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Volumes and geodesic ball packings to the regular prism tilings in $\widetilde{\mathbf{SL_2R}}$ space

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Dedicated to Professor Lajos Tamássy on his 90th birthday

Abstract. After having investigated the regular prisms and prism tilings in the $\widetilde{\mathbf{SL_2R}}$ space in the previous work [15] of the second author, we consider the problem of geodesic ball packings related to those tilings and their symmetry groups $\mathbf{pq2_1}$. $\widetilde{\mathbf{SL_2R}}$ is one of the eight Thurston geometries that can be derived from the 3-dimensional Lie group of all 2×2 real matrices with determinant one.

In this paper we consider geodesic spheres and balls in $\widetilde{\mathbf{SL_2R}}$ (even in $\mathbf{SL_2R}$), if their radii $\rho \in [0, \frac{\pi}{2})$, and determine their volumes. Moreover, we consider the prisms of the above space, compute their volumes and define the notion of the geodesic ball packing and its density. We develop a procedure to determine the densities of the densest geodesic ball packings for the tilings, or in this paper more precisely, for their generating groups $\mathbf{pq2_1}$ (for integer rotational parameters p, q; $3 \leq p$, $\frac{2p}{p-2} < q$). We look for those parameters p and q above, where the packing density large enough as possible. Now our record is 0.567362 for $(p,q) = (8,10)$. These computations seem to be important, since we do not know optimal ball packing, namely in the hyperbolic space $\mathbf{H}^3$. We know only the density upper bound 0.85326, realized by horoball packing of $\mathbf{H}^3$ to its ideal regular simplex tiling. Surprisingly, for the so-called translation ball packings under the same groups $\mathbf{pq2_1}$ in [8] we have got larger density 0.841700 for $(p,q) = (5, 10000 \to \infty)$ close to the above upper bound.

We use for the computation and visualization of the $\widetilde{\mathbf{SL_2R}}$ space its projective model introduced by the first author in [4].

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1. On $\widetilde{\mathbf{SL_2R}}$ geometry

The real 2×2 matrices $\left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right)$ with unit determinant $ad - bc = 1$ constitute a Lie transformation group by the usual product operation, taken to act on row matrices as on point coordinates on the right as follows

$$\begin{gathered}(z^0, z^1)\begin{pmatrix} d & b \\ c & a \end{pmatrix} = (z^0 d + z^1 c, z^0 b + z^1 a) = (w^0, w^1) \\ \text{with } w = \frac{w^1}{w^0} = \frac{b + \frac{z^1}{z^0} a}{d + \frac{z^1}{z^0} c} = \frac{b + za}{d + zc}\end{gathered} \tag{1.1}$$

as right action on the complex projective line $\mathbf{C}^\infty$. This group is a 3-dimensional manifold, because of its 3 independent real coordinates and with its usual neighbourhood topology [9], [17]. In order to model the above structure in the projective sphere $\mathcal{PS}^3$ and in the projective space $\mathcal{P}^3$ (see [4]), we introduce the new projective coordinates (x^0, x^1, x^2, x^3) where

$$a := x^0 + x^3, \quad b := x^1 + x^2, \quad c := -x^1 + x^2, \quad d := x^0 - x^3,$$

with positive, then the non-zero multiplicative equivalence as a projective freedom in $\mathcal{PS}^3$ and in $\mathcal{P}^3$, respectively. Meanwhile we turn to the proportionality $\mathbf{SL_2R} < \mathbf{PSL_2R}$, natural in this context. Then it follows that

$$0 > bc - ad = -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 \tag{1.2}$$

describes the interior of the above one-sheeted hyperboloid solid $\mathcal{H}$ in the usual Euclidean coordinate simplex, with the origin $E_0(1;0;0;0)$ and the ideal points of the axes $E_1^\infty(0;1;0;0)$, $E_2^\infty(0;0;1;0)$, $E_3^\infty(0;0;0;1)$. We consider the collineation group $\mathbf{G}_*$ that acts on the projective sphere $\mathcal{SP}^3$ and preserves a polarity, i.e. a scalar product of signature $(--++)$, this group leaves the one sheeted hyperboloid solid $\mathcal{H}$ invariant. We have to choose an appropriate subgroup $\mathbf{G}$ of $\mathbf{G}_*$ as isometry group, then the universal covering group and space $\widetilde{\mathcal{H}}$ of $\mathcal{H}$ will be the hyperboloid model of $\widetilde{\mathbf{SL_2R}}$ (see Figure 1 and [4]).

The specific isometries $\mathbf{S}(\phi)$ $(\phi \in \mathbf{R})$ constitute a one parameter group given by the matrices

$$\mathbf{S}(\phi) : (s_i^j(\phi)) = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}. \tag{1.3}$$

The elements of $\mathbf{S}(\phi)$ are the so-called *fibre translations*. We obtain a unique fibre line to each $X(x^0;x^1;x^2;x^3) \in \widetilde{\mathcal{H}}$ as the orbit by right action of $\mathbf{S}(\phi)$ on X. The coordinates of points lying on the fibre line through X can be expressed as the images of X by $\mathbf{S}(\phi)$:

$$(x^0;x^1;x^2;x^3) \xrightarrow{\mathbf{S}(\phi)} (x^0\cos\phi - x^1\sin\phi; x^0\sin\phi + x^1\cos\phi; \\ x^2\cos\phi + x^3\sin\phi; -x^2\sin\phi + x^3\cos\phi) \tag{1.4}$$

for the Euclidean coordinates $x := \frac{x^1}{x^0}$, $y := \frac{x^2}{x^0}$, $z := \frac{x^3}{x^0}$, $x^0 \neq 0$ as well. The π periodicity for the above coordinates in the above maps can be seen from the formula (1.4). In (1.3) and (1.4) we can see the 2π periodicity of ϕ. Moreover, we

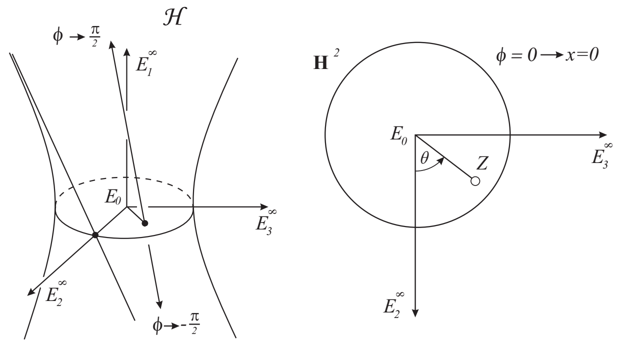

Figure 1. The hyperboloid model

see the (logical) extension to $\phi \in \mathbf{R}$, as real parameter, to have the universal covers $\widetilde{\mathcal{H}}$ and $\widetilde{\mathbf{SL_2R}}$, respectively, through the projective sphere $\mathcal{PS}^3$. The elements of the isometry group of $\mathbf{SL_2R}$ (and so by the above extension the isometries of $\widetilde{\mathbf{SL_2R}}$) can be described by the matrix (a_i^j) (see [4] and [5])

$$(a_i^j) = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 & a_0^3 \\ \mp a_0^1 & \pm a_0^0 & \pm a_0^3 & \mp a_0^2 \\ a_2^0 & a_2^1 & a_2^2 & a_2^3 \\ \pm a_2^1 & \mp a_2^0 & \mp a_2^3 & \pm a_2^2 \end{pmatrix} \quad \text{where}$$

$$-(a_0^0)^2 - (a_0^1)^2 + (a_0^2)^2 + (a_0^3)^2 = -1, \quad -(a_2^0)^2 - (a_2^1)^2 + (a_2^2)^2 + (a_2^3)^2 = 1, \\ -a_0^0a_2^0 - a_0^1a_2^1 + a_0^2a_2^2 + a_0^3a_2^3 = 0 = -a_0^0a_2^1 + a_0^1a_2^0 - a_0^2a_2^3 + a_0^3a_2^2, \tag{1.5}$$

and we allow positive proportionality, of course, as projective freedom. We define the *translation group* $\mathbf{G}_T$, as a subgroup of the isometry group of $\mathbf{SL_2R}$, those isometries acting transitively on the points of $\mathcal{H}$ and by the above extension on the points of $\widetilde{\mathcal{H}}$. $\mathbf{G}_T$ maps the origin $E_0(1;0;0;0)$ onto $X(x^0;x^1;x^2;x^3)$. These isometries and their inverses (up to a positive determinant factor) can be given by

$$\mathbf{T}:(t_i^j)=\begin{pmatrix} x^0 & x^1 & x^2 & x^3\\ -x^1 & x^0 & x^3 & -x^2\\ x^2 & x^3 & x^0 & x^1\\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix},$$

$$\mathbf{T}^{-1}:(T_j^k)=\begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3\\ x^1 & x^0 & -x^3 & x^2\\ -x^2 & -x^3 & x^0 & -x^1\\ -x^3 & x^2 & x^1 & x^0 \end{pmatrix}. \tag{1.6}$$

The rotation about the fibre line through the origin $E_0(1;0;0;0)$ by angle ω $(-\pi<\omega\leq\pi)$ can be expressed by

$$\mathbf{R}_{E_O}(\omega):\ (r_i^j(E_0,\omega))=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos\omega & \sin\omega\\ 0 & 0 & -\sin\omega & \cos\omega \end{pmatrix}, \tag{1.7}$$

and the rotation $\mathbf{R}_X(\omega)$ with matrix : $(r_i^j(X,\omega))$ about the fibre line through $X(x^0;x^1;x^2;x^3)$ by angle ω can be derived by formulas (1.6) and (1.7) by conjugacy $\mathbf{R}_X(\omega)=\mathbf{T}^{-1}\mathbf{R}_{E_O}(\omega)\mathbf{T}$. Thus the above rotation $\mathbf{R}_X(\omega)$, with a specific X $(\cosh r,0,\sinh r,0)\sim(1,0,\tanh r,0)$ has the important matrix (see [15])

$$\begin{pmatrix} \begin{matrix}1+\sinh^2 r-\\ -\sinh^2 r\cos\omega\end{matrix} & \sinh^2 r\sin\omega & \begin{matrix}\frac{1}{2}\sinh 2r-\\ -\frac{1}{2}\sinh 2r\cos\omega\end{matrix} & -\frac{1}{2}\sinh 2r\sin\omega\\ -\sinh^2 r\sin\omega & \begin{matrix}1+\sinh^2 r-\\ -\sinh^2 r\cos\omega\end{matrix} & -\frac{1}{2}\sinh 2r\sin\omega & \begin{matrix}-\frac{1}{2}\sinh 2r+\\ +\frac{1}{2}\sinh 2r\cos\omega\end{matrix}\\ \begin{matrix}-\frac{1}{2}\sinh 2r+\\ +\frac{1}{2}\sinh 2r\cos\omega\end{matrix} & -\frac{1}{2}\sinh 2r\sin\omega & \begin{matrix}1-\cosh^2 r+\\ +\cosh^2 r\cos\omega\end{matrix} & \cosh^2 r\sin\omega\\ -\frac{1}{2}\sinh 2r\sin\omega & \begin{matrix}\frac{1}{2}\sinh 2r-\\ -\frac{1}{2}\sinh 2r\cos\omega\end{matrix} & -\cosh^2 r\sin\omega & \begin{matrix}1-\cosh^2 r+\\ +\cosh^2 r\cos\omega\end{matrix} \end{pmatrix} \tag{1.8}$$

Horizontal intersection of the hyperboloid solid $\mathcal{H}$ with the plane $E_0E_2^\infty E_3^\infty$ provides the *base plane* of the model $\widetilde{\mathcal{H}}=\widetilde{\mathbf{SL_2R}}$. The fibre through X intersects the hyperbolic ($\mathbf{H}^2$) base plane $z^1 = x = 0$ in the foot point

$$Z(z^0 = x^0x^0 + x^1x^1; z^1 = 0; z^2 = x^0x^2 - x^1x^3; z^3 = x^0x^3 + x^1x^2). \tag{1.9}$$

We generally introduce a so-called hyperboloid parametrization by [4] as follows

$$\begin{aligned} x^0 &= \cosh r \cos\phi, \\ x^1 &= \cosh r \sin\phi, \\ x^2 &= \sinh r \cos(\theta - \phi), \\ x^3 &= \sinh r \sin(\theta - \phi), \end{aligned} \tag{1.10}$$

where (r, θ) are the polar coordinates of the $\mathbf{H}^2$ base plane, and ϕ is the fibre coordinate. We note that

$$-x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 = -\cosh^2 r + \sinh^2 r = -1 < 0.$$

The inhomogeneous coordinates in (1.11), which will play an important role in the later $\mathbf{E}^3$-visualization of the prism tilings in $\widetilde{\mathbf{SL_2R}}$, are given by

$$\begin{aligned} x &= \frac{x^1}{x^0} = \tan\phi, \\ y &= \frac{x^2}{x^0} = \tanh r \frac{\cos(\theta-\phi)}{\cos\phi}, \\ z &= \frac{x^3}{x^0} = \tanh r \frac{\sin(\theta-\phi)}{\cos\phi}. \end{aligned} \tag{1.11}$$

The infinitesimal arc-length-square can be derived by the standard pull back method. By T^{-1}-action of (1.6) on the differentials $(\mathrm{d}x^0; \mathrm{d}x^1; \mathrm{d}x^2; \mathrm{d}x^3)$, we obtain that in this parametrization the infinitesimal arc-length-square at any point of $\widetilde{\mathbf{SL_2R}}$ is the following:

$$(\mathrm{d}s)^2 = (\mathrm{d}r)^2 + \cosh^2 r \sinh^2 r (\mathrm{d}\theta)^2 + \big[(\mathrm{d}\phi) + \sinh^2 r (\mathrm{d}\theta)\big]^2. \tag{1.12}$$

Hence we get the symmetric metric tensor field g_{ij} on $\widetilde{\mathbf{SL_2R}}$ by components:

$$g_{ij} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 r(\sinh^2 r + \cosh^2 r) & \sinh^2 r \\ 0 & \sinh^2 r & 1 \end{pmatrix}, \tag{1.13}$$

and

$$\mathrm{d}V = \sqrt{\det(g_{ij})}\ dr\ \mathrm{d}\theta\ \mathrm{d}\phi = \frac{1}{2}\sinh(2r)\mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi$$

as the volume element in hyperboloid coordinates. The geodesic curves of $\widetilde{\mathbf{SL_2R}}$ are generally defined as having locally minimal arc length between any two of their (close enough) points.

By (1.13) the second order differential equation system of the $\widetilde{\mathbf{SL_2R}}$ geodesic curve is the following:

$$\begin{aligned} \ddot{r} &= \sinh(2r)\,\dot{\theta}\,\dot{\phi} + \frac{1}{2}\big(\sinh(4r) - \sinh(2r)\big)\dot{\theta}\,\dot{\theta}, \\ \ddot{\phi} &= 2\dot{r}\tanh(r)(2\sinh^2(r)\,\dot{\theta} + \dot{\phi}), \end{aligned} \tag{1.14}$$

$$\ddot{\theta} = \frac{2\dot{r}}{\sinh(2r)}\big((3\cosh(2r) - 1)\dot{\theta} + 2\dot{\phi}\big). \tag{1.15}$$

We can assume, by the homogeneity, that the starting point of a geodesic curve is the origin $(1,0,0,0)$. Moreover, $r(0) = 0$, $\phi(0) = 0$, $\theta(0) = 0$, $\dot{r}(0) = \cos(\alpha)$, $\dot{\phi}(0) = \sin(\alpha) = -\dot{\theta}(0)$ are the initial values in Table 1 for the solution of (1.14), and so the unit velocity will be achieved.

Types	
$0 \le \alpha < \frac{\pi}{4}$ ($\mathbf{H}^2$ − like direction)	$r(s,\alpha) = \operatorname{arsinh}\Big(\frac{\cos\alpha}{\sqrt{\cos 2\alpha}}\sinh(s\sqrt{\cos 2\alpha})\Big)$ $\theta(s,\alpha) = -\arctan\Big(\frac{\sin\alpha}{\sqrt{\cos 2\alpha}}\tanh(s\sqrt{\cos 2\alpha})\Big)$ $\phi(s,\alpha) = 2\sin\alpha s + \theta(s,\alpha)$
$\alpha = \frac{\pi}{4}$ (light direction)	$r(s,\alpha) = \operatorname{arsinh}\Big(\frac{\sqrt{2}}{2}s\Big)$ $\theta(s,\alpha) = -\arctan\Big(\frac{\sqrt{2}}{2}s\Big)$ $\phi(s,\alpha) = \sqrt{2}s + \theta(s,\alpha)$
$\frac{\pi}{4} < \alpha \le \frac{\pi}{2}$ (fibre-like direction)	$r(s,\alpha) = \operatorname{arsinh}\Big(\frac{\cos\alpha}{\sqrt{-\cos 2\alpha}}\sin(s\sqrt{-\cos 2\alpha})\Big)$ $\theta(s,\alpha) = -\arctan\Big(\frac{\sin\alpha}{\sqrt{-\cos 2\alpha}}\tan(s\sqrt{-\cos 2\alpha})\Big)$ $\phi(s,\alpha) = 2\sin\alpha s + \theta(s,\alpha)$

Table 1

The equation of the geodesic curve in the hyperboloid model has been determined in [2], with the usual geographical sphere coordinates (λ, α), as longitude and altitude, respectively, from the general starting position of (1.10), (1.11), $(-\pi < \lambda \leq \pi,\ -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2})$, and the arc-length parameter $0 \leq s \in \mathbf{R}$. The Euclidean coordinates $X(s,\lambda,\alpha)$, $Y(s,\lambda,\alpha)$, $Z(s,\lambda,\alpha)$ of the geodesic curves can be determined by substituting the results of Table 1 (see [2]) into the equations (1.10) and (1.11) as follows

$$\begin{aligned} X(s,\lambda,\alpha) &= \tan(\phi(s,\alpha)), \\ Y(s,\lambda,\alpha) &= \frac{\tanh(r(s,\alpha))}{\cos(\phi(s,\alpha)}\cos\big[\theta(s,\alpha)-\phi(s,\alpha)+\lambda\big], \\ Z(s,\lambda,\alpha) &= \frac{\tanh(r(s,\alpha))}{\cos(\phi(s,\alpha)}\sin\big[\theta(s,\alpha)-\phi(s,\alpha)+\lambda\big]. \end{aligned} \tag{1.16}$$

2. Geodesic balls in $\widetilde{\mathbf{SL_2R}}$

Definition 2.1. The *distance* $d(P_1,P_2)$ between the points P_1 and P_2 is defined by the arc length of the geodesic curve from P_1 to P_2.

The numerical approximation of the distance $d(O,P)$, by Table 1 and (1.15) for given $P(X,Y,Z)$ from the origin O, will not be detailed here.

Definition 2.2. The *geodesic sphere* of radius ρ (denoted by $S_{P_1}(\rho)$) with the center in point P_1 is defined as the set of all points P_2 with the condition $d(P_1,P_2)=\rho$. Moreover, we require that the geodesic sphere is a simply connected surface without selfintersection.

Definition 2.3. The body of the geodesic sphere of centre P_1 and with radius ρ is called *geodesic ball*, denoted by $B_{P_1}(\rho)$, i.e., $Q \in B_{P_1}(\rho)$ iff $0 \leq d(P_1,Q) \leq \rho$.

Figure 2.a shows a geodesic sphere of radius $\rho = 1.3$ with centre O and Figure 2.b shows its intersection with the (x,z) plane. From (1.15) it follows that $S(\rho)$ is a simply connected surface in $\mathbf{E}^3$ and $\widetilde{\mathbf{SL_2R}}$, respectively, if $\rho \in [0,\frac{\pi}{2})$. If $\rho \geq \frac{\pi}{2}$ then the universal cover should be discussed. *Therefore, we consider geodesic spheres and balls only with radii $\rho \in [0,\frac{\pi}{2})$ in the following.* These will be satisfactory for our cases.

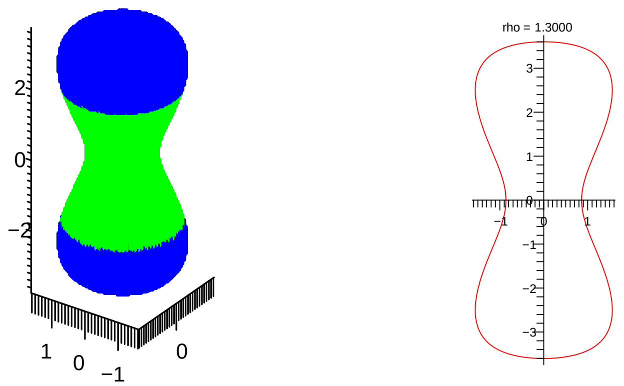

Figure 2. a, b

2.1. The volume of a geodesic ball. The volume formula of the geodesic ball $B(\rho)$ follows from the metric tensor g_{ij}. We obtain the connection between the hyperboloid coordinates (r, θ, ϕ) and the geographical coordinates (s, λ, α) in a standard way by Table 1 and by (1.15). Therefore, the volume of the geodesic ball of radius ρ can be computed by the following

Theorem 2.1.

$$\mathrm{Vol}(B(\rho)) = \int_B \frac{1}{2}\sinh(2r)\ dr\ d\theta\ d\phi = 4\pi \int_0^\rho \int_0^{\frac{\pi}{4}} \frac{1}{2}\sinh(2r(s,\alpha))|J_1|\ d\alpha\ ds$$
$$+ 4\pi \int_0^\rho \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\sinh(2r(s,\alpha))|J_2|\ d\alpha\ ds \tag{2.1}$$

where $|J_1| = \begin{vmatrix} \frac{\partial r}{\partial s} & \frac{\partial r}{\partial \alpha} \\ \frac{\partial \phi}{\partial s} & \frac{\partial \phi}{\partial \alpha} \end{vmatrix}$ *and similarly* $|J_2|$*(by Table 1 and* $\frac{\partial \theta}{\partial \lambda} = 1$*) are the corresponding Jacobians.*

The complicated formulas above need numerical approximations by computer (see Figure 3).

3. Regular prism tilings and their space groups $\mathbf{pq2_1}$

In [15] we have defined and described the regular prisms and prism tilings with a space group class $\Gamma = \mathbf{pq2_1}$ of $\widetilde{\mathbf{SL_2R}}$. These will be summarized in this section.

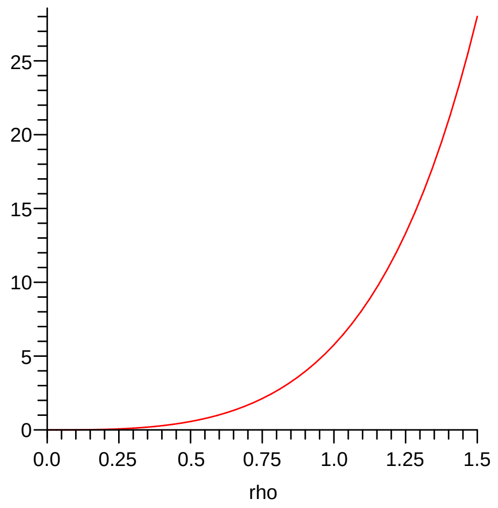

Figure 3. The increasing function $\rho \mapsto \mathrm{Vol}(B(\rho))$.

Definition 3.1. Let $\mathcal{P}^i$ be an infinite solid that is bounded by certain surfaces that can be determined (in [15]) by "side fibre lines" passing through the vertices of a regular p-gon $\mathcal{P}^b$ lying in the base plane. The images of solids $\mathcal{P}^i$ by $\widetilde{\mathbf{SL_2R}}$ isometries are called *infinite regular p-sided prisms.* Here regular means that the side surfaces are congruent to each other under rotations about a fiber line (e.g. through the origin).

The common part of $\mathcal{P}^i$ with the base plane is the *base figure* of $\mathcal{P}^i$ that is denoted by $\mathcal{P}$ and its vertices coincide with the vertices of $\mathcal{P}^b$, **but $\mathcal{P}$ is not assumed to be a polygon**.

Definition 3.2. A bounded regular p-sided prism is analogously defined if the face of the base figure $\mathcal{P}$ and its translated copy $\mathcal{P}^t$, under a fibre translation by (1.3) and so (1.5), are also introduced. The faces $\mathcal{P}$ and $\mathcal{P}^t$ are called *cover faces.*

Remark 3.1. All cross-sections of a prism generated by fibre translations from the base plane are congruent. Prisms are named for their base, e.g. the prism in Figure 4 is a trigonal prism.

We consider regular prism tilings $\mathcal{T}_p(q)$ by prisms $\mathcal{P}_p(q)$ where q pieces regularly meet at each side edge by q-rotation.

The following theorem has been proved in [15]:

Theorem 3.1. *There exist regular infinite prism tilings $\mathcal{T}^i_p(q)$ in $\widetilde{\mathbf{SL_2R}}$ for each $3 \le p \in \mathbb{N}$ where $\frac{2p}{p-2} < q \in \mathbb{N}$. For bounded prisms, these are not face-to-face.*

We assume that the prism $\mathcal{P}_p(q)$ is a *topological polyhedron* having at each vertex one p-gonal cover face (it is not a polygon at all) and two *skew quadrangles* which lie on certain side surfaces in the model. Let $\mathcal{P}_p(q)$ be one of the tiles of $\mathcal{T}_p(q)$, $\mathcal{P}^b$ is centered in the origin with vertices $A_1A_2A_3\dots A_p$ in the base plane (Figure 4). It is clear that the side curves $c_{A_iA_{i+1}}$ $(i = 1\dots p,\ A_{p+1} \equiv A_1)$ of the base figure are derived from each other by $\frac{2\pi}{p}$ rotation about the vertical x axis, so there are congruent in $\widetilde{\mathbf{SL_2R}}$ sense. The corresponding vertices $B_1B_2B_3\dots B_p$ are generated by a fibre translation τ given by (1.3) with parameter $0 < \Phi \in \mathbb{R}$. The fibre lines through the vertices A_iB_i are denoted by f_i, $(i = 1,\dots,p)$ and the

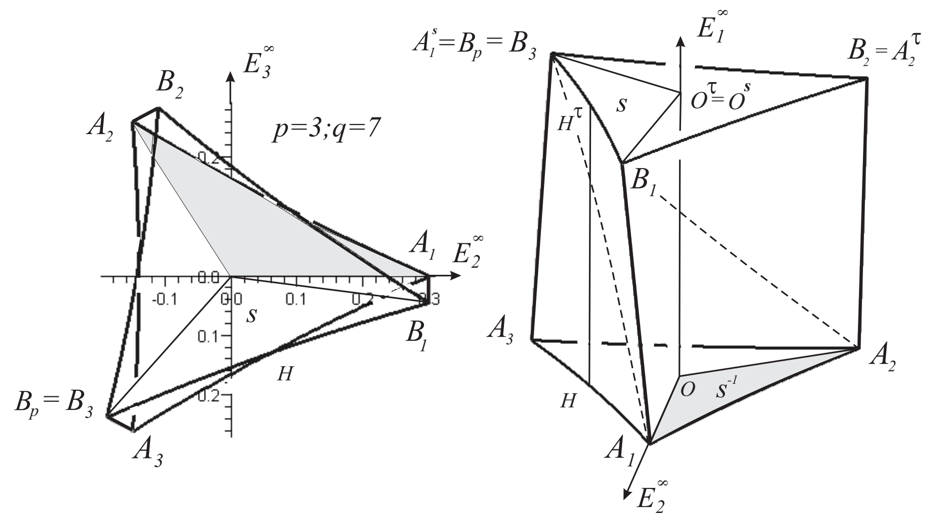

Figure 4

fibre line through the "midpoint" H of the curve $c_{A_1A_p}$ is denoted by f_0. This f_0 will be a half-screw axis as follows below.

The tiling $\mathcal{T}_p(q)$ is generated by a discrete isometry group $\Gamma_p(q) = \mathbf{pq2_1} \subset Isom(\widetilde{\mathbf{SL_2R}})$ which is given by its fundamental domain $A_1A_2OA_1^{\mathbf{s}}A_2^{\mathbf{s}}O^{\mathbf{s}}$ a *topological polyhedron* and the group presentation (see Figure 4 for $p = 3$ and [15] for details):

$$\begin{aligned}\mathbf{pq2_1} &= \{\mathbf{a}, \mathbf{b}, \mathbf{s} : \mathbf{a}^p = \mathbf{b}^q = \mathbf{asa}^{-1}\mathbf{s}^{-1} = \mathbf{babs}^{-1} = \mathbf{1}\} \\ &= \{\mathbf{a}, \mathbf{b} : \mathbf{a}^p = \mathbf{b}^q = \mathbf{ababa}^{-1}\mathbf{b}^{-1}\mathbf{a}^{-1}\mathbf{b}^{-1} = \mathbf{1}\}. \end{aligned} \tag{3.1}$$

Here $\mathbf{a}$ is a p-rotation about the fibre line through the origin (x axis), $\mathbf{b}$ is a q-rotation about the fibre line trough A_1 and $\mathbf{s} = \mathbf{bab}$ is a screw motion $\mathbf{s} : OA_1A_2 \to O^{\mathbf{s}}B_pB_1$. All these can be obtained by formulas (1.7) and (1.8). Then we get the second presentation in (3.1), i.e. $\mathbf{abab} = \mathbf{baba} =: \tau$ is a fibre translation. Then $\mathbf{ab}$ is a $\mathbf{2_1}$ half-screw motion about $f_0 = HH^\tau$ (look at Figure 4) that also determines the fibre translation τ above. This group in (3.1) surprisingly occurred in § 6 of our paper [7] at double links $K_{p,q}$. The coordinates of the vertices $A_1A_2A_3 \dots A_p$ of the base figure and the corresponding vertices $B_1B_2B_3 \dots B_p$ of the cover face can be computed for all given parameters p, q by

$$\tanh(OA_1) = b := \sqrt{\frac{1 - \tan\frac{\pi}{p}\tan\frac{\pi}{q}}{1 + \tan\frac{\pi}{q}\tan\frac{\pi}{q}}}. \tag{3.2}$$

Moreover, the equation of the curve $c_{A_1A_2}$ can be determined as the foot points (see (1.4) and (1.9)) of the corresponding fibre lines. For example, the data of $\mathcal{P}_3(q)$ for some $6 < q \in \mathbb{N}$ are collected in Table 2 by Maple computations.

(p, q)	b
$(3, 7)$	≈ 0.30007426
$(3, 8)$	≈ 0.40561640
$(3, 9)$	≈ 0.47611091
$(3, 10)$	≈ 0.50289355
$(3, 50)$	≈ 0.89636657
$(3, 1000)$	≈ 0.99457331
$(3, \infty)$	1

Table 2

3.1. The volume of the bounded regular prism $\mathcal{P}_p(q)$. The volume formula of a *sector-like* 3-dimensional domain $\mathrm{Vol}(D(\Phi))$ can standardly be computed by the metric tensor g_{ij} (1.13) in hyperboloid coordinates. This defined by the base figure D $(= s^{-1})$ lying in the base plane (see Figure 4) and by fibre translation τ given by (1.3) with the height parameter $\Phi = \pi - \frac{2\pi}{p} - \frac{2\pi}{q}$.

Theorem 3.2. *Suppose we are given a sector-like region D (illustrated in Figure 4), so a continuous function $r = r(\theta)$ where the radius r depends upon the polar angle θ. The volume of domain $D(\Phi))$ is derived by the following integral:*

$$\mathrm{Vol}(D(\Phi)) = \int_D \frac{1}{2}\sinh(2r(\theta))\mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi$$

$$= \int_0^{\Phi}\int_{\theta_1}^{\theta_2}\int_0^{r(\theta)} \frac{1}{2}\sinh(2r(\theta))\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi = \Phi\int_{\theta_1}^{\theta_2}\frac{1}{4}(\cosh(2r(\theta))-1)\ \mathrm{d}\theta. \quad (3.3)$$

Let $\mathcal{T}_p(q)$ be the regular prism tiling above and let $\mathcal{P}_p(q)$ be one of its tiles. We get the following

Theorem 3.3. *The volume of the bounded regular prism $\mathcal{P}_p(q)$ ($3 \le p \in \mathbb{N}$, $\frac{2p}{p-2} < q \in \mathbb{N}$ can be computed by the following simple formula:*

$$\mathrm{Vol}(\mathcal{P}_p(q)) = \mathrm{Vol}(D(p,q,\Phi)) \cdot p, \quad (3.4)$$

where $\mathrm{Vol}(D(p,q,\Phi))$ is the volume of the sector-like 3-dimensional domain that is given by the sector region $OA_1A_2 \subset \mathcal{P}$ (see Figure 4) and by $\Phi = A_1B_1 = \pi - \frac{2\pi}{p} - \frac{2\pi}{q}$, the $\widetilde{\mathbf{SL_2R}}$ height of the prism, depending on p, q.

4. The optimal geodesic ball packings under $\mathbf{pq2_1}$

Sphere packing problems concern arrangements of non-overlapping equal spheres, rather balls, which fill a space. Space is the usual three-dimensional Euclidean space. However, ball packing problems can be generalized to the other 3-dimensional Thurston geometries. But sometimes a difficult problem is – similarly to the hyperbolic space – the exact definition of the packing density. In [16] we extended the problem of finding the densest geodesic ball packing for the other 3-dimensional homogeneous geometries (Thurston geometries). In this paper we study the problem in $\widetilde{\mathbf{SL_2R}}$ and develop a procedure for regular prism tilings and their above group $\mathbf{pq2_1}$ in (3.1).

Let $\mathcal{T}_p(q)$ be a regular prism tiling and let $\mathcal{P}_p(q)$ be one of its tiles which is given by its base figure $\mathcal{P}$ that is centered in the origin with vertices $A_1A_2A_3\dots A_p$ in the base plane of the model (see Figure 5). The corresponding vertices B_1B_2 $B_3\dots B_p$ and $C_1C_2C_3\dots C_p$ are generated by fibre translations $\tau :=$ **abab** $=$ **baba** and its inverse, given by (1.3) (1.8) and (3.1) with parameter Φ at (3.3) also to the above group $\mathbf{pq2_1}$.

It can be assumed by symmetry arguments that the optimal geodesic ball is centered in the origin. Denote by $B(E_0,\rho)$ the geodesic ball of radius ρ centered in $E_0(1;0;0;0)$. The volume $\mathrm{vol}(\mathcal{P}_p(q))$ is given by the parameters p, q and $\Phi \geq 2\rho_{\mathrm{opt}}$. The images of $\mathcal{P}_p(q)$ under the discrete group $\mathbf{pq2_1}$ cover the $\widetilde{\mathbf{SL_2R}}$ space without overlap. For the density of the packing it is sufficient to relate the volume of the optimal ball to that of the solid $\mathcal{P}_p(q)$ (see Definition 3.1).

We study only one case of the multiply transitive geodesic ball packings where the fundamental domains of the $\widetilde{\mathbf{SL_2R}}$ space groups $\mathbf{pq2_1}$ are not prisms. Let the fundamental domains be derived by the Dirichlet—Voronoi cells (D-V cells) where their centers are images of the origin. The volume of the p-times fundamental domain and of the D–V cell is the same, respectively, as in the prism case (for any above (p,q) fixed).

These locally densest geodesic ball packings can be determined for all possible fixed integer parameters (p,q). The optimal radius ρ_{opt} is

$$\rho_{\mathrm{opt}} = \min\Big\{\operatorname{artanh}(OA_1),\ \frac{\Phi}{2} = \frac{\pi}{2} - \frac{\pi}{p} - \frac{\pi}{q},\ \frac{d(O,O^{\mathbf{ab}})}{2}\Big\},$$

where $d(O,O^{\mathbf{ab}})$ is the geodesic distance between O and $O^{\mathbf{ab}}$ by Definition 2.1.

The maximal density of the above ball packings can be computed for any possible parameters p, q. In Table 3 we have summarized some numerical results. The best density that we found ≈ 0.567362 for parameters $p=8, q=10$.

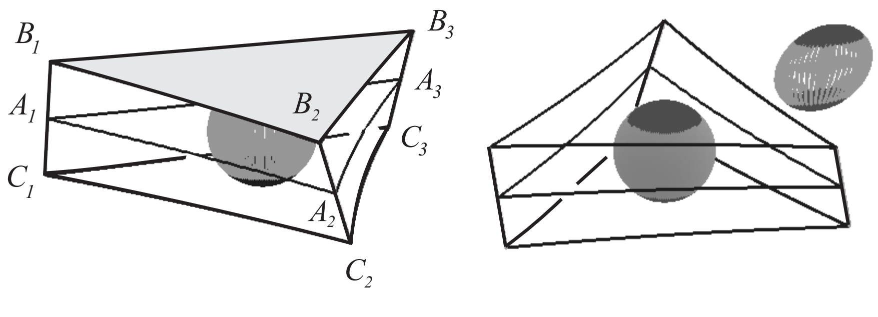

Figure 5. The optimal prism and ball configuration for parameters $p=3$ and $q=7$.

(p, q)	$\rho(K_{\text{opt}})$	$\text{Vol}(B_K)$	$\text{Vol}(\mathcal{P}_p(q))$	$\delta(K_{\text{opt}})$
$(3, 11)$	0.237999	0.057543	0.169931	0.338626
$(3, 12)$	0.261799	0.076892	0.205617	0.373960
$(3, 13)$	0.279134	0.093489	0.238467	0.392044
$(3, 14)$	0.287083	0.101857	0.268561	0.379271
$(3, 50)$	0.350810	0.188371	0.636918	0.295754
$(3, 1000)$	0.370822	0.223543	0.812627	0.275087
$(5, 7)$	0.493679	0.546132	1.218594	0.448165
$(6, 8)$	0.654498	1.350812	2.570209	0.525565
$(6, 9)$	0.692287	1.624770	2.924327	0.555605
$(7, 9)$	0.772932	2.347696	4.181962	0.561386
$(7, 10)$	0.789635	2.523909	4.568217	0.552493
$(\mathbf{8}, \mathbf{10})$	**0.860471**	**3.387783**	**5.971111**	**0.567362**
$(9, 11)$	0.930662	4.456867	7.887074	0.565085
$(9, 3000)$	1.003711	5.838784	13.410609	0.435385
$(20, 60)$	1.361357	18.712577	37.065848	0.504847
$(20, 2000)$	1.387192	20.205264	39.883121	0.506612

Table 3

Remark 4.1. Surprisingly (at the first glance), the analogous translation ball packings led to larger densities, e.g. at $(p, q) = (5, 10000 \to \infty)$ we obtained the density 0.841700 close enough to the $\mathbf{H}^3$ upper bound 0.85326.

Our projective method gives a way of investigating similar problems in Thurston geometries (see e.g. [5], [10]–[14], [16]).

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