A_i}{\partial \dot{x}^j} + \frac{\partial A_j}{\partial \dot{x}^i}\right), \quad (4.21)$$

accounting (2.11), we get the remaining two identities.

Conversely, let (1.1) satisfy conditions of the theorem. Put $A_i = a_{ik}\dot{x}^k$. We have to check identities (4.13), (2.11) and (2.19). (4.13) is obvious due to the skew symmetry of a_{ik}. Next,

$$\frac{\partial A_i}{\partial \dot{x}^k} + \frac{\partial A_k}{\partial \dot{x}^i} = a_{ik} + \frac{\partial a_{ij}}{\partial \dot{x}^k}\dot{x}^j + a_{ki} + \frac{\partial a_{kj}}{\partial \dot{x}^i}\dot{x}^j = 2\frac{\partial B_{ik}}{\partial x^j}\dot{x}^j, \quad (4.22)$$

and

$$\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial A_i}{\partial \dot{x}^k} - \frac{\partial A_k}{\partial \dot{x}^i}\right)\dot{x}^j$$
$$= \frac{\partial \alpha_{ij}}{\partial x^k}\dot{x}^j - \frac{\partial \alpha_{kj}}{\partial x^i}\dot{x}^j - \frac{\partial \alpha_{ik}}{\partial x^j}\dot{x}^j - \frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial \alpha_{ip}}{\partial \dot{x}^k} - \frac{\partial \alpha_{kp}}{\partial \dot{x}^i}\right)\dot{x}^j\dot{x}^p = 0, \quad (4.23)$$

as desired. □

A different form of necessary and sufficient conditions of variationality and positive 1-homogeneity can be found in [19].

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