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(Para)quaternionic geometry, harmonic forms, and stochastical relaxation

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This paper is dedicated to Professor Lajos Tamássy on the occasion of his ninetieth birthday

Abstract. Both quaternionic and para-quaternionic geometry are important when studying harmonic forms and stochastical relaxation with the help of Fokker–Planck-type or Oguchi-type parabolic equations. In a recent paper the first-named author and H. M. Polatoglou (2012) have shown that the five-dimensional case is the simplest case that the use of para-quaternions is more convenient that the use of quaternions. Now we discuss that case in some detail.

1. Introduction and preliminaries

Quaternionic geometry was studied e.g. in [1], [8], [18]-[22], including the twistor aspect; para-quaternionic geometry was investigated e.g. in [27]–[29], [7], [17]. The initial difference is due to the replacement of matrix units $\mathbf{1}, i\sigma_1, i\sigma_2, i\sigma_3$ of the usual quaternions, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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are generators of the Pauli algebra, by the units

$$\mathbf{1},\ \tilde{\mathbf{i}} = i\sigma_2, \quad \tilde{\mathbf{j}} = \sigma_1, \quad \tilde{\mathbf{k}} = \sigma_3$$

of para-quaternions, so that our $\tilde{\mathbf{i}}$ $\tilde{\mathbf{j}}$, and $\tilde{\mathbf{k}}$ mean $\mathbf{j}, (1/i)\mathbf{i}$, and $(1/i)\mathbf{k}$ in [8], respectively. This is due to our definition of the real Clifford *algebra* $\tilde{\mathbb{H}}$ of *para-quaternions* as generated by 1 and imaginary units $\tilde{i}$, $\tilde{j}$, $\tilde{k}$ satisfying

$$-\tilde{i}^2 = \tilde{j}^2 = \tilde{k}^2 = 1, \quad \tilde{i}\tilde{j} = -\tilde{j}\tilde{i} = \tilde{k}. \tag{1}$$

For a para-quaternionic structure the left module structure is defined up to conjugation in $\tilde{\mathbb{H}}$.

In a more general setting, let V be a real vector space. A *complex structure* on V^{2n} is an endomorphism $J \in \mathrm{End}(V)$ such that $J^2 = -\mathrm{Id}$. A *hypercomplex structure* H on V^{4n} is a triple $(J_\alpha) = (J_1, J_2, J_3)$ of anticommuting complex structures on V satisfying $J_1 J_2 = J_3$; it defines on V the structure of left vector space over quaternions $\mathbb{H} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$ such that multiplications by i, j and k are given by J_1, J_2 and J_3. A *quaternionic structure* on V^{4n} is the 3-dimensional subspace $Q \subset \mathrm{End}(V)$ spanned by a hypercomplex structure H, i.e. $Q = \mathrm{span}_{\mathbb{R}}\{J_1, J_2, J_3\}$.

A triple $\tilde{J}_1$, $\tilde{J}_2$, $\tilde{J}_3$ of anticommuting endomorphisms of V satisfying the relations

$$-\tilde{J_1}^2 = \tilde{J_2}^2 = \tilde{J_3}^2 = \mathrm{Id}, \quad \tilde{J}_1\tilde{J}_2 = \tilde{J}_3$$

is called a *para-hypercomplex structure* on V. Observe that ($\tilde{J}_1$ is a complex structure and) $\tilde{J}_2$ and $\tilde{J}_3$ are para-complex structures on V. A Lie subalgebra $\tilde{Q} \subset \mathfrak{gl}(V)$ is called a *para-quaternionic structure* on V if there exists a basis $\tilde{J}_1$, $\tilde{J}_2$, $\tilde{J}_3$ satisfying the above relations. A para-hypercomplex structure ($\tilde{J}_1$, $\tilde{J}_2$, $\tilde{J}_3$) defines on V the structure of a left module over the Clifford algebra generated by unity 1 and generators $\tilde{i}$, $\tilde{j}$, $\tilde{k}$ satisfying (1).

The Hurwitz twistors are deduced from quaternions and Clifford structures as follows. Let $\mathbb{C}^4(I_2, I_2)$ be the 4-dimensional complex space with the indefinite hermitian metric

$$\kappa = I_{2,2} = \mathrm{diag}^*(I_2, -I_2) = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix},\ I_2 = \mathbf{1},$$

and $\mathbb{R}^5(I_{2,3})$ – the 5-dimensional real space with the indefinite symmetric metric $I_{2,3} = \mathrm{diag}(I_2, -I_3)$. Let $(e_1, \ldots, e_4)$ and $(e_1, \ldots, e_5)$ denote the corresponding canonical bases, and $\circ$ the multiplication acting from $\mathbb{R}^3(I_{2,3}) \otimes_{\mathbb{R}} \mathbb{C}^4(I_{2,2})$ to $\mathbb{C}^4(I_{2,2})$. Let us set

$$\epsilon_\alpha \circ \epsilon_k = C^1_{\alpha k}\epsilon_1 + \cdots + C^4_{\alpha k}\epsilon_4, \quad C_\alpha + (C^j_{\alpha k}), \quad j = 1, \ldots, 5.$$

Consider the algebra $\mathcal{A}_{2,3}$ generated by $\{C^\#_\alpha C_\beta : \alpha \le \beta\}$ where $C^\#_\alpha = \kappa C^*_j \kappa^{-1}$.

An element $x \in \mathcal{A}_{2,3}$ is called *Hurwitz twistor* [12], [13] whenever x has the form

$$x = \sum_{\alpha<\beta} \xi_{\alpha,\beta} C^\#_\alpha, \quad \xi_{\alpha,\beta} \in \mathbb{C}, \tag{2}$$

and im $x^2 = 0$, where $x \in \mathcal{A}_{2,3}$ is defined in the following manner: $x \in \mathcal{A}_{2,3}$ can be written uniquely as

$$x = \sum_{k=0}^{4} x_k, \quad x = \sum_{\alpha_1<\beta_1<\ldots<\alpha_k<\beta_k} \xi_{\alpha_1,\beta_1,\ldots,\alpha_k,\beta_k} C^\#_{\alpha_1} C_{\beta_1} \ldots C^\#_{\alpha_k} C_{\beta_k}, \tag{3}$$

with $x_0 = \xi_0 I_4$ for $k = 0$. We set im $x := x - x_0$ and denote the collection of Hurwitz twistors by H:

$$\mathrm{H} = \Big\{ x = \sum_{\alpha<\beta} \xi_{\alpha,\beta} C^\#_\alpha C_\beta : \mathrm{im}\ x^2 = 0 \Big\}.$$

Traditionally, the 5-dimensional space-time is $\mathbb{R}^5(I_{1,4})$; when speaking on Hurwitz twistors, it seems convenient to associate them with $\mathbb{R}^5(I_{3,2})$ instead of $\mathbb{R}^5(I_{2,3})$. It appears that the expression (2) is an element of H, if and only if the following $\binom{5}{4}$ equations hold:

$$\xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23} = 0,$$

$$\xi_{12}\xi_{35} - \xi_{13}\xi_{25} + \xi_{15}\xi_{23} = 0,$$

$$\xi_{12}\xi_{45} - \xi_{14}\xi_{25} + \xi_{15}\xi_{24} = 0,$$

$$\xi_{13}\xi_{45} - \xi_{14}\xi_{35} + \xi_{15}\xi_{34} = 0,$$

$$\xi_{23}\xi_{45} - \xi_{24}\xi_{35} + \xi_{25}\xi_{34} = 0.$$

In analogous way the anti-objects, called *anti-Hurwitz twistors,* correspond to $\mathbb{R}^5(1,4)$ and are determined by $\binom{5}{4}$ similar equations as well; we denote the collection of those anti-objects by aH. Still in analogy we consider $\mathbb{C}^{16}(I_{8,8})$ and $\mathbb{R}^9(8,1)$ replacing it by $\mathbb{R}^9(1,8)$ which leads to *pseudotwistors* [15]:

$$\mathsf{p} = \Big\{ x = \textstyle\sum_{\alpha<\beta<9} \xi_{\alpha,\beta} C^\#_\alpha C_\beta : \mathrm{im}\ x^2 = 0 \Big\},$$

determined by $\binom{9}{4} = 126$ algebraic equations; we denote the collection of corresponding anti-objects by ap. We may also consider $\mathbb{C}^{64}(I_{32,32})$ and $\mathbb{R}^{13}(6,7)$ replaced by $\mathbb{R}^{13}(7,6)$ which leads to *bitwistors* determined by $\binom{13}{4} = 715$ algebraic equations; we denote their collection by b and the collection of their anti-objects – by ab. The above leads to the so-called *Cartan-like triality* [6].

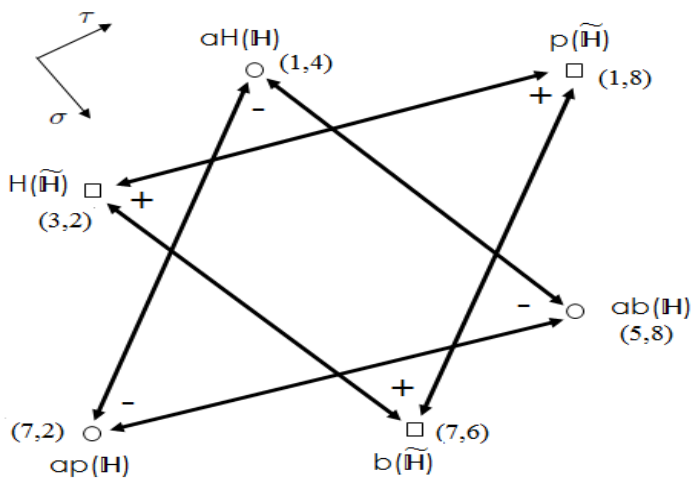

Figure 1. Double Cartan-like triality of Hurwitz twistors, pseudotwistors, and bitwistors.

2. Some relationship with traditional harmonicity and holomorphy

Before we start to use quaternions or para-quaternions for investigating parabolic equations responsible for relaxation, we recall some known results on relations with traditional harmonicity and holomorphy.

2.1. Relationship with harmonic forms. Let $Z_{\mathcal{A}}^{(n)}(U)$ be the space of real-analytic solutions of the structure spinor equations (of spin $\frac{1}{2}n$) on an open set $U \subset \mathbb{C}^{2k}$, $k = 1, 2$. Then [14] they can be written as harmonic forms, i.e., there exists a one-to-one correspondence between spinor solutions and harmonic forms

with respect to:
the $(1,1)$-metric

$$ds^2 := dz^1 d\bar{z}^1 - dz^2 d\bar{z}^2 \; for \;\; k = 1 \;\; \text{(Hurwitz twistors)};$$

the $(0,4)$-metric

$$ds^2 := -dz^1 d\bar{z}^1 - dz^2 d\bar{z}^2 - dz^3 d\bar{z}^3 - dz^4 d\bar{z}^4 \quad \text{for } k = 2 \text{ (pseudotwistors)}.$$

This correspondence can be expressed as:

$$\mathrm{Z}^{(n)}_{\mathcal{A}}(U) \simeq \mathbb{H}^1\left(U, \mathbb{C}^{2^{2k-1}(n-1)}\right) \quad \text{for } k = 1, 2,$$

where

$$\mathbb{H}^1\big(U, \mathbb{C}^{2^{2k-1}(n-1)}\big) = \left\{\phi \in \Gamma^{1,0}\big(U, \mathbb{C}^{2^{2k-1}(n-1)}\big) : \partial\phi = 0 \text{ and } \vartheta\phi = 0\right\}$$

and ϑ is the formally adjoint operator of ∂ with respect to the indefinite fibre $(2^{2k-1}, 0)$-metric

$$d\rho^2 := d\zeta^1 d\bar{\zeta}^1 + d\zeta^2 d\bar{\zeta}^2 + \ldots + d\zeta^{2k-1} d\bar{\zeta}^{2k-1}.$$

2.2. Relationship with the one-dimensional Dolbeault cohomology group. Set

$$\mathcal{P}^1 := \{L^1_1 : L^1_1 \subset \mathbb{C}^4, \text{ linear subspace, } \dim L^1_1 = 1\} \; (\simeq \mathbb{P}^3(\mathbb{C})),$$

$$\mathcal{U}^1 := \{L^1_2 : L^1_2 \subset \mathbb{C}^4, \text{ linear subspace}, \dim L^1_2 = 2\} \;\; (\simeq \mathrm{G}(2,4),$$

$$\mathcal{P}^2 := \{L^2_1 : L^2_1 \subset \mathbb{C}^8, \text{ linear subspace, } \dim L^2_1 = 1\} \; (\simeq \mathbb{P}^7(\mathbb{C})),$$

$$\mathcal{U}^2 := \{L^2_2 : L^2_2 \subset \mathbb{C}^8, \text{ linear subspace, } \dim L^2_2 = 2\} \; (\simeq \mathrm{G}(2,8)),$$

where $\mathbb{P}^3(\mathbb{C})$, $\mathbb{P}^7(\mathbb{C})$, $\mathrm{G}(2,4)$, $\mathrm{G}(2,8)$ are the corresponding complex projective and Grassmannian spaces, respectively. Then we have the following correspondences:

$$\begin{array}{ccc} & \mathrm{H} & \\ \mu_1 \swarrow & & \searrow \nu_1, \\ \mathcal{P}^1 & & \mathcal{U}^1 \end{array} \quad \begin{array}{ccc} & \mathrm{p} & \\ \mu_2 \swarrow & & \searrow \nu_2, \\ \mathcal{P}^2 & & \mathcal{U}^2 \end{array} \quad \begin{array}{ccc} & \mathrm{ab} & \\ \mu_2 \swarrow & & \searrow \nu_2. \\ \mathcal{P}^2 & & \mathcal{U}^2 \end{array} \qquad (4)$$

Let $Z^n_{\mathcal{H}}(U_k)$ be the space of holomorphic solutions of the structure spinor equations (of spin $\frac{1}{2}n$) on an open set U_k, whereas μ_k and ν_k be the related fibre bundles forming the diagrams (4). We set

$$U'_k = \nu_k^{-1}(U_k) \text{ and } U''_k = \mu_k \circ \nu_k^{-1}(U_k) \quad \text{for } k = 1, 2.$$

Then, if every fibre of μ_k is connected, there exists a one-to-one correspondence [14]:

$$Z^n_{\mathcal{H}}(U_k) \simeq \mathrm{H}^1(U''_k, \mathcal{O}(-\alpha_k n - \beta_k)),$$

where H^1 denotes the one-dimensional Dolbeault cohomology group,

$$\mathcal{O}(-\alpha_k n - \beta_k) = \mathcal{O}([e]^{-\alpha_k n - \beta_k}),$$

$[e]$ being the canonical effective divisor of $\mathbb{P}^3(\mathbb{C})$, *while* α_k and β_k are positive integers. Moreover,

$$\alpha_1 = 1,\ \beta_1 = 2; \quad \beta_2 \geq 2.$$

2.3. Relationship with traditional holomorphy. Consider the holomorphic embeddings

$$\mathbb{C}^2 \simeq \mathbb{R}^4 \overset{\iota}{\longrightarrow} G(2,4), \quad \mathbb{R}^4 \ni x \overset{\iota}{\longmapsto} \sum_{\alpha=1}^{3} x^\alpha S_\alpha + x^4 I_4, \tag{5}$$

$$\mathbb{C}^4 \simeq \mathbb{R}^8 \overset{\iota}{\longrightarrow} G(8,16), \quad \mathbb{R}^8 \ni x \overset{\iota}{\longmapsto} \sum_{\alpha=1}^{7} x^\alpha S_\alpha + x^8 I_8, \tag{6}$$

where $G(\tau, \nu)$ stands for a τ-dimensional Grassmannian submanifold, while S_α and I_4 or I_8 are generators of the corresponding algebra, proposed explicitly first in [13], so that they are real parts of holomorphic mappings in the classical sense. The result we are going to quote was first published without specification of the quaternionic or para-quaternionic dependence in [14], [15] and with specifying this dependence – in [7]. In the case $(\sigma - 1, \tau) = (0, 4)$ resp. $(0, 8)$ we are interested, it states that *there exists a complex structure* $I = I[\iota(\sigma - 1, \tau)]$ *on the holomorphic embedding* (5) *resp.* (6) *with properties*

$$\iota(0,4) = \iota(0,4)(\mathbb{H}),\ \textit{resp.}\ \iota(0,8) = \iota(0,8)(\widetilde{\mathbb{H}}) \tag{7}$$

and the each embedding concerned is the real part of a holomorphic mapping in the classical sense.

We introduce seven 2×2-complex matrices which we call *atoms*:

$$A_0 = A_0(\mathbb{H}) = \begin{pmatrix} u & v_{\mathbb{H}} \\ -v_{\mathbb{H}} & u \end{pmatrix}, \quad A_1 = \begin{pmatrix} \hat{u} & \hat{v} \\ v & -u \end{pmatrix}, \quad A_2 = \begin{pmatrix} -u & -\hat{v} \\ -v & \hat{u} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}, \quad A_4 = \begin{pmatrix} \hat{w} & 0 \\ 0 & \hat{w} \end{pmatrix}$$

$$A_5 = A_5(\tilde{\mathbb{H}}) \begin{pmatrix} t_{\tilde{\mathbb{H}}} & 0 \\ 0 & t_{\tilde{\mathbb{H}}} \end{pmatrix}, \quad A_6 = A_6(\tilde{\mathbb{H}}) = \begin{pmatrix} t_t\hat{\mathbb{H}} & 0 \\ 0 & t\hat{\mathbb{H}} \end{pmatrix},$$

where u in A_0 and $v_{\mathbb{H}}$ are given in

$$u = x_4 + ix_3 \in \mathbb{C}, \quad v_{\mathbb{H}} = x_2 + ix_1 \in \mathbb{C}, \quad v_{\tilde{\mathbb{H}}} = x_1 + ix_2 \in \mathbb{C},$$

whereas u in A_1, A_2; v, w, and $t_{\tilde{\mathbb{H}}}$ are given in

$$u = x_3 + ix_8 \in \mathbb{C}, \quad v = x_1 + ix_2 \in \mathbb{C}, \quad w = x_4 + ix_5 \in \mathbb{C},$$

$$t_{\mathbb{H}} = x_7 + ix_6 \in \mathbb{C}, \quad t_{\tilde{\mathbb{H}}} = x_6 + ix_7 \in \mathbb{C}.$$

Let

$$\zeta_{\mathbb{H}} = (u, v_{\mathbb{H}}) \in \mathbb{C}^2 \text{ and } z_{\tilde{\mathbb{H}}} = (u, v, w, t_{\tilde{\mathbb{H}}}) \in \mathbb{C}^4.$$

The *atomization method* allows us to find the following explicit formulae for the embedding in question:

$$\iota(\zeta_{\mathbb{H}}) = A_0 \text{ for } (0,4), \quad \iota(z_{\tilde{\mathbb{H}}}) = \begin{pmatrix} A_1 & A_3 & A_5 & 0 \\ A_4 & A_2 & 0 & A_5 \\ A_6 & 0 & A_2 & -A_3 \\ 0 & A_6 & -A_4 & A_5 \end{pmatrix} \text{ for } (0,8).$$

2.4. Pseudotwistros of degree 3 vs. those of degree 1. Consider *quaternal embeddings*, like $i_{\mathbb{A}} = \operatorname{diag}(\mathbb{A}, \mathbb{A}, \mathbb{A}, \mathbb{A})$, $\mathbb{A} = A_3$, acting from $G(2,4)$ to $G(8,16)$, instead of (2), *pseudotwistors of degree* k with x as in the second formula in (3), and collections $\mathcal{J}^{(k)}$ of all such x with $\operatorname{im} x^2 = 0$. Consider the following analogues of (4):

$$\begin{array}{ccc} & \mathcal{J}^{(k)} & \\ \swarrow & & \searrow \\ J_-^{(k)} & & J_+^{(k)} \end{array} \quad k = 1, 3; \quad \begin{array}{ccc} & \mathcal{J}^{(1)} & \\ \swarrow & & \searrow \\ J_{-\mathbb{A}}^{(1)} & & J_{+\mathbb{A}}^{(1)} \end{array}, \quad J_{\pm\mathbb{A}}^{(1)} = \left(J_{\pm\mathbb{A}}^{(1)}\right). \tag{8}$$

If $k = 1$, then for any quaternal embedding $i_{\mathbb{A}}$ of some $V_{\mathbb{A}}$ in $G(2,4)$ to $G(8,16)$ we have

$$\iota_{\mathbb{A}}^* \mathcal{J}^{(1)} = \mathcal{J}_{\mathbb{A}}^{(1)}, \quad \iota_{\mathbb{A}}^* \mathcal{J}_+^{(1)} = \mathcal{J}_{+\mathbb{A}}^{(1)}, \quad \iota_{\mathbb{A}}^* \mathcal{J}_-^{(1)} = \mathcal{J}_{-\mathbb{A}}^{(1)}$$

and the diagram in (7) related with $\mathcal{J}_{\mathbb{A}}^{(1)}$. If $k = 3$, we have

$$\mathcal{J}_-^{(3)} \subseteq \bigoplus_{\mathbb{A}} \mathcal{J}_{\mathbb{A}-}^{(1)} \cdot \mathcal{J}_{\mathbb{A}_+^c}^{(1)}, \quad \mathcal{J}_+^{(3)} \subseteq \bigoplus_{\mathbb{A}} \mathcal{J}_{\mathbb{A}+}^{(2)} \cdot \mathcal{J}_{\mathbb{A}_+^c}^{(2)}, \tag{9}$$

where $\mathbb{A} \cup \mathbb{A}^c = \mathbb{A}^*$. The addends in (9) depend on the quaternionic or para-quaternionic structure according to dependence of $\mathbb{A}$ expressed in terms of $A_1, \ldots, A_6$.

3. Stochastical relaxation and the specific role of dimension 5

3.1. Setting of the problem. We consider a *modified Oguchi equation* [25], [4], [11]

$$\frac{\partial}{\partial t}\langle s(t,\bar{\tau})\rangle = -\frac{1}{\hat{\tau}}[\langle s(t,\bar{\tau})\rangle - \langle s(t,\bar{\tau})\rangle_{\text{l.e.}}] \tag{10}$$

where $\bar{\tau}$ is the spin-lattice relaxation time related to a spin on **R**-site, $\mathbf{R} = (x_1, \ldots, x_\tau)$ in $\mathbb{R}^r, r = 1, 2, \ldots; \bar{\tau} = x_{\tau+1}$ stands for the *stochastic variable* responsible for the stochastic behaviour of the lattice, describing thermal oscillations of spin, and $\langle s(t,\bar{\tau})\rangle$ denotes the canonical average of spin; $\langle s(t,\bar{\tau})\rangle_{\text{l.e.}}$ being its local equilibrium value. $\langle s(t,\tau)\rangle$ does not depend on the positions in a fixed layer $x_\tau = \hat{x}_\tau$. Set

$$\Gamma = \frac{1}{\hat{\tau}}\left[1 - \frac{1}{2}(1 - 4\langle s\rangle^2)\frac{\hat{x}_\tau J}{k_B T}\right], \tag{11}$$

$$\Lambda = \frac{a^2}{\hat{\tau}} \cdot \frac{1}{2}(1 - 4\langle s\rangle^2)\frac{\hat{x}_\tau J}{k_B T}, \tag{12}$$

where J is the parameter of the theory responsible for the interaction between two neighbouring spins, and a is the lattice constant. The equation (10) can be transformed to

$$\frac{\partial}{\partial t}\langle s(t,\bar{\tau})\rangle = -\Gamma\langle s(t,\bar{\tau})\rangle + \Lambda\left(\sum_{\nu=1}^{\tau}\frac{\partial^2}{\partial x_\nu^2} - \frac{\hat{a}^2}{a^2}\frac{\partial^2}{\partial\bar{\tau}^2}\right)\langle s(t,\bar{\tau})\rangle,$$

where Γ and Λ are given by (11) and (12), respectively, while $\hat{a}$ is the amplitude of stochastic movement. Then the substitution $\bar{\tau} = (\hat{a}/a)\tilde{\tau}$ brings the above equation to

$$\frac{\partial}{\partial t}\langle s(t,\tilde{\tau})\rangle = -\Gamma\langle s(t,\tilde{\tau})\rangle + \Lambda\left(\sum_{\nu=1}^{\tau}\frac{\partial^2}{\partial x_\nu^2} - \frac{\partial^2}{\partial\tilde{\tau}^2}\right)\langle s(t,\tilde{\tau})\rangle. \tag{13}$$

In [4], for solving (13), $\tau = 2$ and 3, the quaternionic approach was used systematically.

By (7), the 8- (resp. 4-)dimensional stochastical relaxation problem may be considered in relation with the pseudotwistors in p (resp. anti-Hurwitz twistors in aH) in terms of para-quaternions (resp. quaternions) [11]. By restriction of solution of (13) an analogous conclusion holds for the 7-, 6-, and 5- (resp. 3-, 2-, and 1-) dimensional stochastical relaxation problems as well as for the 8-, 7-, 6-, and 5- (resp. 4-, 3-, 2-, and 1-) dimensional relaxation problems related with (13)

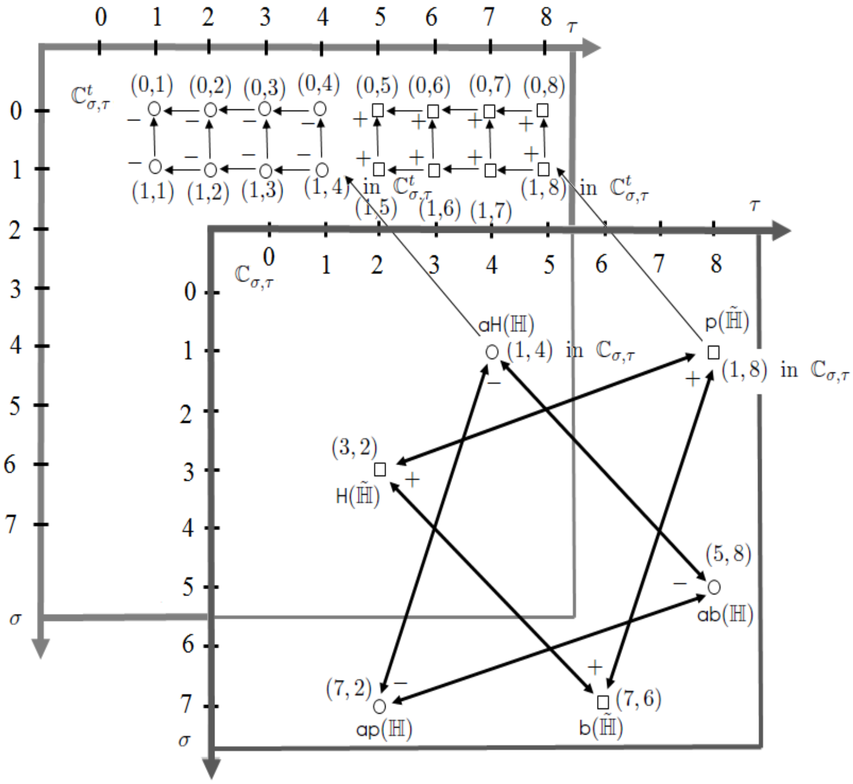

Figure 2. Applicability of para-quaternions (resp. quaternions) and pseudo-twistors (resp. anti-Hurwitz twistors) for 5-, 6-, 7-, 8- (resp. 1-, 2-, 3-, 4-)dimensional relaxation and stochastical relaxation problems.

for $\langle s(t,\bar{\tau})\rangle = \langle s(t)\rangle$, $\langle s(t,\bar{\tau})\rangle_{\text{l.e.}} = 0$. The reasoning is illustrated by Figure 2; the family of solutions to (13) for $\tau = 8$ is represented by the point $(1,8)$ on the projection plane $\mathbb{C}_{\sigma,\tau} = \{(\sigma,\tau)\}$.

It seems interesting to consider, with help of para-quaternions, the simplest proper case of equation (13), i.e. for $\tau = 5$. Let

$$s(t,\tilde{\tau}) = s(x,y,z;\xi,\eta,\tilde{\tau};t), \quad (x,y,z,\xi,\eta,\tilde{\tau}) \in \mathbb{R}^6 \simeq \mathbb{C}^3 \quad t \in \mathbb{R}^+.$$

Then the equation (13) reads

$$\frac{\partial}{\partial t}\langle s(t,\tilde{\tau})\rangle = -\Gamma\langle s(t,\tilde{\tau})\rangle + \Lambda\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}-\frac{\partial^2}{\partial \tilde{\tau}^2}\right)\langle s(t,\tilde{\tau})\rangle. \tag{14}$$

Mathematically, a specific position of this equation is connected with the fact that $\mathbb{C}^2 \simeq \mathbb{R}^4$ in (5) and $\mathbb{C}^4 \simeq \mathbb{R}^8$ in (6). We are going to discuss the equation (13) in detail.

3.2. Setting of a linearization procedure. As in [4], in relation with (13) we concentrate on the Fokker–Planck type [26] equation

$$\frac{\partial}{\partial t}s(t) = -\Gamma s_*(t) + \Lambda\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}-\frac{\partial^2}{\partial \tilde{\tau}^2}\right)s(t), \quad (x,y,z,\xi,\eta)\in\mathbb{R}^4,\ t\in\mathbb{R}^+, \tag{15}$$

where $s_*(t)$ is an arbitrary admissible function; in particular we may take [3]:

$$s_* = s_0 \equiv -\int_0^t\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \times \exp\left[\frac{(x-x')^2+(y-y')^2+(z-z')^2+(\xi-\xi')^2+(\eta-\eta')^2-(\tilde{\tau}-\tilde{\tau}')^2}{4\Lambda(x',y',z',\xi',\eta',\tilde{\tau}',t')(t-t')}\right] \times \frac{(\Gamma s_0)(x-x',y-y',z-z',\xi-\xi',\eta-\eta',\tilde{\tau}-\tilde{\tau}',t-t')}{2\sqrt{\Lambda(x',y',z',\xi',\eta',\tilde{\tau}',t')(t-t')}} \times dx'dy'dz'd\xi'd\eta'd\tilde{\tau}'dt'. \tag{16}$$

According to [23], [24] we need an 8-dimensional vector

$$\mathbf{s} = (s, s_0, s_1, \ldots, s_6) \in \mathbb{R}^8 \simeq \mathbb{C}^4 \tag{17}$$

and two bases:

$$(\varepsilon, \varepsilon_\alpha) = (\varepsilon, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_5) \tag{18}$$

say, for the space S of variables x, y, z, ξ, η, $\tilde{\tau}$, t, and

$$(e, e_j) = (e, e_0, e_1, \ldots, e_6) \tag{19}$$

for the space V of solution (17). Hence, in our case, (ε_α) consists of complex 8×8-matrices. They have to satisfy the relations

$$\varepsilon^2 = -\varepsilon_0, \quad \varepsilon_\alpha^2 = \varepsilon_0, \quad \alpha = 1,\ldots,4; \quad \varepsilon_5^2 = 0,$$

$$\varepsilon\varepsilon_\alpha + \varepsilon_\alpha\varepsilon = 0, \quad \varepsilon_\alpha\varepsilon_\beta + \varepsilon_\beta\varepsilon_\alpha = 0, \quad \alpha, \beta = 1, \dots, 5. \tag{20}$$

The explicit formulae for (18) in terms of para-quaternions can be deduced from the corresponding formulae obtained for $\tau = 1$ and 2 in [4] (after converting quaternions to para-quaternions) with the use of interaction procedure of [18], formulae (1) and (18), expressed already in terms of para-quaternions. The explicit formulae will be published in a subsequent paper. The algebra determined by the basis (18) satisfying the conditions (20) is known as the *Clifford-Grassmann algebra* $Cl^{*0}_{1,4}(\mathbb{C})$.

Then we find analogues of the familiar operators $\partial_{\bar z}$ and ∂_z: $\bar\partial$ and ∂ (say):

$$\bar\partial \mathbf{s} = P\mathbf{s} - \mathbf{v}, \quad \Lambda\partial(P\mathbf{s}) = (\partial/\partial t)\mathbf{s} \quad \text{with } \Lambda\partial\mathbf{v} = -\Gamma\mathbf{s}. \tag{21}$$

Here

$$\mathbf{v} \in V, \quad \Lambda\partial(P\mathbf{s}) = \partial(Q\mathbf{s}), \tag{22}$$

Q being a polynomical of $\varepsilon, \varepsilon_0, \varepsilon_1, \dots, \varepsilon_4$:

$$\begin{aligned} Q\mathbf{s} = {} & s^1_1 \odot \varepsilon + s^1_2 \odot \varepsilon_0 + s^1_3 \odot \varepsilon_1 + \dots + s^1_6 \odot \varepsilon_4 + s^1_7 \odot \varepsilon\varepsilon_0 \\ & + s^2_1 \odot \varepsilon\varepsilon_1 + \dots + s^2_4 \odot \varepsilon\varepsilon_4 + s^2_5 \odot \varepsilon_0\varepsilon_1 + \dots + s^2_7 \odot \varepsilon_0\varepsilon_3 \\ & + s^3_1 \odot \varepsilon_0\varepsilon_4 + s^3_2 \odot \varepsilon_1\varepsilon_2 + \dots \\ & + s^3_4 \odot \varepsilon_1\varepsilon_4 + s^3_5 \odot \varepsilon_2\varepsilon_3 + s^3_6 \odot \varepsilon_2\varepsilon_4 + s^3_7 \odot \varepsilon_3\varepsilon_4, \end{aligned} \tag{23}$$

where

$$s^k_j, \ j = 1, \dots, 7; \quad k = 1, \dots, 4, \text{ belong to } V \text{ and are } \mathbb{C}^4\text{-valued}, \tag{24}$$

while $\odot$ is the multiplication $\odot : V \otimes S \to V$ in the algebra $Cl^{*0}_{1,4}(\mathbb{C})$. Indeed, from (18) and (19) we infer that

$$\begin{aligned} \Lambda\partial\bar\partial\mathbf{s} &= \Lambda\partial(P\mathbf{s}) - \Lambda\partial\mathbf{v} = \frac{\partial}{\partial t}s - \Gamma s \\ &= \Lambda\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \tilde\tau^2}\right)\mathbf{s}, \end{aligned}$$

so

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \tilde\tau^2}\right)\mathbf{s} = \partial\bar\partial\mathbf{s}$$

with s being in V and $\mathbb{C}^4$-valued. Hence by (18)-(20) we arrive at the formulae

$$\mathbf{s} = s^4_1 \odot \varepsilon + s^4_2 \odot \varepsilon_0 + s^4_3 \odot \varepsilon_1 + \dots + s^4_6 \odot \varepsilon_4 + s^4_7 \odot \varepsilon\varepsilon_0 + s^5_1 \odot \varepsilon\varepsilon_1 + \dots$$

$$
\begin{aligned}
&+ s_4^5 \odot \varepsilon\varepsilon_4 + s_5^5 \odot \varepsilon_0\varepsilon_1 + \dots + s_7^5 \odot \varepsilon_0\varepsilon_3 + s_1^6 \odot \varepsilon_0\varepsilon_4 + s_2^6 \odot \varepsilon_1\varepsilon_2 + \dots \\
&+ s_4^6 \odot \varepsilon_1\varepsilon_4 + s_5^6 \odot \varepsilon_2\varepsilon_3 + s_6^6 \odot \varepsilon_2\varepsilon_4 + s_7^6 \odot \varepsilon_3\varepsilon_4 + (Q\mathbf{s}) \odot \upsilon \\
&= se + s_0e_0 + s_1e_1 + \dots + s_6e_6 \qquad (25)
\end{aligned}
$$

with

$$s, s_0, s_1, \dots, s_6, \text{ being in } V \text{ and } \mathbb{C}^4\text{-valued.} \qquad (26)$$

3.3. The fundamental solution. In order to find s effectively (in principle we do not need $s, s_0, s_1, \dots, s_6$) we have to find the system of fundamental solutions of the equation (15) and to be able to compare on both sides of (25) the coordinates with respect to e. Here we have to remember that

$$s_j^k, j = 1, \dots, 7; \quad ; k = 1, \dots, 6,$$

are linear combinations of $e, e_0, e_1, \dots, e_6$, so we need to determine the multiplication scheme for $e \odot \varepsilon_\alpha, e_0 \odot \varepsilon_\alpha, e_j \odot \varepsilon_\alpha$ in the algebra $Cl_{1,4}^{*0}$; the multiplication $\odot$ has to be compatible with the problem of solving the equation (15).

As far as the first question is concerned, we have

$$
\begin{aligned}
\mathbf{s} = {} & c_1^1\mathbf{s}^x + c_2^1\mathbf{s}^y + c_3^1\mathbf{s}^z + c_4^1\mathbf{s}^\xi + c_5^1\mathbf{s}^\eta + c_6^1\mathbf{s}^{\tilde\tau} + c_1^2\mathbf{s}^{xy} + c_2^2\mathbf{s}^{xz} + c_3^2\mathbf{s}^{x\xi} \\
&+ c_4^2\mathbf{s}^{x\eta} + c_5^2\mathbf{s}^{x\tilde\tau} + c_6^2\mathbf{s}^{yz} + c_1^3\mathbf{s}^{y\xi} + c_2^3\mathbf{s}^{y\eta} + c_3^3\mathbf{s}^{y\tilde\tau} + c_4^3\mathbf{s}^{z\xi} + c_5^3\mathbf{s}^{z\eta} + c_6^3\mathbf{s}^{z\tilde\tau} \\
&+ c_1^4\mathbf{s}^{\xi\eta} + c_2^4\mathbf{s}^{\xi\tilde\tau} + c_3^4\mathbf{s}^{\eta\tilde\tau} + c_4^4\mathbf{s}^{xt} + c_5^4\mathbf{s}^{yt} + c_6^4\mathbf{s}^{zt} + c_1^5\mathbf{s}^{\xi t} + c_2^5\mathbf{s}^{\eta t} + c_3^5\mathbf{s}^{\tilde\tau t} \\
&+ c_4^5\mathbf{s}^{xyt} + c_5^5\mathbf{s}^{xzt} + c_6^5\mathbf{s}^{x\xi t} + c_1^6\mathbf{s}^{x\eta t} + c_2^6\mathbf{s}^{x\tilde\tau t} + c_3^6\mathbf{s}^{yzt} + c_4^6\mathbf{s}^{y\xi t} + c_5^6\mathbf{s}^{y\eta t} \\
&+ c_6^6\mathbf{s}^{y\tilde\tau t} + c_1^7\mathbf{s}^{z\xi t} + c_2^7\mathbf{s}^{z\eta t} + c_3^7\mathbf{s}^{z\tilde\tau t} + c_4^7\mathbf{s}^{\xi\eta t} + c_5^7\mathbf{s}^{\xi\tilde\tau t} + c_6^7\mathbf{s}^{\eta\tilde\tau t}, \qquad (27)
\end{aligned}
$$

where

$$c_k^j, \ j = 1, \dots, 7; \quad k = 1, \dots, 6,$$

are complex constants to be determined from the initial conditions

$$
\begin{aligned}
&s_{\mathbf{R}=0}(0) = s_0, \ s(t) \to \infty \quad \text{as} \quad (\mathbf{R}, \tilde\tau, t) \to (\infty, \tilde\tau_0, t_0), \\
&s(t) \to 0 \text{ as } (\mathbf{R}, \tilde\tau, t) \to (\mathbf{R}_0, \tilde\tau_0, t_0) \text{ for some } \mathbf{R}_0, \tilde\tau_0, t_0, \qquad (28)
\end{aligned}
$$

and the boundary conditions, and

$$
\begin{gathered}
\mathbf{s}^x, \mathbf{s}^y, \mathbf{s}^z, \mathbf{s}^\xi, \mathbf{s}^\eta, \mathbf{s}^{\tilde\tau}, \mathbf{s}^{xy}, \mathbf{s}^{xz}, \mathbf{s}^{x\xi}, \mathbf{s}^{x\eta}, \mathbf{s}^{x\tilde\tau}, \mathbf{s}^{yz}, \mathbf{s}^{y\xi}, \mathbf{s}^{y\eta}, \\
\mathbf{s}^{y\tilde\tau}, \mathbf{s}^{z\xi}, \mathbf{s}^{z\eta}, \mathbf{s}^{z\tilde\tau}, \mathbf{s}^{\xi\eta}, \mathbf{s}^{\xi\tilde\tau}, \mathbf{s}^{\eta\tilde\tau}, \mathbf{s}^{xt}, \mathbf{s}^{yt}, \mathbf{s}^{zt}, \mathbf{s}^{\xi t}, \mathbf{s}^{\eta t}, \mathbf{s}^{\tilde\tau t}, \mathbf{s}^{xyt}, \\
\mathbf{s}^{xzt}, \mathbf{s}^{x\xi t}, \mathbf{s}^{x\eta t}, \mathbf{s}^{x\tilde\tau t}, \mathbf{s}^{yzt}, \mathbf{s}^{y\xi t}, \mathbf{s}^{y\eta t}, \mathbf{s}^{y\tilde\tau t}, \mathbf{s}^{z\xi t}, \mathbf{s}^{z\eta t}, \mathbf{s}^{z\tilde\tau t}, \mathbf{s}^{\xi\eta t}, \mathbf{s}^{\xi\tilde\tau t}, \mathbf{s}^{\eta\tilde\tau t},
\end{gathered}
$$

are fundamental solutions of the equation (15). Explicitly, we get

$$\begin{aligned}
\mathbf{s}^x &= (\partial s_0)\varepsilon, & \mathbf{s}^{xy} &= (\partial s_0)\varepsilon\varepsilon_0, & \mathbf{s}^{y\xi} &= (\partial s_0)\varepsilon_0\varepsilon_2,\\
\mathbf{s}^y &= (\partial s_0)\varepsilon_0, & \mathbf{s}^{xz} &= (\partial s_0)\varepsilon\varepsilon_1, & \mathbf{s}^{y\eta} &= (\partial s_0)\varepsilon_0\varepsilon_3\\
\mathbf{s}^z &= (\partial s_0)\varepsilon_1, & \mathbf{s}^{x\xi} &= (\partial s_0)\varepsilon\varepsilon_2, & \mathbf{s}^{y\tilde{\tau}} &= (\partial s_0)\varepsilon_0\varepsilon_4,\\
\mathbf{s}^\xi &= (\partial s_0)\varepsilon_2, & \mathbf{s}^{x\eta} &= (\partial s_0)\varepsilon\varepsilon_3, & \mathbf{s}^{z\xi} &= (\partial s_0)\varepsilon_1\varepsilon_2,\\
\mathbf{s}^\eta &= (\partial s_0)\varepsilon_3, & \mathbf{s}^{x\tilde{\tau}} &= (\partial s_0)\varepsilon\varepsilon_4, & \mathbf{s}^{z\eta} &= (\partial s_0)\varepsilon_1\varepsilon_3,\\
\mathbf{s}^{\tilde{\tau}} &= (\partial s_0)\varepsilon_4, & \mathbf{s}^{yz} &= (\partial s_0)\varepsilon_0\varepsilon_1, & \mathbf{s}^{z\tilde{\tau}} &= (\partial s_0)\varepsilon_1\varepsilon_4,
\end{aligned} \tag{29}$$

$$\begin{aligned}
\mathbf{s}^{\xi\eta} &= (\partial s_0)\varepsilon_2\varepsilon_3, & \mathbf{s}^{\xi t} &= s_0\varepsilon_2 + (\partial s_0)\varepsilon_2\varepsilon_5,\\
\mathbf{s}^{\xi\tilde{\tau}} &= (\partial s_0)\varepsilon_2\varepsilon_4, & \mathbf{s}^{\eta t} &= s_0\varepsilon_3 + (\partial s_0)\varepsilon_3\varepsilon_5,\\
\mathbf{s}^{\eta\tilde{\tau}} &= (\partial s_0)\varepsilon_3\varepsilon_4, & \mathbf{s}^{\tilde{\tau}t} &= s_0\varepsilon_4 + (\partial s_0)\varepsilon_4\varepsilon_5,\\
\mathbf{s}^{xt} &= s_0\varepsilon + (\partial s_0)\varepsilon\varepsilon_5, & \mathbf{s}^{xyt} &= -s_0\varepsilon\varepsilon_0 + (\partial s_0)\varepsilon\varepsilon_0\varepsilon_5,\\
\mathbf{s}^{yt} &= s_0\varepsilon_0 + (\partial s_0)\varepsilon_0\varepsilon_5, & \mathbf{s}^{xzt} &= -s_0\varepsilon\varepsilon_1 + (\partial s_0)\varepsilon\varepsilon_1\varepsilon_5,\\
\mathbf{s}^{zt} &= s_0\varepsilon_1 + (\partial s_0)\varepsilon_1\varepsilon_5, & \mathbf{s}^{x\xi t} &= -s_0\varepsilon\varepsilon_2 + (\partial s_0)\varepsilon\varepsilon_2\varepsilon_5,
\end{aligned} \tag{30}$$

$$\begin{aligned}
\mathbf{s}^{x\eta t} &= -s_0\varepsilon\varepsilon_3 + (\partial s_0)\varepsilon\varepsilon_3\varepsilon_5, & \mathbf{s}^{z\xi t} &= -s_0\varepsilon_1\varepsilon_2 + (\partial s_0)\varepsilon_1\varepsilon_2\varepsilon_5,\\
\mathbf{s}^{x\tilde{\tau}t} &= -s_0\varepsilon\varepsilon_4 + (\partial s_0)\varepsilon\varepsilon_4\varepsilon_5, & \mathbf{s}^{z\eta t} &= -s_0\varepsilon_1\varepsilon_3 + (\partial s_0)\varepsilon_1\varepsilon_3\varepsilon_5,\\
\mathbf{s}^{yzt} &= -s_0\varepsilon_0\varepsilon_1 + (\partial s_0)\varepsilon_0\varepsilon_1\varepsilon_5, & \mathbf{s}^{z\tilde{\tau}t} &= -s_0\varepsilon_1\varepsilon_4 + (\partial s_0)\varepsilon_1\varepsilon_4\varepsilon_5,\\
\mathbf{s}^{y\xi t} &= -s_0\varepsilon_0\varepsilon_2 + (\partial s_0)\varepsilon_0\varepsilon_2\varepsilon_5, & \mathbf{s}^{\xi\eta t} &= -s_0\varepsilon_2\varepsilon_3 + (\partial s_0)\varepsilon_2\varepsilon_3\varepsilon_5,\\
\mathbf{s}^{y\eta t} &= -s_0\varepsilon_0\varepsilon_3 + (\partial s_0)\varepsilon_0\varepsilon_3\varepsilon_5, & \mathbf{s}^{\xi\tilde{\tau}t} &= -s_0\varepsilon_2\varepsilon_4 + (\partial s_0)\varepsilon_2\varepsilon_4\varepsilon_5,\\
\mathbf{s}^{y\tilde{\tau}t} &= -s_0\varepsilon_0\varepsilon_4 + (\partial s_0)\varepsilon_0\varepsilon_4\varepsilon_5, & \mathbf{s}^{\eta\tilde{\tau}t} &= -s_0\varepsilon_3\varepsilon_4 + (\partial s_0)\varepsilon_3\varepsilon_4\varepsilon_5,
\end{aligned} \tag{31}$$

where s_0 is determined by (16).

As far as second question is concerned, we have the multiplication rules

$$e_j \odot \varepsilon_\alpha = (e, e_j \odot \varepsilon_\alpha)e + \sum_{k=0}^{6}(e_k, e_j \odot \varepsilon_\alpha)e_k, \quad e_k = (e, e_k)e + \sum_{j=0}^{6}(e_j, e_k)e_j,$$
$$[(e^j e^k)] = [(e_j, e_k)]^{-1},\ j, k = 0, \dots, 6; \quad \alpha = 0, \dots, 5, \tag{32}$$

and analogous rules for (e, e), (e, e_j) and (e_k, e).

Here $[(e_j, e_k)]$ denotes the matrix of all elements (e_j, e_k) including (e, e), (e_j, e) and (e, e_k)-the scalar product of ej and e_k etc., where we follow the convection (19).

3.4. Concluding the proof. Formulation of the result. Thanks to Sections 3.2 and 3.3 we have proved

Theorem 1. *Let* $s = s(t)$ *be a solution of the Fokker–Planck type equation* (15)*, where* Γ *and* Λ *are* C^1*-scalar functions of* $\mathbf{R} = (x, y, z, \xi, \eta) \in \mathbb{R}^5$, $\tau \in \mathbb{R}$, *and* $t \in \mathbb{R}^+$. *Consider the space* S *of variables* x, y, z, ξ, η, $\tilde{\tau}$, t *with a basis* (18) *of complex* 8×8*-matrices specified the relations* (20) *and the space* V *of solutions* (17) *forming the algebra* $Cl^{*0}_{1,4}(\mathbb{C})$, *with a basis* (19) *of the linearized problem*

$$\bar{\partial}\mathbf{s} = P\mathbf{s} - \mathbf{v}, \quad \Lambda\partial(P\mathbf{s}) = (\partial/\partial t)\mathbf{s}, \quad \text{with} \quad \Lambda\partial\mathbf{v} = -\Gamma\mathbf{s}, \tag{33}$$

where

$$\mathbf{v} \in V, \quad \Lambda\partial(P\mathbf{s}) = \partial(Q\mathbf{s}), \tag{34}$$

and Q *being a polynomial of* $\varepsilon, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_4$, *where* s *is related to* $\mathbf{s}$ *by* (17).

(a) *Explicitly, a solution* $\mathbf{s}$ *of* (33) *can be expressed by* (25) *with coefficients as in* (26) *and* (23)*, and the multiplication* $\odot : V \times S \to V$ *in the algebra* $Cl^{*0}_{1,4}(\mathbb{C})$, *which has to satisfy the rules* (32)*. The polynomial* Q *is given by* (23) *and the matrices* (18) *can be satisfied according to* (20) *in the terms of para-quaternions.*

(b) *Then the general solution of the system* (33) *is a linear combination* (27) *with complex coefficients of* 42 *fundamental solutions which, in the case of* (16)*, are explicitly given by the formulae* (29)–(31).

Differently speaking, we have an equivalent

Theorem 2. (i) *Let* Q *be a polynomial of* $\varepsilon, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_4$, *given by* (23)*, where* (18) *are complex* 8×8*-matrices satisfying the relations* (20) *and forming a basis of the space* V *of variables* x, y, z, ξ, η, $\tilde{\tau}$, t *with* P *as in* (34)*, where* V *is the space of solutions* (17) *forming the algebra* $Cl^{*0}_{1,4}(\mathbb{C})$ *with a basis* (19)*, of the linearized problem* (33)*, corresponding to the Fokker–Planck equation* (15)*,* Γ *and* Λ *are* C^1*-scalar functions of* $\mathbf{R} = (x, y, z, \xi, \eta) \in \mathbb{R}^5$, $\tau \in \mathbb{R}$, *and* $t \in \mathbb{R}^+$, *and a solution* $s = s(t)$ *of* (15) *is related to* $\mathbf{s}$ *by* (17)*. The multiplication* $\odot : V \times S \to V$ *in algebra* $Cl^{*0}_{1,4}(\mathbb{C})$ *has to satisfy the rules* (32).

(ii) *Then a solution* $\mathbf{s}$ *of* (33) *can be expressed as in* (25) *with coefficients as in* (26) *and* (24)*, and the matrices* (18) *can be specified according to* (20) *in terms of para-quaternions. Moreover, the general solution of the system* (33) *is a linear combination* (27) *with complex coefficients of* 42 *fundamental solutions which, in the case of* (16)*, are given explicitly by the formulae* (29)–(31)*, where* s_0 *is determined by* (16).

The results obtained have a clear physical significance [2], [5], [9], [11], [16], [17], [25], [26].

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