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On subgroups of the multiplicative group of the positive real numbers associated to information functions

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Dedicated to Professor Lajos Tamássy on his 90th birthday

Abstract. In this note, we solve two problems formulated in the book Aczél–Daróczy [1]. These problems refer to the connection between information functions and certain subgroups of the multiplicative group of positive real numbers. Our main result can also be considered as a new characterization of the Shannon information function.

1. Introduction

This paper is motivated by two problems, raised in the book Aczél–Daróczy [1, p. 88]. These problems have not been solved so far. Zoltán Daróczy called my attention to this fact (personal communication). These problems refer to the connection between information functions and certain subgroups of the multiplicative group of positive real numbers $\mathbb{R}_+$. An *information function* is a function $f : [0, 1] \to \mathbb{R}$ (the set of all real numbers) if $f(0) = f(1)$, $f\left(\frac{1}{2}\right) = 1$, and

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f satisfies the functional equation

$$f(x)+(1-x)f\left(\frac{y}{1-x}\right)=f(y)+(1-y)f\left(\frac{x}{1-y}\right) \tag{1.1}$$

for all $x, y \in [0,1[\,,\ x+y \leq 1$. Equation (1.1) is called the *fundamental equation of information* and plays a significant role in characterizing the Shannon entropy. The most important information function is the Shannon information function $S : [0,1] \to \mathbb{R}$ defined by

$$S(x) = -x\log_2 x - (1-x)\log_2(1-x) \quad (x \in [0,1])$$

which is the Shannon entropy belonging to the complete probability distribution $\{x, 1-x\}$, $x \in [0,1]$. Here the convention $-0\log_2 0 = 0$ is adapted (see [1]).

For a function $f : [0,1] \to \mathbb{R}$, define the set G_f in the following way. The positive real number u belongs to the set G_f if, and only if, there exist real numbers $0 < \delta_u < 1$ and $k_u > 0$ such that

$$|f(ux) - uf(x)| < k_u \quad (0 \leq x < \delta_u).$$

This concept (in somewhat different but equivalent form) was originally introduced by Lee [4] only for information functions and he also proved that this set is a multiplicative subgroup of $\mathbb{R}_+$, but later on the same was proved in [1] for any $f : [0,1] \to \mathbb{R}$.

In this paper, we give some examples for such a group and solve the following problems in [1] mentioned before.

Problem 1. If G is a proper subgroup of the multiplicative group $\mathbb{R}_+$, does there always exist an information function f such that $G = G_f$?

Problem 2. Is it true that, for any information function f different from S, we always have $G_f = \{1\}$?

2. Examples and preliminaries

Although a proof of the following statement can be found in [1, p. 85], for the sake of completeness, we give here a short proof for it, as well.

Lemma 2.1. *Let $f : [0,1] \to \mathbb{R}$ be an arbitrary function. Then G_f is a multiplicative subgroup of $\mathbb{R}_+$.*

PROOF. Obviously $1 \in G_f$. We show that $u_1 u_2^{-1} \in G_f$ whenever $u_1, u_2 \in G_f$. Indeed, by the assumption and the definition of G_f, there exist $0 < \delta_{u_i} < 1$ and $k_{u_i} \in \mathbb{R}_+$ such that $|f(u_i x) - u_i f(x)| < k_{u_i}$ for all $0 \le x < \delta_{u_i}$, $i = 1, 2$. In particular,

$$|f(u_1 x) - u_1 f(x)| < k_{u_1} \text{ if } 0 \le x < \delta_{u_1} \text{ and}$$
$$|f(u_2(u_1 u_2^{-1} x)) - u_2 f(u_1 u_2^{-1} x)| < k_{u_2} \text{ if } 0 \le x < u_2 u_1^{-1} \delta_{u_2}.$$

Thus, with the notations $u = u_1 u_2^{-1}$, $\delta = \min\{\delta_{u_1}, u^{-1}\delta_{u_2}\}$, $k = u_2^{-1}(k_{u_1} + k_{u_2})$, we have, for all $0 \le x < \delta$, that

$$\begin{aligned}
|f(ux) - uf(x)| &= u_2^{-1} |f(u_1 x) - u_1 f(x) - \big(f(u_2(u_1 u_2^{-1} x)) - u_2 f(u_1 u_2^{-1} x)\big)| \\
&\le u_2^{-1} |f(u_1 x) - u_1 f(x)| + u_2^{-1} |f(u_2(u_1 u_2^{-1} x)) - u_2 f(u_1 u_2^{-1} x)| \\
&\le u_2^{-1} k_{u_1} + u_2^{-1} k_{u_2} = k,
\end{aligned}$$

that is, $u = u_1 u_2^{-1} \in G_f$ which implies that G_f is a subgroup, indeed. □

In what follows we present examples for subgroups of type G_f corresponding to some function $f : [0, 1] \to \mathbb{R}$. Let us begin with a very simple one.

Example 1. If $f : [0, 1] \to \mathbb{R}$ is bounded then $G_f = \mathbb{R}_+$, consequently $G_S = \mathbb{R}_+$ where S is the Shannon information function.

In the next three examples, the basic properties of the real additive functions play an important role. We say that a function $a : \mathbb{R} \to \mathbb{R}$ is *additive* if

$$a(x + y) = a(x) + a(y) \quad \text{for all } x, y \in \mathbb{R}.$$

It is known (see e.g. [3, pp. 121, 206, 207, 223]) that, if the additive function $a : \mathbb{R} \to \mathbb{R}$ is bounded below or above on a set of positive Lebesgue measure then $a(x) = a(1)x$ for all $x \in \mathbb{R}$. On the other hand, there are discontinuous additive functions, as well. Especially there is an additive function a that vanishes at all rational points but $a(\sqrt{2}\,) = 1$ (see [3, pp. 78, 121]).

Example 2. Let a be a discontinuous additive function and

$$f(x) = \begin{cases} x a(\log_2 x), & \text{if } x \in]0, 1[\\ 0, & \text{if } x = 0. \end{cases}$$

It is easy to see that f is not bounded (neither below nor above) on any subinterval of positive length, nevertheless $G_f = \mathbb{R}_+$.

Example 3. Let a be a discontinuous additive function again and

$$f(x) = \begin{cases} a(\log_2 x), & \text{if } x \in]0,1[\\ 0, & \text{if } x = 0. \end{cases}$$

Then f is not bounded (neither below nor above) on any subinterval of positive length, nevertheless $G_f = \{1\}$. Otherwise, if a were bounded on an interval of positive length then it would be continuous.

All the above examples represent extreme cases. Our last example shows an intermediate case.

Example 4. Let a be an additive function vanishing at the rational points and satisfying the additional requirement $a(\sqrt{2}\,) = 1$, and let f be the restriction of a to the interval $[0,1]$. Then, of course, G_f contains the multiplicative subgroup of positive rational numbers. We show that $\sqrt{2} \notin G_f$. Otherwise, we would have that the additive function $x \mapsto a(\sqrt{2}x) - \sqrt{2}a(x)$ is bounded on an interval of positive length. Thus

$$a(\sqrt{2}x) - \sqrt{2}a(x) = \big(a(\sqrt{2}) - \sqrt{2}a(1)\big)x = x$$

would follow for all $x \in \mathbb{R}$. However this does not hold at the point $x = \sqrt{2}$. This argument shows that G_f is a proper subgroup of $\mathbb{R}_+$ that contains the multiplicative subgroup of the positive rational numbers.

In the next section, we prove our main result. To do this, we shall apply the following two known theorems. The first one is proved in Didderich [2] and also in Maksa [5] while the proof of the second one is in [1, p. 100].

Theorem 2.2. *If an information function f is bounded on a set of positive Lebesgue measure then $f = S$ on $[0,1]$.*

Theorem 2.3. *A function $f : [0,1] \to \mathbb{R}$ is an information function if, and only if, there exists a function $\varphi : [0,+\infty[\to \mathbb{R}$ such that $\varphi\big(\frac{1}{2}\big) = \frac{1}{2}$,*

$$\varphi(xy) = x\varphi(y) + y\varphi(x) \quad (x, y \in [0,+\infty[) \tag{2.1}$$

and

$$f(x) = \varphi(x) + \varphi(1-x) \quad (x \in [0,1])\,. \tag{2.2}$$

The functional equation (2.1) is well-known and it is easy to see that a function $\varphi : [0,+\infty[\to \mathbb{R}$ is a solution to it if, and only if, $\varphi(0) = 0$ and the function defined on $\mathbb{R}$ by $x \mapsto 2^{-x}\varphi(2^x)$ is additive. In the computations of the next section, we often use the equality

$$\varphi\left(\frac{x}{y}\right) = \frac{y\varphi(x) - x\varphi(y)}{y^2} \quad (x \geq 0,\ y > 0) \tag{2.3}$$

which is a simple consequence of (2.1).

3. The main result

The main result of this note is contained in the following

Theorem 3.1. *Suppose that $f : [0,1] \to \mathbb{R}$ is an information function and $G_f \neq \{1\}$. Then $f = S$, consequently $G_f = G_S = \mathbb{R}_+$.*

Proof. Because of the assumption there are real numbers $0 < u \neq 1, 0 < k$, and $0 < \delta < 1$ such that

$$|f(ux) - uf(x)| < k \quad (0 \leq x < \delta). \tag{3.1}$$

Since G_f is a group we may (and do) suppose that $u < 1$. On the other hand, by Theorem 2.2., there exists a function $\varphi : [0, +\infty[\to \mathbb{R}$ such that $\varphi\big(\frac{1}{2}\big) = \frac{1}{2}$ and φ has the properties (2.1) and (2.2). Therefore (3.1) and (2.1) imply that

$$-k - x\varphi(u) < \varphi(1-ux) - u\varphi(1-x) < k - x\varphi(u) \quad (0 \leq x < \delta). \tag{3.2}$$

Let $x, y \in \mathbb{R}_+$ such that $\frac{x}{x+y} < \delta$ and substitute x in (3.2) by $\frac{x}{x+y}$ to obtain that

$$-k - \frac{x}{x+y}\varphi(u) < \varphi\left(1 - \frac{ux}{x+y}\right) - u\varphi\left(\frac{y}{x+y}\right) < k - \frac{x}{x+y}\varphi(u)$$

which, by using (2.3), can be written in the form

$$-k(x+y) - x\varphi(u) < \varphi\left((1-u)x + y\right) - u\varphi(y) - (1-u)\varphi(x+y) < k(x+y) - x\varphi(u),$$

and this line of inequalities holds provided that $x, y \in \mathbb{R}_+$, $\frac{x}{x+y} < \delta$. In the next step, suppose that $x > y > 0$, $y > (1-\delta)x$, and replace x by $x - y$ in the above inequalities. Then we get that

$$-kx - (x-y)\varphi(u) < \varphi\left((1-u)x + uy\right) - u\varphi(y) - (1-u)\varphi(x) < kx - (x-y)\varphi(u).$$

Substitute here x by $\frac{x}{1-u}$ and y by $\frac{y}{u}$, respectively and use (2.3) again to simplify the inequalities so obtained. Thus, after some calculations, we arrive at the inequalities

$$
\begin{aligned}
-k\frac{x}{1-u} + \frac{x}{1-u}\left(\varphi(u)+\varphi(1-u)\right) &< -\varphi(x+y)+\varphi(x)+\varphi(y) \\
&< k\frac{x}{1-u} + \frac{x}{1-u}\left(\varphi(u)+\varphi(1-u)\right)
\end{aligned}
$$

which hold for all $x, y \in \mathbb{R}_+$ satisfying the requirement $(1-\delta)ux < (1-u)y < ux$. Let finally $1-u < x < \frac{1-u}{1-\delta u}$ and $y = 1-x$ in the above inequalities. Then, by (2.2), we have that

$$
\frac{x}{1-u}(f(u)-k) < f(x) < \frac{x}{1-u}(f(u)+k)
$$

for all $x \in \left]1-u, \frac{1-u}{1-\delta u}\right[$. This shows that f is bounded on an interval of positive length therefore, by Theorem 2.2., we have that $f = S$. Since S is bounded $G_f = G_S = \mathbb{R}_+$ follows. $\square$

Remark. Our theorem gives the solutions of the motivating problems. Concerning *Problem 1.*, it is not true that, if G is any proper subgroup of the multiplicative group $\mathbb{R}_+$ then there exists an information function f such that $G = G_f$. Moreover, there are only two subgroups ($\{1\}$ and $\mathbb{R}_+$ itself) which occur as a subgroup G_f with some information function f. We may discuss however the modified question: If G is any proper subgroup of the multiplicative group $\mathbb{R}_+$ then does there exist *a function* $f : [0,1] \to \mathbb{R}$ such that $G = G_f$? Concerning *Problem 2.*, it is true that, for any information function f different from S, we always have $G_f = \{1\}$.

References

[1] J. Aczél and Z. Daróczy, On measures of information and their characterizations, *Academic Press, New York, San Francisco, London*, 1975.

[2] G. T. Diderrich, Boundedness on a set of positive measure and the fundamental equation of information, *Publ. Math. Debrecen* **33** (1986), 1–7.

[3] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities, *Państwowe Wydawnictwo Naukowe – Uniwersytet Śląski, Warszawa, Kraków, Katowice*, 1985.

[4] P. M. Lee, On the axioms of information theory, *Ann. Math. Statist.* **35** (1964), 415–418.

[5] Gy. Maksa, Bounded symmetric information functions, *C.R. Math. Rep. Acad. Sci. Canada* **2** (1980), 247–252.

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