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# On functional equations involving means

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*Dedicated to Professor Lajos Tamássy on his 90th birthday*

**Abstract.** Let $I \subset \mathbb{R}$ be a non-void open interval and let $M, N : I^2 \to I$ be two mean values on $I$. We consider functional equations of type

$$K(M(x,y)) = K(N(x,y)) \quad (x, y \in I)$$

where $K : I \to \mathbb{R}$ is an unknown function.

## 1. Introduction

In the sequel $I \subset \mathbb{R}$ will be a non-void open interval. The function $M : I^2 \to I$ is said to be a mean value on $I$ if

$$\min\{x, y\} \le M(x,y) \le \max\{x, y\}$$

holds for all $x, y \in I$. Obviously $M(x,x) = x$ for all $x \in I$.

Let $M, N : I^2 \to I$ be two mean values and $K : I \to \mathbb{R}$ an unknown function for which the functional equation

$$K\left(M(x,y)\right) = K\left(N(x,y)\right) \tag{1}$$

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holds for all $x, y \in I$. In this paper we study some particular cases of functional equations of type (1).

## 2. Elementary cases

(a) Let $I \subset \mathbb{R}_+ := \{x \mid x \in \mathbb{R},\ x > 0\}$ and

$$M(x,y) := \frac{x+y}{2}, \quad N(x,y) := \sqrt{xy} \quad (x, y \in I)$$

be the arithmetic and geometric mean value. In this case equation (1) is of the form

$$K\left(\frac{x+y}{2}\right) = K\left(\sqrt{xy}\right)) \quad (x, y \in I), \tag{2}$$

where $K : I \to \mathbb{R}$ is an unknown function.

(b) Let $I \subset \mathbb{R}_+ := \{x : x \in \mathbb{R},\ x > 0\}$ and

$$M(x,y) := px + (1-p)y, \quad N(x,y) := qx + (1-q)y \quad (x, y \in I),$$

where $0 < p < 1$, $0 < q < 1$ and $p \neq q$. Now (1) has the form

$$K\left(px + (1-p)y\right) = K\left(qx + (1-q)y\right)) \quad (x, y \in I) \tag{3}$$

where $K : I \to \mathbb{R}$ is an unknown function.

(c) Let $I \subset \mathbb{R}_+$ and

$$M(x,y) := \frac{x+y}{2}, \quad N(x,y) := \frac{-1 + \sqrt{1 + 4(x + y^2)}}{2} \quad (x, y \in I).$$

It is easy to see that $N : I^2 \to I$ is a mean value and (1) is of the form

$$K\left(\frac{x+y}{2}\right) = K\left(\frac{-1 + \sqrt{1 + 4(x + y^2)}}{2}\right) \quad (x, y \in I) \tag{4}$$

where $K : I \to \mathbb{R}$ is an unknown function.

The means appearing in the cases (a), (b) and (c) all belong to the class of generalized weighted quasi arithmetic means introduced by Matkowski [5].

## 3. Generalized weighted quasi arithmetic means

Denote by $\mathcal{CM}(I)$ the class of continuous and strictly monotone increasing real functions defined on the interval $I$.

*Definition 1* (Matkowski [5]). The function $M : I^2 \to I$ is said to be a *generalized weighted quasi arithmetic mean* (briefly Matkowski mean) on $I$ if there exist functions $f, g \in \mathcal{CM}(I)$ such that

$$M(x,y) = (f+g)^{-1}\left(f(x)+g(y)\right) =: M_{f,g}(x,y) \tag{5}$$

for all $x, y \in I$.

For $x < y$, $x, y \in I$ we have

$$f(x)+g(x) < f(x)+g(y) < f(y)+g(y)$$

therefore $M_{f,g}$ is a mean indeed.

If $0 < p < 1$ and $\varphi : I \to \mathbb{R}$ is a continuous and strictly monotone function,

$$f(x) := \varepsilon_\varphi p\varphi(x), \quad g(x) := \varepsilon_\varphi (1-p)\varphi(x) \quad (x \in I)$$

with

$$\varepsilon_\varphi := \begin{cases} 1 & \text{if } \varphi \text{ is increasing} \\ -1 & \text{if } \varphi \text{ is decreasing} \end{cases}$$

then from (5)

$$M_{f,g}(x,y) = \varphi^{-1}\left(p\varphi(x)+(1-p)\varphi(y)\right) := A_{\varphi,p}(x,y) \quad (x, y \in I). \tag{6}$$

The mean (6) is the well-known weighted quasi arithmetic mean on $I$ which has a rich literature Hardy–Littlewood–Pólya [3], Aczél [1], Kuczma [4], Daróczy–Páles [2], Matkowski [5].

In case (a) with $I \subset \mathbb{R}_+$, $\varphi(x) := x$, $(x \in I)$, $p := \frac{1}{2}$ we get that $A_{\varphi,p}(x,y) = \frac{x+y}{2}$, and with $\varphi(x) := \log x$, $(x \in I)$, $p := \frac{1}{2}$ we get that $A_{\varphi,p}(x,y) = \sqrt{xy}$.

In the case (b) it is obvious that the means in the equation are weighted quasi arithmetic means.

In the case (c) $I \subset \mathbb{R}_+$ and

$$N(x,y) := \frac{-1+\sqrt{1+4(x+y^2)}}{2} = M_{f,g}(x,y) \quad (x, y \in I) \tag{7}$$

is a Matkowski mean with $f(x) = x$, $g(x) = x^2$ $(x \in I)$.

The Matkowski mean (7) is however not a weighted quasi arithmetic mean.

## 4. The main result

Our main result concerns equation (1) in the case when the means $M$, $N$ are Matkowski means.

**Theorem 1.** *Let $f, g, F, G : I \to \mathbb{R}$ be continuous and strictly monotone increasing functions, for which*

$$M_{f,g}(x,y) \neq M_{F,G}(x,y) \quad \text{if } x, y \in I,\ x \neq y. \tag{8}$$

*If $K : I \to \mathbb{R}$ satisfies the functional equation*

$$K\left(M_{f,g}(x,y)\right) = K\left(M_{F,G}(x,y)\right) \quad (x, y \in I) \tag{9}$$

*then there exists a constant $c \in \mathbb{R}$ such that*

$$K(t) = c \quad (t \in I). \tag{10}$$

Proof. For an arbitrary $\xi \in I$ choose $A, B \in I$ such that $A < \xi < B$. Let

$$E_1 := \{(x,y) \in ]A,B[^2 : M_{f,g}(x,y) = \xi\}.$$

If $x \in ]A,B[$ then we search for those $y \in ]A,B[$ for which $(x,y) \in E_1$. Then by

$$y = g^{-1}\left((f+g)(\xi) - f(x)\right)$$

we have

$$A < g^{-1}\left((f+g)(\xi) - f(x)\right) < B,$$

hence

$$f^{-1}\left((f+g)(\xi) - g(B)\right) < x < f^{-1}\left((f+g)(\xi) - g(A)\right)$$

therefore

$$\begin{aligned}\alpha := \max\{A, f^{-1}\left((f+g)(\xi) - g(B)\right)\} < x \\ < \min\{B, f^{-1}\left((f+g)(\xi) - g(A)\right\} := \beta.\end{aligned}$$

This means that for all $\alpha < x < \beta$ we have

$$\left(x, y = g^{-1}\left((f+g)(\xi) - f(x)\right)\right) \in E_1.$$

This and equation (9) imply that

$$\begin{aligned} c := K(\xi) &= K\left(M_{f,g}(x,y)\right) = K\left(M_{F,G}(x,y)\right) \\ &= K\left(M_{F,G}\left(x, g^{-1}\left((f+g)(\xi) - f(x)\right)\right)\right) = K\left(\tau_1(x)\right), \end{aligned} \tag{11}$$

where

$$\tau_1(x) := (F+G)^{-1}\left(F(x)+G\left(g^{-1}\left((f+g)(\xi)-f(x)\right)\right)\right) \quad (\alpha < x < \beta).$$

It is clear that $\alpha < \xi < \beta$ and $\tau_1(\xi) = \xi$.

We show that the function $\tau_1 : ]\alpha,\beta[ \to \mathbb{R}$ is not constant. Otherwise it were $\tau_1(x) = \xi$ for all $x \in ]\alpha,\beta[$ thus

$$(F+G)^{-1}\left(F(x)+G\left(g^{-1}\left((f+g)(\xi)-f(x)\right)\right)\right) = \xi \quad \text{for all } x \in ]\alpha,\beta[\,.$$

Hence

$$y := g^{-1}\left((f+g)(\xi)-f(x)\right) = G^{-1}\left((F+G)(\xi)-F(x)\right) \quad \text{for all } x \in ]\alpha,\beta[$$

or

$$M_{f,g}(x,y) = \xi = M_{F,G}(x,y),$$

and by the property (8)

$$x = y = g^{-1}\left((f+g)(\xi)-f(x)\right).$$

Since $f+g$ is strictly increasing this implies that $x = \xi$ which is a contradiction.

Concerning the function $\tau_1$ there are two possibilities.

(i) Either $\tau_1(]\alpha,\beta[)$ is an interval whose interior contains the point $\xi$,

(ii) or the function $\tau_1$ has an extremum in the point $\xi$.

In the case (i) by (11) there exists an open interval $I_\xi \subset I$ containing the point $\xi$ such that the function $K$ is constant (=c) on $I_\xi$.

In the case (ii) we may assume, without restricting the generality, that $\tau_1(x) \geq \xi$ if $x \in ]\alpha,\beta[$. Let

$$E_2 := \{(x,y) \in ]A,B[^2 \colon M_{F,G}(x,y) = \xi\}.$$

We easily get that $\left(x, y = G^{-1}\left((F+G)(\xi)-F(x)\right)\right) \in E_2$ if and only if $x \in ]\alpha',\beta'[$ where

$$\alpha' := \max\{A, F^{-1}\left((F+G)(\xi)-G(B)\right)\}$$
$$\beta' := \max\{B, F^{-1}\left((F+G)(\xi)-G(A)\right)\}.$$

From this by (9)

$$\begin{aligned} c := K(\xi) &= K\left(M_{F,G}(x,y)\right) = K\left(M_{f,g}(x,y)\right) \\ &= K\left(M_{f,g}\left(x, G^{-1}\left((F+G)(\xi)-F(x)\right)\right)\right) = K\left(\tau_2(x)\right), \end{aligned} \tag{12}$$

where

$$\tau_2(x) := (f+g)^{-1}\left(f(x) + g\left(G^{-1}\left((F+G)(\xi) - F(x)\right)\right)\right) \quad (\alpha' < x < \beta').$$

It is clear that $\alpha' < \xi < \beta'$ and $\tau_2(\xi) = \xi$. Again, similarly to previous discussions the function $\tau_2 : ]\alpha', \beta'[ \to \mathbb{R}$ is not constant.

Let $\alpha^* := \max\{\alpha, \alpha'\}$ and $\beta^* := \max\{\beta, \beta'\}$ then $\alpha^* < \xi < \beta^*$ and $\tau_i : ]\alpha^*, \beta^*[ \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions with $\tau_i(x) = \xi$ $(i = 1, 2)$. For $x \in ]\alpha^*, \beta^*[$ we have $\tau_1(x) \geq \xi$, or

$$(F+G)^{-1}\left(F(x) + G\left(g^{-1}\left((f+g)(\xi) - f(x)\right)\right)\right) \geq \xi.$$

From this we obtain after some calculations that

$$\xi \geq (f+g)^{-1}\left(f(x) + g\left(G^{-1}\left((F+G)(\xi) - F(x)\right)\right)\right) = \tau_2(x)$$

for all $x \in ]\alpha^*, \beta^*[$. Thus for the function

$$\tau(x) := \begin{cases} \tau_1(x) & \text{if } \xi \leq x < \beta^* \\ \tau_2(x) & \text{if } \alpha^* < x \leq \xi \end{cases}$$

by (11) and (12) we have $c = K(\tau(x))$ provided that $x \in ]\alpha^*, \beta^*[$ and $\tau\left(]\alpha^*, \beta^*[\right)$ is such a proper interval whose interior contains $\xi$.

With this we proved that for every $\xi \in I$ there is an open interval $I_\xi \subset I$ such that $\xi \in I_\xi$ and for all $t \in I_\xi$ we have $K(t) = K(\xi) = c$.

Let now $J \subset I$ be the maximal open interval on which the function $K$ is constant i.e. $K(t) = c$ for all $t \in J$. If $J \neq I$ then $J$ has and endpoint $\xi$ for which $\xi \in I$. Then there exists a nonempty open interval $I_\xi \subset I$ such that $\xi \in I_\xi$ and $K$ is constant on the interval $I_\xi$. As $\xi$ is an.endpoint of $J$, the constant $c_\xi$ can only be $c$ which contradicts to the maximality of $J$. Therefore $J = I$ proving our theorem, i.e. proving (10). □

## 5. Remarks and problems

From our theorem it follows that the functional equations in the elementary cases (a)–(c) have only constant solutions.

The following question arises: what (other than constant) solutions can equation (1) have?

It is clear, that if $M(x, y) = N(x, y)$ for all $x, y \in I$ than equation (1) gives no information on the unknown function $K : I \to \mathbb{R}$, i.e. $K$ is arbitrary.

*Remark 1.* Let $M : I^2 \to I$ be a mean on $I$ and $c \in I$ be a fixed constant. Define the function $M^{[c]} : I^2 \to I$ by

$$M^{[c]}(x,y) := \begin{cases} c & \text{if } \min\{x,y\} \le c \le \max\{x,y\} \\ M(x,y) & \text{otherwise} \end{cases} \qquad (x, y \in I.)$$

It is easy to see that $M^{[c]}$ is a mean on $I$. Let $M, N : I^2 \to I$ be two means on $I$, $c \in I$ and consider the functional equation

$$K(M^{[c]}(x,y)) = K(N^{[c]}(x,y)) \quad (x, y \in I)$$

of type (1). It is easy to check that the function

$$K(t) := \begin{cases} c_1 & \text{if } t \in I \text{ and } t < c \\ c_2 & \text{if } t \in I \text{ and } t > c \\ c_3 & \text{if } t = c \end{cases} \quad (t \in I),$$

where $c_1$, $c_2$, $c_3$ are arbitrary constants is a solution of our equation.

*Remark 2.* Obviously, the means $M^{[c]}$ and $N^{[c]}$ do not have the property

$$M^{[c]}(x,y) \ne N^{[c]}(x,y) \qquad \text{if } x \ne y \ x, y \in I.$$

Thus, we have the following question. Are there means $M$ and $N$ with the property

$$M(x,y) \ne N(x,y) \quad if x \ne y \ x, y \in I$$

such that (1) has non-constant solution.
The answer is "yes", as the following example shows.
Let $C, D \subset \mathbb{R}$ be dense subsets of $\mathbb{R}$ with $C \cap D = \emptyset$ such that the set $H := \mathbb{R} \backslash C \cap D$ is also dense in $\mathbb{R}$. Such sets exist, for example, take $C = \mathbb{Q}$, $D =$ the set of transcendental numbers and $H =$ the set of non-rational algebraic numbers. Define the functions $M, N : I^2 \to I$ as follows: If $\min\{x,y\} < \max\{x,y\}$ then choose an (arbitrary) element $c_{x,y} \in C$ such that $\min\{x,y\} < c_{x,y} < \max\{x,y\}$ and let $M(x,y) = c_{x,y}$ while for $x = y$ let $M(x,x) = x$.
Likewise, if $\min\{x,y\} < \max\{x,y\}$ then choose an (arbitrary) element $d_{x,y} \in D$ such that $\min\{x,y\} < d_{x,y} < \max\{x,y\}$ and let $N(x,y) = d_{x,y}$ while for $x = y$ let $N(x,x) = x$.

Obviously $M, N : I^2 \to I$ are means on $I$ and

$$M(x,y) \ne N(x,y) \quad \text{if } x, y \in I, \ x \ne y.$$

It is easy to check that the function

$$K(t) := \begin{cases} c & \text{if } t \in (C \cup D) \cap I, \\ \text{arbitrary} & \text{if } t \in H \cap I, \end{cases}$$

is a solution of the equation (1) thus in this case (1) has a non-constant solution $K$.

**Problem 1.** *Find necessary and sufficient conditions for the functions $f, g, F, G \in \mathcal{CM}(I)$ such that the condition*

$$M_{f,g}(x, y) \neq M_{F,G}(x, y), \quad \text{if } x, y \in I, \ x \neq y$$

*is satisfied.*

The means constructed in Remark 2 are not continuous.

**Problem 2.** *Let $M, N : I^2 \to I$ be continuous means on $I^2$, such that*

$$M(x, y) \neq N(x, y) \quad \text{if } x \neq y; \ x, y \in I.$$

*Is it true that the the functional equation $K(M(x, y)) = K(N(x, y))$ has only constant solution $K$?*

Problem 2 was mentioned in a talk at the 50th ISFE (International Symposium on Functional Equations, Hajdúszoboszló, Hungary, June 17–24, 2012). During the Symposium Antal Járai solved the problem affirmatively. His solution is to appear in the Report on the 50th ISFE, in Aequationes Math.

## References

[1] J. Aczél, Lectures on Functional Equations and Their Applications, *Academic Press, New York – London*, 1966.

[2] Z. Daróczy and Zs. Páles, Gauss-composition of means and the solution of the Matkowski–Suto problem, *Publ. Math. Debrecen* **61** (2002), 157–218.

[3] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, (first edition), 1952 (second edition), *Cambridge University Press, Cambridge*, 1934.

[4] M. Kuczma, An introduction to the theory of functional equations and inequalities, Cauchy's equation and Jensen's inequality, Panstwowe Wydawnictwo Naukowe (PWN), Warsaw, *Birkhäuser Verlag, Basel*, 1985 (first edition), 2009 (second edition).

[5] J. Matkowski, Generalized weighted and quasi-arithmetic means, *Aequationes Math.* **79** (2010), 203–212.

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