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Real hypersurfaces of e-($J^4 = 1$)-Kaehler manifolds

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§0. Introduction

($J^4 = 1$)-Kaehler manifolds were introduced in [3] as a natural generalization of both Kaehler manifolds and para-Kaehler manifolds. Several interesting results on the topic can be found in [3–5, 7].

We shall consider e-($J^4 = 1$)-Kaehler manifolds. As the metric in that case is necessarily semi-Riemannian, the theory of submanifolds has to deal with both degenerate and non-degenerate submanifolds. Such a study is initiated in the present paper for real hypersurfaces of e-($J^4 = 1$)-Kaehler manifolds. First we obtain the geometric structure induced on a real hypersurface of an e-metric ($J^4 = 1$)-manifolds. Then we give characterizations of the integrable distributions on a real hypersurface of an e-($J^4 = 1$)-Kaehler manifold by means of their second fundamental forms.

§1. Preliminaries

Let M be a real $2m$-dimensional differentiable manifold endowed with a semi-Riemannian metric g and a tensor field J of type (1,1) satisfying

$$g \circ (J \times I) + g \circ (I \times J) = 0; \quad J^4 = I,$$

where I is the identity map on TM. The characteristic polynomial of J is supposed to be $P_J(\lambda) = (\lambda^2 - 1)^r (\lambda^2 + 1)^s$, with $r + s = m$. Then (M, g, J) is called an *e-metric ($J^4 = 1$)-manifold* (cf. [3]). If moreover $\nabla J = 0$, where ∇ is the Levi-Civita connection on M with respect to g, then (M, g, J) is called an e-($J^4 = 1$)-*Kaehler manifold*. The letter e appears in these names as a consequence of the fact that the fundamental 2-form $F = g \circ (I \times J)$ of M, in the case $m = 2$, is related to the electromagnetic field.

Throughout the paper we shall denote by $\Gamma(E)$ the $F(M)$-module of differentiable sections of a vector bundle E over M, where $F(M)$ is the algebra of differentiable functions on M. We shall use $\oplus$ and $\perp$ for the Whitney sum and the orthogonal Whitney sum of vector bundles, respectively.

Consider the tensor fields $P = \frac{1}{2}(I+J^2)$ and $Q = \frac{1}{2}(I-J^2)$ and express the tangent bundle of M as follows: $TM = U \perp V$, where $U = \operatorname{Im} P$ and $V = \operatorname{Im} Q$. It is easy to check that U and V are para-holomorphic and holomorphic distributions respectively, i.e., J acts as an almost product operator on U and an almost complex operator on V and satisfies

$$g \circ (J \times J) = -g \circ (I \times I) \quad \text{and} \quad g \circ (J \times J) = g \circ (I \times I)$$

on U and V, respectively. Moreover, U and V are orthogonal distributions and (see [3]) M is locally the product of a $2r$-dimensional para-Kaehler manifold and a $2s$-dimensional Kaehler manifold, provided that M is an e-$(J^4 = 1)$-Kaehler manifold.

Let N be a real hypersurface on an e-metric $(J^4 = 1)$-manifold (M, g, J). For any $x \in N$ consider the perp-vector space $T_xN^\perp$ to T_xN in T_xM (cf. O'Neill [6]) and construct the *perp-vector bundle* $TN^\perp = \bigcup_{x \in N} T_xN^\perp$ over N. As our metric is necessarily semi-Riemannian, the induced metric on N, denoted by the same letter g, is either non-degenerate or degenerate according as the perp-vector bundle is non-degenerate or degenerate, respectively.

First we consider the case in which g is non-degenerate and call N a *non-degenerate real hypersurface* of M. In this situation the perp-vector bundle is just the normal bundle of N. Suppose ξ is a unit vector field normal to N and put

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi \tag{1.1}$$

and

$$\bar{\nabla}_X \xi = -A_\xi X, \tag{1.2}$$

for any $X, Y \in \Gamma(TN)$, where ∇ and $\bar{\nabla}$ are the Levi-Civita connections on N and M respectively, h is the second fundamental form of N and A_ξ is the shape operator of N.

Next, suppose that the induced metric is degenerate on TN, i.e., there exists $\xi \in \Gamma(TN)$, $\xi \neq 0$, such that $g(\xi, X) = 0$, for any $X \in \Gamma(TN)$.

Then we call N a *degenerate (null) real hypersurface* of M. In this case the perp-vector bundle $TN^{\perp}$ becomes a distribution on N and thus the above theory of non-degenerate real hypersurfaces is useless. In order to study the geometry of the degenerate immersion of N in M we need a vector bundle transversal to TN in TM. The construction of such a vector bundle was performed by Bejancu and Duggal [2] for degenerate hypersurfaces of semi-Riemannian manifolds as follows. First, consider the so-called *screen distribution* SN on N, which is a distribution complementary of $TN^{\perp}$ in TN. It is proved in [2] that there exists a unique vector bundle tN with rank $tN = 1$ such that for any $\xi \in \Gamma(TN^{\perp})$ there exists a unique $\bar{\xi} \in \Gamma(tN)$ satisfying

$$g(\bar{\xi}, \bar{\xi}) = g(\bar{\xi}, X) = 0, \quad g(\xi, \bar{\xi}) = 1, \quad \forall X \in \Gamma(SN). \tag{1.3}$$

In this case, Gauss and Weingarten equations for the degenerate immersion of N in M become

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)\bar{\xi} \tag{1.4}$$

and

$$\bar{\nabla}_X \bar{\xi} = -A'_{\bar{\xi}} X + \tau(X)\bar{\xi}, \tag{1.5}$$

for any $X, Y \in \Gamma(TN)$, where B is the second fundamental form of the immersion, ∇ is a torsion-free connection induced by $\bar{\nabla}$ on N, τ is a 1-form locally defined on N and $A'_{\bar{\xi}}$ is the shape operator of N. We have to note that B does not depend on the screen distribution and, in general, ∇ is not a metric connection. In this theory we need also the equations [2]:

$$\nabla_X hY = \overset{*}{\nabla}_X hY + C(X, hY)\xi \tag{1.6}$$

and

$$\nabla_X \xi = -\overset{*}{A}_{\xi} X - \tau(X)\xi, \tag{1.7}$$

for any $X, Y \in \Gamma(TN)$, where h is the projection of TN on SN, $\overset{*}{\nabla}$ is the linear connection induced by ∇ on SN, C is an $F(M)$-bilinear form on $TN \times SN$ and $\overset{*}{A}_{\xi}$ is a linear operator on $\Gamma(TN)$. Note that, in general, C is not symmetric on $SN \times SN$ but it satisfies

$$C(X, hY) = g(A'_{\bar{\xi}} X, hY), \quad \forall X, Y \in \Gamma(TN). \tag{1.8}$$

Finally, note that B is symmetric and satisfies

$$B(X, hY) = g(\overset{*}{A}_{\xi} X, hY), \quad \forall X, Y \in \Gamma(TN),$$

and

$$B(\xi, Y) = 0, \quad \forall Y \in \Gamma(TN).$$

§2. Geometric structure induced on a real hypersurface on an e-metric $(J^4 = 1)$-manifold

2.I. The case of a non-degenerate real hypersurface.

First, suppose that N is a non-degenerate real hypersurface of (M, g, J). In order to get the geometric structure induced on N we have to analyse two cases.

Case 2. I. 1. ($TN^\perp$ is a *vector sub-bundle* of U or of V). In this case $J^2(TN^\perp) = TN^\perp$ and therefore $J(TN^\perp) = J^3(TN^\perp)$. As $J(TN^\perp)$ is orthogonal to $TN^\perp$ it follows that the tangent bundle of N decomposes as follows:

$$TN = J(TN^\perp) \perp D_1, \tag{2.1}$$

where D_1 is the orthogonal distribution comlementary of $J(TN^\perp)$ in TN. Moreover, it is easy to check that D_1 is J-invariant, and therefore N is the analogous of a CR-submanifold for Kaehler manifolds (cf. [1]).

Case 2. I. 2. ($TN^\perp \cap U = \{0\}$ and $TN^\perp \cap V = \{0\}$). As g is semi-Riemannian we have to analyse two subcases:

Case 2. I. 2. a. ($J(TN^\perp)$ is a *null vector bundle*). In this case $J(TN^\perp)$ and $J^3(TN^\perp)$ are degenerate distributions on N while $J^2(TN^\perp)$ is non-degenerate. Moreover, $\{J\xi, J^2\xi, J^3\xi\}$ is a set of linearly independent local vector fields for any $\xi \in \Gamma(TN^\perp)$. Therefore, the tangent bundle of N is decomposed as follows:

$$TN = \{J(TN^\perp) \oplus J^3(TN^\perp)\} \perp J^2(TN^\perp) \perp D_2. \tag{2.2}$$

It is important to note that D_2 is J-invariant. In particular, we state:

Proposition 1. *Let N be an orientable non-degenerate real hypersurface of a 4-dimensional e-metric $(J^4 = 1)$-manifold M such that $J(TN^\perp)$ is a null vector bundle. Then N is a parallelizable manifold.*

Proof. It follows from the fact that ξ is globally defined and $\{J\xi, J^2\xi, J^3\xi\}$ is a set of linearly independent vector fields on N.

Case 2. I. 2. b. ($J(TN^\perp)$ is a *non-null vector bundle*). In this case $J^2(TN^\perp)$ is neither tangent nor normal to N at any point. However, $J(TN^\perp)$ and $J^3(TN^\perp)$ are distributions on N and we have the decomposition

$$TN = \{J(TN^\perp) \oplus J^3(TN^\perp)\} \perp D_3. \tag{2.3}$$

2. II. The case of a degenerate real hypersurface.

Now, suppose N is a degenerate real hypersurface of an e-metric (J^4=1)-manifold. Then $TN^\perp$ becomes a distribution on N and we have

$$TN = TN^\perp \perp SN, \tag{2.4}$$

where SN is a screen distribution on N. In this case we may use the e-metric structure (g, J) in order to get some particular distributions.

Case 2. II. 1. ($TN^\perp$ is a *vector sub-bundle* of U but $J(TN^\perp) \cap TN^\perp = \{0\}$). As in the non-degenerate case we have $J^2(TN^\perp) = TN^\perp$ and $J(TN^\perp) = J^3(TN^\perp)$. It follows that $J(TN^\perp)$ is a null distribution on N. Thus we can choose SN as a vector sub-bundle complementary of $TN^\perp$ in TN but such that $J(TN^\perp) \subset SN$. Then $J(tN)$ is a vector subbundle of SN too. Therefore we have the decomposition

$$TN = TN^\perp \perp \{J(TN^\perp) \oplus J(tN)\} \perp D_4, \tag{2.5}$$

where D_4 is the orthogonal distribution complementary of $J(TN^\perp) \oplus J(tN)$ in SN. Note that $\{\xi, J\xi, J\bar{\xi}\}$ is a set of linearly independent vector fields for any $\xi \in \Gamma(TN^\perp)$ and $\bar{\xi} \in \Gamma(tN)$ satisfying (1.3).

Case 2. II. 2. ($J\xi = \pm\xi$, $\forall \xi \in \Gamma(TN^\perp)$). In this case the decomposition is as in (2.4) and $J(tN)$ is not tangent to N.

Case 2. II. 3. ($TN^\perp \cap U = \{0\}$ and $TN^\perp \cap V = \{0\}$). We have to analyse the following two subcases:

Case 2. II. 3. a. ($J(TN^\perp)$ is a *null vector bundle*). Choose SN such that it contains the null distributions $J(TN^\perp)$, $J^2(TN^\perp)$ and $J^3(TN^\perp)$. It follows that ξ, $J\xi$, $J^2\xi$ and $J^3\xi$ are linearly independent on N and therefore $m > 2$. The tangent bundle of N can be decomposed as follows:

$$TN = TN^\perp \perp J(TN^\perp) \perp J^2(TN^\perp) \perp J^3(TN^\perp) \perp D_5. \tag{2.6}$$

Case 2. II. 3. b. ($J(TN^\perp)$ is a *non-null vector bundle*). It follows that $J(TN^\perp)$ and $J^3(TN^\perp)$ are non-null distributions on N while $J^2(TN^\perp)$ is a null vector bundle which is not tangent to N. Moreover ξ, $J\xi$ and $J^3\xi$ are linearly independent vector fields and therefore we have

$$TN = TN^\perp \perp J(TN^\perp) \perp J^3(TN^\perp) \perp D_6. \tag{2.7}$$

§3. Integrability of distributions on a real hypersurface of an e-($J^4 = 1$)-Kaehler manifold

Suppose N is a non-degenerate real hypersurface of an e-($J^4 = 1$)-Kaehler manifold M such that TN decomposes as in (2.1). Then, taking into account that J is parallel with respect to $\bar{\nabla}$ and using (1.1), we obtain

$$J([X,Y]) = \nabla_X JY - \nabla_Y JX + \{h(X,JY) - h(Y,JX)\}\xi \tag{3.1}$$

for any $X, Y \in \Gamma(D_1)$. Applying J^3 to (3.1) and taking into account that $TN^{\perp}$ is a vector sub-bundle either of U or of V, we obtain

$$[X,Y] = J^3(\nabla_X JY - \nabla_Y JX) \pm \{h(X,JY) - h(Y,JX)\}J\xi. \tag{3.2}$$

It follows that $J^3(\nabla_X JX - \nabla_Y JX)$ is tangent to N. Moreover we have

$$g(J^3(\nabla_X JY - \nabla_Y JX), J\xi) = -g(\nabla_X JY - \nabla_Y JX, \xi) = 0,$$

and hence $J^3(\nabla_X JY - \nabla_Y JX)$ belongs to D_1. Therefore from (3.2) we obtain:

Theorem 1. *Let N be a non-degenerate real hypersurface of an e-($J^4 = 1$)-Kaehler manifold M whose tangent vector bundle decomposes as in* (2.1). *Then the distribution D_1 is integrable iff the second fundamental form of N satisfies*

$$h(X,JY) = h(Y,JX), \quad \forall X, Y \in \Gamma(D_1).$$

In a similar way, using (1.1), (1.2) and the decompositions (2.2) and (2.3), we obtain the following characterizations of the integrability of distributions on N.

Theorem 2. *Let N be a non-degenerate real hypersurface of an e-($J^4 = 1$)-Kaehler manifold M whose tangent bundle decomposes as in* (2.2). *Then we have:*

(i) $J(TN^{\perp}) \oplus J^3(TN^{\perp})$ *is integrable iff h satisfies*

$$h(J\xi, J\xi) + h(J^3\xi, J^3\xi) = 0$$

and

$$h(J\xi, J^3X) = h(JX, J^3\xi), \quad \forall X \in \Gamma(D_2); \tag{3.3}$$

(ii) $\{J(TN^{\perp}) \oplus J^3(TN^{\perp})\} \perp J^2(TN^{\perp})$ *is integrable iff h satisfies*

$$h(J\xi, JX) + h(J^3\xi, X) = 0, \ h(J^3\xi, JX) + h(J^3\xi, X) = 0$$

and (3.3)*;*

(iii) D_2 *is integrable iff h satisfies*

$$h(JX, JY) = h(J^2X, Y) = h(X, J^2Y), \quad \forall X, Y \in \Gamma(D_2).$$

Theorem 3. *Let N be a non-degenerate real hypersurface of an e-($J^4 = 1$)-Kaehler manifold M whose tangent bundle decomposes as in* (2.3). *Then we have:*

(i) *$J(TN^\perp) \oplus J^3(TN^\perp)$ is integrable iff h satisfies*

$$h(J\xi, J^3X) = h(J^3\xi, JX) = 0, \quad \forall X \in \Gamma(D_3);$$

(ii) *D_3 is integrable iff h satisfies*

$$h(X, JY) = h(Y, JX), \quad \forall X, Y \in \Gamma(D_3).$$

Next, we consider a totally umbilical non-degenerate real hypersurface N of M, i.e., the second fundamental form is expressed as follows:

$$h(X, Y) = \lambda g(X, Y), \quad \forall X, Y \in \Gamma(TN),$$

where λ is a differentiable function on N. If λ vanishes on N we say that N is totally geodesically immersed in M. Then, using Theorems 1, 2 and 3 we obtain the following results:

Corollary 1. *Let N be a totally umbilical non-degenerate real hypersurface of an e-($J^4 = 1$)-Kaehler manifold M whose tangent bundle admits the decomposition* (2.1). *Then D_1 is involutive iff N is totally geodesic.*

Corollary 2. *Let N be a totally umbilical non-degenerate real hypersurface of an e-($J^4 = 1$)-Kaehler manifold M whose tangent bundle decomposes as in* (2.2). *Then we have:*

(i) *The distributions $J(TN^\perp) \oplus J^3(TN^\perp)$ and $\{J(TN^\perp) \oplus J^3(TN^\perp)\}\perp J^2(TN^\perp)$ are integrable;*

(ii) D_2 is integrable iff N is totally geodesic.

Corollary 3. *Let N be a totally umbilical non-degenerate real hypersurface of an e-($J^4 = 1$)-Kaehler manifold M whose tangent bundle decomposes as in* (2.3). *Then we have:*

(i) *The distribution $J(TN^\perp) \oplus J^3(TN^\perp)$ is integrable;*

(ii) *If N is totally geodesic then D_3 is integrable. If N is not totally geodesic, D_3 is integrable iff D_3 and $J(D_3)$ are orthogonal distributions.*

We now consider a degenerate real hypersurface N of an e-($J^4 = 1$)-Kaehler manifold M whose tangent bundle satisfies (2.5). Using (1.3)–(1.8) we obtain

$$g([J\xi, J\bar{\xi}], \bar{\xi}) = C(J\xi, J\bar{\xi}) - B(J\bar{\xi}, J\bar{\xi})$$

and

$$g([J\xi, J\bar{\xi}], X) = C(J\xi, JX) - B(J\bar{\xi}, JX), \quad \forall X \in \Gamma(D_4).$$

Therefore we may state:

Theorem 4. *Let N be a degenerate real hypersurface of an e-(J^4=1)-Kaehler manifold M whose tangent bundle satisfies* (2.5)*. Then we have:*

(i) $J(TN^{\perp}) \oplus J(tN)$ *is integrable iff the second fundamental forms B and C satisfy*

$$C(J\xi, J\bar{\xi}) = B(J\bar{\xi}, J\bar{\xi})$$

and

$$C(J\xi, JX) = B(J\bar{\xi}, JX), \quad \forall X \in \Gamma(D_4); \tag{3.4}$$

(ii) $TN^{\perp} \perp \{J(TN^{\perp}) \oplus J(tN)\}$ *is integrable iff B and C satisfy* (3.4) *and*

$$C(\xi, JX) = -B(J\xi, X) = -B(J\bar{\xi}, X), \quad \forall X \in \Gamma(D_4).$$

(iii) D_4 *is integrable iff C is symmetric on D_4 and satisfies*

$$C(X, JY) = C(Y, JX), \quad \forall X, Y \in \Gamma(D_4)$$

and B satisfies

$$B(X, JY) = B(Y, JX), \quad \forall X, Y \in \Gamma(D_4).$$

Similar results follow from decompositions (2.6) and (2.7). Therefore we may conclude that integrability of distributions on real hypersurfaces of e-($J^4 = 1$)-Kaehler manifolds is characterized by means of the second fundamental forms.

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